W-JAFFARD DOMAINS IN PULLBACKS

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Abstract. In this paper we study the class of w-Jaffard domains in pullback constructions, and give new examples of these domains. In particular we give examples to show that the two classes of w-Jaffard and Jaffard domains are incomparable. As another application, we establish that for each pair of positive integers (n, m) with $n+1 \leq m \leq 2n+1$, there is an (integrally closed) integral domain R such that w-$\dim(R) = n$ and $w[X]$-$\dim(R[X]) = m$.

1. Introduction

Throughout this paper, R denotes a (commutative integral) domain with identity with quotient field $qf(R)$. Let X be an algebraically independent indeterminate over R. In [26, Theorem 2] Seidenberg proved that if R has finite Krull dimension, then

$$\dim(R) + 1 \leq \dim(R[X]) \leq 2(\dim(R)) + 1.$$

Moreover, Krull [18] showed that if R is any finite-dimensional Noetherian ring, then $\dim(R[X]) = 1 + \dim(R)$ (cf. also [26, Theorem 9]). Seidenberg subsequently proved the same equality in case R is any finite-dimensional Prüfer domain. To unify and extend such results on Krull-dimension, Jaffard [17] introduced and studied the *valuative dimension*, denoted by $\dim_v(R)$, for a domain R. This is the maximum of the ranks of the valuation overrings of R. Jaffard proved in [17, Chapitre IV] that, if R has finite valuative dimension, then $\dim_v(R[X]) = 1 + \dim_v(R)$, and that if R is a Noetherian or a Prüfer domain, then $\dim(R) = \dim_v(R)$. In [1] Anderson, Bouvier, Dobbs, Fontana and Kabbaj introduced the notion of *Jaffard domains*, as finite dimensional integral domains R such that $\dim(R) = \dim_v(R)$, and studied this class of domain systematically (see also [6]).

The v, t and w-operations in integral domains are of special importance in multiplicative ideal theory and were investigated by many authors in the 1980's. Ideal w-multiplication converts ring notions such as Dedekind, Noetherian, Prüfer, and quasi-Prüfer, respectively to Krull, strong Mori, PvMD, and UMt. As the w-counterpart of Jaffard domains, in [22], we introduced the class of *w-Jaffard domains*, as integral domains R such that $w\text{-}\dim(R) = w\text{-}\dim_v(R) < \infty$. In this paper we study the transfer of w-Jaffard domains in pullback constructions, in order to provide original examples.

We need to recall some notions about star operations. Let $F(R)$ denotes the set of nonzero fractional ideals, and $f(R)$ be the set of all nonzero finitely generated

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fractional ideals of R. Let $*$ be a star operation on the domain R. For every $A \in F(R)$, put $A^{*_f} := \bigcup F^*$, where the union is taken over all $F \in f(R)$ with $F \subseteq A$. It is easy to see that $*_f$ is a star operation on R. A star operation $*$ is said to be *of finite character* if $* = *_f$. We say that a nonzero ideal I of R is a *$*$-ideal* if $I^* = I$, a *$*$-prime* if I is a prime $*$-ideal of R, a *$*$-maximal* if I is maximal in the set of $*$-prime ideals of R. The set of $*$-maximal ideals of R is denoted by $*$-Max(R). It has become standard to say that a star operation $*$ is *stable* if $(A \cap B)^* = A^* \cap B^*$ for all A, $B \in F(R)$.

Given a star operation $*$ on an integral domain R, it is possible to construct a star operation $\widetilde{*}$, which is stable and of finite character defined as follows: for each $A \in F(R)$,

$$A^{\widetilde{*}} := \{x \in qf(R) | xJ \subseteq A, \text{ for some } J \subseteq R, J \in f(R), J^* = R\}.$$

The most widely studied star operations on R have been the identity d, v, $t := v_f$, and $w := \widetilde{v}$ operations, where $A^v := (A^{-1})^{-1}$, with $A^{-1} := (R : A) := \{x \in qf(R) | xA \subseteq R\}$. In this work we mostly deal with the w-operation.

It is well-known that t-Max$(R) = w$-Max(R), every t-prime ideal is a w-prime ideal, and that every prime subideal of a prime w-ideal of R is also a w-ideal.

Let $*$ be a star operation on a domain R. The *$*$-Krull dimension of R* is defined as

$$*\text{-}\dim(R) := \sup\{n | P_1 \subset \cdots \subset P_n \text{ where } P_i \text{ is } *\text{-prime}\}.$$

If the set of $*$-prime ideals is an empty set then pose $*\text{-}\dim(R) = 0$. Note that, the notions of $\widetilde{*}$-dimension, t-dimension, and of w-dimension have received a considerable interest by several authors (cf. for instance, [22, 23, 24, 14, 15, 28, 29]).

Now we recall a special case of a general construction for semistar operations (see [22]). Let X, Y be two indeterminates over R, and let $K := qf(R)$. Set $R_1 := R[X]$, $K_1 := K(X)$ and take the following subset of Spec(R_1):

$$\Theta_1^w := \{Q_1 \in \mathrm{Spec}(R_1) |\ Q_1 \cap R = (0) \text{ or } (Q_1 \cap R)^w \subsetneq R\}.$$

Set $\mathfrak{S}_1^w := R_1[Y] \backslash (\bigcup \{Q_1[Y] | Q_1 \in \Theta_1^w\})$ and:

$$E^{\circlearrowleft_{\mathfrak{S}_1^w}} := E[Y]_{\mathfrak{S}_1^w} \cap K_1, \text{ for all } E \in F(R_1).$$

It is proved in [22, Theorem 2.1] that, the mapping $w[X] :=\circlearrowleft_{\mathfrak{S}_1^w}: F(R_1) \to F(R_1)$, $E \mapsto E^{w[X]}$ is a stable star operation of finite character on $R[X]$, i.e., $\widetilde{w[X]} = w[X]$. If $X_1, \cdots, X_r$ are indeterminates over R, for $r \geq 2$, we let

$$w[X_1, \cdots, X_r] := (w[X_1, \cdots, X_{r-1}])[X_r].$$

For an integer r, let $w[r]$ denote $w[X_1, \cdots, X_r]$, and $R[r]$ to denote $R[X_1, \cdots, X_r]$.

Proposition 1.1. *(*[22, Theorem 3.1]*) For each positive integer r and for $n := w\text{-}\dim(R)$ we have*

$$r + n \leq w[r]\text{-}\dim(R[r]) \leq r + (r + 1)n.$$

Proposition 1.2. *(*[24, Lemma 4.4]*) Let R be an integral domain and n be an integer. Then*

$$w[n]\text{-}\dim(R[n]) = \sup\{\dim(R_M[n]) | M \in w\text{-}\mathrm{Max}(R)\}.$$

A valuation overring V of R, is called a *w-valuation overring of R*, provided $F^w \subseteq FV$, for each $F \in f(R)$. Following [22], the *w-valuative dimension* of R is defined as:

$$w\text{-}\dim_v(R) := \sup\{\dim(V) | V \text{ is } w\text{-valuation overring of } R\}.$$

Proposition 1.3. *(*[24, Lemma 2.5]*) For each domain R,*

$$w\text{-}\dim_v(R) = \sup\{\dim_v(R_P) | P \in w\text{-}\mathrm{Max}(R)\}.$$

Proposition 1.4. *(*[24, Theorem 4.2]*) Let R be an integral domain, and n be a positive integer. Then the following statements are equivalent:*

(1) $w\text{-}\dim_v(R) = n$.
(2) $w[n]\text{-}\dim(R[n]) = 2n$.
(3) $w[r]\text{-}\dim(R[r]) = r + n$ *for all* $r \geq n - 1$.

It is observed in [22] that $w\text{-}\dim(R) \leq w\text{-}\dim_v(R)$. We say that R is a *w-Jaffard domain*, if $w\text{-}\dim(R) = w\text{-}\dim_v(R) < \infty$. It is proved in [22], that R is a w-Jaffard domain if and only if

$$w[n]\text{-}\dim(R[n]) = w\text{-}\dim(R) + n,$$

for each positive integer n.

Recall that an integral domain is called a *strong Mori domain* if it satisfies the ascending chain condition on w-ideals (cf. [30]). Also recall that an integral domain R is called a *UMt-domain*, if every upper to zero in $R[X]$ is a maximal t-ideal [16, Section 3]. It is shown in [22, Corollary 4.6 and Theorem 4.14] that a strong Mori domain or a UMt domain of finite w-dimension is a w-Jaffard domain. In particular every Krull domain is a w-Jaffard domain (of w-dimension 1).

If $F \subseteq K$ are fields, then $\mathrm{tr.deg.}(K/F)$ stands for the *transcendence degree* of K over F. Let T be an integral domain, M a maximal ideal of T, $k = T/M$ and $\varphi : T \to k$ the canonical surjection. Let D be a proper subring of k and $R = \varphi^{-1}(D)$ be the pullback of the following diagram:

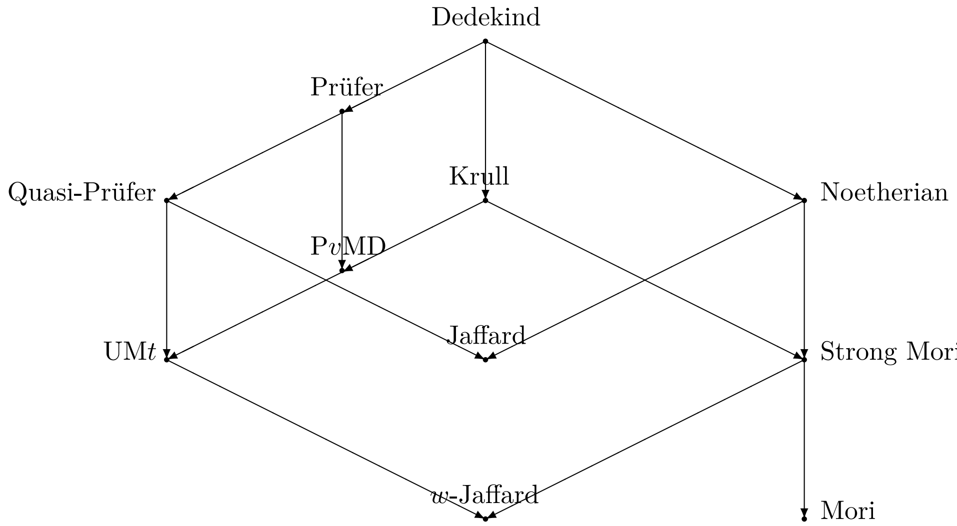
$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

In Section 2 we prove that if $F := qf(D)$ then:

(1) $w\text{-}\dim(R) = \max\{w\text{-}\dim(T), w\text{-}\dim(D) + \dim(T_M)\}$.
(2) $w\text{-}\dim_v(R) = \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(D) + \dim_v(T_M) + \mathrm{tr.deg.}(k/F)\}$.
(3) If T is quasilocal, then R is a w-Jaffard domain if and only if D is a w-Jaffard domain, T is a Jaffard domain, and k is algebraic over F.

Using these results, in Section 3 we give examples to show that the two classes of w-Jaffard and Jaffard domains are incomparable, and an example of a w-Jaffard domain which is not a strong Mori nor a UMt domain. Also we observe that a Mori domain need not be a w-Jaffard domain. As another application in Section 4 we prove that for any pair of positive integers (n, m) with $n + 1 \leq m \leq 2n + 1$, there is an integrally closed integral domain R such that $w\text{-}\dim(R) = n$ and $w[X]\text{-}\dim(R[X]) = m$, which is similar to a result of Seidenberg [27].

For the convenience of the reader, the following displays a diagram of implications summarizing the relations between the main classes of integral domains involved in this work.



A ring-theoretic perspective for w-Jaffard property.

2. Pullbacks

It is shown in [22, Theorem 4.14] that, a UMt domain of finite w-dimension is a w-Jaffard domain. Now we give an example of a w-Jaffard non UMt domain. Recall that recently Houston and Mimouni in [15, Theorem 4.2] proved that, if m, n are integers with $1 \leq m \leq n$, and $B \subseteq \{2, \cdots, n\}$ with $|B| = n - m$, then there exists a local Noetherian domain R such that $\dim(R) = n$, t-$\dim(R) = m$, and for each $i \in B$, every prime ideal of height i is a non-t-prime. Now let $n = 3$, $m = 2$ and $B = \{3\}$. Then there exists a local Noetherian domain $(R, \mathfrak{m})$ such that $\dim(R) = 3 = \mathrm{ht}(\mathfrak{m})$, t-$\dim(R) = 2$, and that $\mathfrak{m}$ is a non-t-prime. Consequently we have w-$\dim(R) = 2$. Since R is Noetherian thus it is strong Mori and hence is a w-Jaffard domain. But R is not a UMt domain since w-$\dim(R) = 2$ (cf. [16, Theorem 3.7]). In Example 3.3 we will give a w-Jaffard domain which is not a strong Mori nor a UMt domain.

To avoid unnecessary repetition, let us fix the notation. Let T be an integral domain, M a maximal ideal of T, $k = T/M$ and $\varphi : T \to k$ the canonical surjection. Let D be a proper subring of k and $R = \varphi^{-1}(D)$ be the pullback of the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

We assume that $R \subsetneq T$, and we refer to this diagram as a diagram of type $(\square)$ and if the quotient field of D is equal to k, we refer to the diagram as a diagram of type $(\square^*)$. The case where $T = V$ is a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V is of crucial interest, known as classical "$D + M$" construction.

Recall that $(R : T) = M$ is a prime ideal of R and therefore M is a divisorial ideal (or a v-ideal) of R. Thus M is a w-prime ideal of R. Also recall that $R/M \simeq D$, and R and T have the same quotient field. Moreover, T is quasilocal if and only if every ideal of R is comparable (under inclusion) to M. For each prime ideal P of R with $P \not\supseteq M$, there is a unique prime ideal Q of T with $Q \cap R = P$ and such that $R_P = T_Q$. For more details on general pullbacks, we refer the reader to [7, 11, 12], and [4] for classical $D + M$ constructions.

Lemma 2.1. *For a diagram of type* ($\square$)*, suppose that P is a prime ideal of D and Q is a prime ideal of R such that $Q = \varphi^{-1}(P)$. Then P is a w-prime (resp. w-maximal) ideal of D if and only if Q is a w-prime (resp. w-maximal) ideal of R*

Proof. By [20, Lemma 3.1] we have $Q^w = \varphi^{-1}(P^w)$. So that if P is a w-prime ideal of D then Q is w-prime ideal of R. Conversely if Q is a w-prime ideal of R, then we have $\varphi^{-1}(P^w) = \varphi^{-1}(P)$. Let $a \in P^w$. Then $\varphi^{-1}(a) \subseteq \varphi^{-1}(P^w) = \varphi^{-1}(P)$. So that $a \in P$ since $\varphi^{-1}(a) \neq \emptyset$. Thus $P^w = P$. The other assertion is clear. $\square$

It is well-known that [7, Proposition 2.1(5)] for a diagram of type ($\square$), if T is quasilocal, we have $\dim(R) = \dim(D) + \dim(T)$. The following proposition gives a satisfactory analogue of this equality.

Proposition 2.2. *For a diagram of type* ($\square$)*, assume that T is quasilocal. Then*

$$w\text{-}\dim(R) = w\text{-}\dim(D) + \dim(T).$$

Proof. Let $n := w\text{-}\dim(R)$, $s := w\text{-}\dim(D)$, and $t := \dim(T)$. Suppose that $P_1 \subset \cdots \subset P_s$ is a chain of w-prime ideals of D. Let $Q_i := \varphi^{-1}(P_i)$ which is a w-prime ideal of R by Lemma 2.1. Thus $M \subset Q_1 \subset \cdots \subset Q_s$. Also consider a chain $L_1 \subset \cdots \subset L_t = M$ of prime ideals of T. Note that each L_j is a w-prime ideal of R. Now we have a chain $L_1 \subset \cdots \subset L_t = M \subset Q_1 \subset \cdots \subset Q_s$ of distinct w-prime ideals of R. This means that $s + t \leq n$. Conversely suppose that $L_1 \subset \cdots \subset L_r = M \subset Q_1 \subset \cdots \subset Q_u$ is a chain of distinct w-prime ideals of R such that $r + u = n$. Thus $L_1 \subset \cdots \subset L_r = M$ is a chain of prime ideals of T and hence $r \leq t$. On the other hand by setting $P_i := Q_i/M$, we have a chain $P_1 \subset \cdots \subset P_u$ of w-prime ideals of D by Lemma 2.1, and hence $u \leq s$. Therefore $n = r + u \leq t + s$ completing the proof. $\square$

Remark 2.3. *For a diagram of type* ($\square$)*, assume that T is quasilocal and $D = F$ is a field. Then by* [21, Theorem 3.1(2)]*, we have that R is a DW-domain (that is the d- and w-operations are the same). Hence* $w\text{-}\dim(R) = \dim(R)$*. So that the equality in Lemma 2.2 would be* $(\dim(R) =) w\text{-}\dim(R) = \dim(T)$.

The following proposition is inspired by [1, Proposition 2.3].

Proposition 2.4. *For a diagram of type* ($\square^*$)*, assume that T is quasilocal. Then:*

(a) $w[r]\text{-}\dim(R[r]) = w[r]\text{-}\dim(D[r]) + \dim(T[r]) - \dim(k[r])$ *for each positive integer r,*
(b) $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + \dim_v(T)$,
(c) *R is a w-Jaffard domain $\Leftrightarrow$ D is a w-Jaffard domain and T is a Jaffard domain.*

Proof. (a) By Proposition 2.2 we have $w\text{-}\dim(R) < \infty$ if and only if $w\text{-}\dim(D) < \infty$ and $\dim(T) < \infty$. Hence $w[r]\text{-}\dim(R[r]) < \infty$ if and only if $w[r]\text{-}\dim(D[r])$ and $\dim(T[r])$ are finite numbers by Proposition 1.1. Thus we can assume that each domain is finite (w-)dimensional.

By [24, Lemma 4.4 and Corollary 4.5], there is a w-maximal ideal $\mathfrak{q}$ of R and a $w[r]$-maximal ideal Q of $R[r]$ such that $\mathfrak{q} = Q \cap R$, and $w[r]\text{-}\dim(R[r]) = \operatorname{ht}(Q) = r + \operatorname{ht}(\mathfrak{q}[r]) = \dim(R_{\mathfrak{q}}[r])$. Note that since M is a w-prime ideal of R we have $M \subseteq \mathfrak{q}$. Thus by Lemma 2.1 we have that $P := \mathfrak{q}/M$ is a w-maximal ideal of D. Next we claim that $\sup\{\dim(D_L[r]) | L \in w\text{-}\operatorname{Max}(D)\} = \dim(D_P[r])$, then Proposition 1.2 will implies that $w[r]\text{-}\dim(D[r]) = \dim(D_P[r])$. Let $L \in w\text{-}\operatorname{Max}(D)$ and set $\mathfrak{q}_0 := \varphi^{-1}(L)$. We have the following diagrams:

$$\begin{array}{ccc} R_{\mathfrak{q}_0} & \longrightarrow & D_L \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k, \end{array} \qquad \begin{array}{ccc} R_{\mathfrak{q}} & \longrightarrow & D_P \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

Therefore by [1, Proposition 2.3] we have $\dim(D_L[r]) + \dim(T[r]) - r = \dim(R_{\mathfrak{q}_0}[r]) \leq \dim(R_{\mathfrak{q}}[r]) = \dim(D_P[r]) + \dim(T[r]) - r$, where the inequality holds by Proposition 1.2. Thus $\dim(D_L[r]) \leq \dim(D_P[r])$ for each $L \in w\text{-}\operatorname{Max}(D)$, and hence $\sup\{\dim(D_L[r]) | L \in w\text{-}\operatorname{Max}(D)\} = \dim(D_P[r])$. Therefore we have

$$w[r]\text{-}\dim(R[r]) = w[r]\text{-}\dim(D[r]) + \dim(T[r]) - \dim(k[r]).$$

(b) First suppose that $w\text{-}\dim_v(R) < \infty$. Then $w\text{-}\dim(D) + \dim(T) = w\text{-}\dim(R) < \infty$, and so both $w\text{-}\dim(D)$ and $\dim(T)$ are finite. In addition we claim that $\dim_v(T) < \infty$. To this end let (V, N) be a valuation overring of T and set $P := N \cap T$. So that $P \subseteq M$ and thus P is a prime ideal of R. Hence P is in fact a w-prime ideal of R. Since $R_P \subseteq T_{R \backslash P} \subseteq T_P \subseteq V$ we obtain that V is a w-valuation overring of R by [10, Theorem 3.9], and consequently $\dim(V) \leq w\text{-}\dim_v(R)$. This means that $\dim_v(T) \leq w\text{-}\dim_v(R) < \infty$. Next we observe that $w\text{-}\dim_v(D) < \infty$. By Proposition 1.3, there exists a w-prime ideal P of D such that $w\text{-}\dim_v(D) = \dim_v(D_P)$. Let $Q := \varphi^{-1}(P)$, which is a w-prime ideal of R. Note that we have $M \subseteq Q$ and thus $(R \backslash Q) \cap M = \emptyset$, and $\varphi(R \backslash Q) = D \backslash P$. Therefore by [7, Proposition 1.9] we have the following pullback diagram:

$$\begin{array}{ccc} R_Q & \longrightarrow & D_P \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

If B is an n-dimensional overring of D_P, then $A := \varphi^{-1}(B)$ is an overring of R_Q, and [7, Proposition 2.1(5)] yields that $n + \dim(T) = \dim(A)$. Thus $n + \dim(T) \leq \dim_v(R_Q)$. This means that $w\text{-}\dim_v(D) = \dim_v(D_P) \leq \dim_v(R_Q) \leq w\text{-}\dim_v(R) < \infty$, where the second inequality holds by Proposition 1.3.

Let r be a positive integer such that $r \geq \max\{w\text{-}\dim_v(R), w\text{-}\dim_v(D), \dim_v(T)\} - 1$. Then by Proposition 1.4 and [2, Theorem 6] we have

$$\begin{aligned} w[r]\text{-}\dim(R[r]) &= w\text{-}\dim_v(R) + r, \\ w[r]\text{-}\dim(D[r]) &= w\text{-}\dim_v(D) + r, \\ \dim(T[r]) &= \dim_v(T) + r. \end{aligned}$$

Then by (a), $w\text{-}\dim_v(R) + r = (w\text{-}\dim_v(D) + r) + (\dim_v(T) + r) - r$, yielding (b) in case $w\text{-}\dim_v(R) < \infty$.

To complete the proof of (b) we show that $w\text{-}\dim_v(R) < \infty$ whenever $w\text{-}\dim_v(D)$ and $\dim_v(T)$ are both finite. Let r be a positive integer such that

$$r \geq \max\{w\text{-}\dim_v(D), \dim_v(T)\} - 1.$$

Then by (a), Proposition 1.4, and [2, Theorem 6] we have $w[r]\text{-}\dim(R[r]) = w[r]\text{-}\dim(D[r]) + \dim(T[r]) - r = (w\text{-}\dim_v(D)+r) + (\dim_v(T)+r) - r = w\text{-}\dim_v(D) + \dim_v(T) + r$. Hence $w\text{-}\dim_v(R) < \infty$ by another appeal to Proposition 1.4.

(c) Since $w\text{-}\dim(R) = w\text{-}\dim(D) + \dim(T)$ and $w\text{-dim}(B) \leq w\text{-dim}_v(B)$ and $\dim(B) \leq \dim_v(B)$ for a domain B, (c) follows directly from (b). ☐

Recall from [5] the notion of CPI (complete pre-image) extension of a domain R with respect to a prime ideal P of R; this is denoted $R(P)$ and is defined by the following pullback diagram:

$$\begin{array}{ccc} R(P) & \longrightarrow & R/P \\ \downarrow & & \downarrow \\ R_P & \xrightarrow{\varphi} & R_P/PR_P. \end{array}$$

Here φ is the canonical homomorphism.

Corollary 2.5. *Let R be an integral domain, and let P be a prime ideal of R. Then the CPI-extension $R(P)$ is a w-Jaffard domain $\Leftrightarrow$ R/P is a w-Jaffard domain and R_P is a Jaffard domain.*

In [1, Theorem 2.6], Anderson, Bouvier, Dobbs, Fontana and Kabbaj proved that for a diagram of type (☐) such that T is quasilocal and $F := qf(D)$ then $\dim_v(R) = \dim_v(D) + \dim_v(T) + \operatorname{tr.deg.}(k/F)$ and hence R is a Jaffard domain if and only if D and T are Jaffard domains and k is algebraic over F. Now we have:

Theorem 2.6. *For a diagram of type (☐), assume that T is quasilocal and let $F = qf(D)$. Then*

(a) $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + \dim_v(T) + \operatorname{tr.deg.}(k/F)$,

(b) *R is a w-Jaffard domain $\Leftrightarrow$ D is a w-Jaffard domain, T is a Jaffard domain, and k is algebraic over F.*

Proof. (a) Split the pullback diagram (☐) into two parts:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S := \varphi^{-1}(D) & \longrightarrow & F \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

Now the upper diagram is of type (☐*), and S is quasilocal. Thus by Proposition 2.4(b) we have $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + \dim_v(S)$. Also from the lower diagram, [1, Proposition 2.5] yields that $\dim_v(S) = \dim_v(T) + \operatorname{tr.deg.}(k/F)$. We thus have the desired equality.

(b) Since $w\text{-}\dim(R) = w\text{-}\dim(D) + \dim(T)$, (b) follows directly from (a). ☐

In Example 3.1 we will give an example of a w-Jaffard domain which is not Jaffard. Using Theorem 2.6 together with [1, Theorem 2.6] we have the following corollary.

Corollary 2.7. *For a diagram of type ($\square$), assume that T is quasilocal and let $F = qf(D)$. Then R is a w-Jaffard domain which is not Jaffard $\Leftrightarrow$ D is a w-Jaffard domain which is not Jaffard, T is a Jaffard domain and k is algebraic over F.*

We pause here to give some concrete applications of the above theory to the classical $D + M$ constructions.

Corollary 2.8. *Let V be a nontrivial valuation domain of the form $V = K + M$, where K is a field and M is the maximal ideal of V. Let $R = D + M$, where D is a proper subring of K and let $F = qf(D)$. Then*

(a) $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + \dim(V) + \text{tr.deg.}(K/F)$,
(b) *R is a w-Jaffard domain $\Leftrightarrow$ D is a w-Jaffard domain, V is finite-dimensional, and K is algebraic over F.*

A "global" type of $D+M$ constructions arise from $T = K[[X]]$, the formal power series ring over a field K, by considering $M = XT$ and a subring D of K.

Corollary 2.9. *Let K be a field, D a subring of K with quotient field F, $R = D + XK[[X]]$. Then*

(a) $w\text{-}\dim(R) = w\text{-}\dim(D) + 1$,
(b) $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + \text{tr.deg.}(K/F) + 1$.
(c) *R is a w-Jaffard domain $\Leftrightarrow$ D is a w-Jaffard domain and K is algebraic over F.*

We next proceed to generalize the previous "quasilocal" theory. In this direction we prove the "global" analogue of Propositions 2.2 and 2.4(b). Before that, we need two lemmas.

Lemma 2.10. *For a diagram of type ($\square^*$) we have:*

(a) $w\text{-}\dim(R) = \max\{w\text{-}\dim(T), w\text{-}\dim(D) + \dim(T_M)\}$,
(b) $w\text{-}\dim_v(R) = \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(D) + \dim_v(T_M)\}$.

Proof. (a) We have $w\text{-}\dim(R) = \sup\{\dim(R_P) | P \in w\text{-}\operatorname{Max}(R)\}$. Now let $P \in w\text{-}\operatorname{Max}(R)$ such that $w\text{-}\dim(R) = \dim(R_P)$. If $P \not\supseteq M$ then $R_P = T_Q$ for some $Q \in \operatorname{Spec}(T)$ such that $P = Q \cap R$. Thus Q and M are incomparable prime ideals of T. Hence using [8, Lemma 3.3], we see that Q is a w-maximal ideal of T. On the other hand if $P \supseteq M$, then $\dim(R_P) = \dim(D_Q) + \dim(T_M)$ for some $Q \in w\text{-}\operatorname{Max}(R)$ such that $P = \varphi^{-1}(Q)$. Then we have the inequality $\leq$ in (a). We have two cases to consider:

1° If $P \not\supseteq M$ then $R_P = T_Q$ for some $Q \in w\text{-}\operatorname{Max}(R)$ such that $P = Q \cap R$. We claim that $w\text{-dim}(T) = \dim(T_Q)$. Suppose, contrary to our claim, that there exists $L \in w\text{-}\operatorname{Max}(R)$ such that $w\text{-dim}(T) = \dim(T_L)$ and $\dim(T_Q) \lneq \dim(T_L)$. Set $P_1 := L \cap R$. Consequently $R_{P_1} = T_L$ by [12, Proposition 1.11]. Thus $w\text{-}\dim(R) = \dim(R_P) = \dim(T_Q) \lneq \dim(T_L) = \dim(R_{P_1})$. This implies that P_1 is not a w-ideal of R contradicting [12, Theorem 2.6(2)]. Thus in this case we have the equality in (a).

2° If $P \supseteq M$, then $\dim(R_P) = \dim(D_Q) + \dim(T_M)$ for some $Q \in w\text{-}\operatorname{Max}(D)$ such that $P = \varphi^{-1}(Q)$ (by Proposition 2.2). We claim that $w\text{-dim}(D) = \dim(D_Q)$.

Suppose, contrary to our claim, that there exists $L \in w\text{-}\operatorname{Max}(D)$ such that w-$\dim(D) = \dim(D_L)$ and $\dim(D_Q) \lneq \dim(D_L)$. Set $P_1 := \varphi^{-1}(L)$. Consequently $\dim(R_{P_1}) = \dim(D_L) + \dim(T_M)$. Thus $w\text{-}\dim(R) = \dim(R_P) = \dim(D_Q) + \dim(T_M) \lneq \dim(D_L)+\dim(T_M) = \dim(R_{P_1})$. This implies that P_1 is not a w-ideal of R contradicting Lemma 2.1. Thus in this case again we have the equality in (a).

(b) We have $w\text{-}\dim_v(R) = \sup\{\dim_v(R_P)|P \in w\text{-}\operatorname{Max}(R)\}$ by Proposition 1.3. The rest of the proof is the same as part (a). □

Lemma 2.11. *For a diagram of type* (□) *assume that $D = F$ is a field and let $d = \operatorname{tr.deg.}(k/F)$. Then*

(a) $w\text{-}\dim(R) = \max\{w\text{-}\dim(T), \dim(T_M)\}$,
(b) $w\text{-}\dim_v(R) = \max\{w\text{-}\dim_v(T), \dim_v(T_M) + d\}$.

Proof. (a) Note that M is a w-prime ideal of R. Then we have $w\text{-}\dim(R) = \max\{\sup\{\dim(R_P)|P \in w\text{-}\operatorname{Max}(R),\text{ and } P \not\supset M\}, \dim(R_M)\}$. Like Lemma 2.10 the inequality $\leq$ holds. Let $P \in w\text{-}\operatorname{Max}(R)$ such that $w\text{-}\dim(R) = \dim(R_P)$. If $P = M$, then we have $\dim(R_P) = \dim(T_M)$ by [7, Proposition 2.1(5)]. If not we have $P \not\supset M$. Then $R_P = T_Q$ for some $Q \in \operatorname{Spec}(T)$ such that $P = Q\cap R$. Using [8, Lemma 3.3], we see that Q is a w-maximal ideal of T. We claim that w-$\dim(T) = \dim(T_Q)$. Suppose, contrary to our claim, that there exists $L \in w\text{-}\operatorname{Max}(R)$ such that w-$\dim(T) = \dim(T_L)$ and $\dim(T_Q) \lneq \dim(T_L)$. Set $P_1 := L \cap R$. If $P_1 \not\supset M$ then $R_{P_1} = T_L$. Thus $w\text{-}\dim(R) = \dim(R_P) = \dim(T_Q) \lneq \dim(T_L) = \dim(R_{P_1})$. This implies that P_1 is not a w-ideal. But if $L \subseteq M$ then $P_1 \subseteq M$ and hence P_1 is a w-prime ideal which is a contradiction. So that $L \not\subset M$. Thus P_1 is a w-prime ideal by [12, Theorem 2.6(2)] which is again a contradiction. Therefore $w\text{-}\dim(R) = \dim(R_P) = \dim(T_Q) = w\text{-}\dim(T)$.

(b) It is the same as part (a) noting that we have

$$w\text{-}\dim_v(R) = \max\{\sup\{\dim_v(R_P)|P \in w\text{-}\operatorname{Max}(R),\text{ and } P \not\supset M\}, \dim_v(R_M)\},$$

by Proposition 1.3 and using [1, Theorem 2.11(b)] instead of [7, Proposition 2.1(5)]. □

By combining Lemmas 2.10 and 2.11 we have:

Theorem 2.12. *For a diagram of type* (□), *let $F = qf(D)$ and $d := \operatorname{tr.deg.}(k/F)$. Then:*

(a) $w\text{-}\dim(R) = \max\{w\text{-}\dim(T), w\text{-}\dim(D) + \dim(T_M)\}$.
(b) $w\text{-}\dim_v(R) = \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(D) + \dim_v(T_M) + d\}$.

An integral domain R is said to be a *w-locally Jaffard domain* if R_P is a Jaffard domain for each w-prime ideal P of R. It is easy to see that a w-locally Jaffard domain of finite w-valuative dimension is a w-Jaffard domain. Now we have the following corollary which is the w-analogue of [1, Corollary 2.12].

Corollary 2.13. *For a diagram of type* (□), *let F be the quotient field of D. Then:*

(a) *R is a w-locally Jaffard domain $\Leftrightarrow$ D and T are w-locally Jaffard domains, and k is algebraic over F.*
(b) *If T is a w-locally Jaffard domain with w-$\dim_v(T) < \infty$, D is a w-Jaffard domain, and k is algebraic over F, then R is a w-Jaffard domain.*

A "global" type of $D+M$ constructions arise from $T=K[X]$, the polynomial ring over a field K, by considering $M=XT$ and a subring D of K. In this case neither T nor R is quasilocal. Theorem 2.12 yields:

Corollary 2.14. *Let K be a field, D a subring of K with quotient field F, $R=D+XK[X]$ and $d=\operatorname{tr.deg.}(K/F)$. Then:*

(a) $w\text{-}\dim(R)=w\text{-}\dim(D)+1$.
(b) $w\text{-}\dim_v(R)=w\text{-}\dim_v(D)+d+1$.
(c) *R is a w-Jaffard domain $\Leftrightarrow$ D is a w-Jaffard domain and K is algebraic over F.*

3. Examples

It is well known that [9, Theorem 6.7.8] a finite dimensional domain R has Prüfer integral closure if and only if each overring of R is a Jaffard domain. Similarly in [25] we showed that a finite w-dimensional domain R has Prüfer integral closure if and only if each overring of R is a w-Jaffard domain. Thus in particular each overring of a finite dimensional domain is Jaffard if and only if each overring is w-Jaffard. In the following two examples we show that the classes of w-Jaffard and Jaffard domains are incomparable.

The next example gives a positive answer to our question in [22, page 238], whether it is possible to find a w-Jaffard non-Jaffard domain? There is an old question (see [6]) asking if it is possible to find a UFD (or a Krull domain) which is not Jaffard. We note that a non-Jaffard Krull domain would be an example of a w-Jaffard domain which is not Jaffard. But to the best of author's knowledge there is not such an example.

Example 3.1. *For each $n\geq 3$ there is an integral domain R_n which is w-Jaffard of dimension n but not a Jaffard domain.*
Let K be a field and let W,X,Y,Z be indeterminates over K. Put $L=K(W,X,Y,Z)$. Now, $V_1=K(W,X,Z)+M_1$, where $M_1=YK(W,X,Z)[Y]_{(Y)}$, is a (discrete) rank 1 valuation domain of L with maximal ideal M_1. Let (V,M) be a rank 1 valuation domain of the form $V=K(W,X,Y)+M$, where $M=ZK(W,X,Y)[Z]_{(Z)}$. With τ denoting the canonical surjection $V\to K(W,X,Y)$, consider the pullback $V'=\tau^{-1}(K(W,X)[Y]_{(Y+1)})=K(W,X)+M'$ where $M'=(Y+1)K(W,X)[Y]_{(Y+1)}+ZK(W,X,Y)[Z]_{(Z)}$. Thus $\dim(V')=2$. Finally with ψ denoting the canonical surjection $V'\to K(W,X)$, consider the pullback $V_2=\psi^{-1}(K(W)[X]_{(X)})=K(W)+M_2$, where $M_2=XK(W)[X]_{(X)}+(Y+1)K(W,X)[Y]_{(Y+1)}+ZK(W,X,Y)[Z]_{(Z)}$, is a valuation domain of L with maximal ideal M_2 and we have $\dim(V_2)=3$. Further, V_1 and V_2 are incomparable. If not, it would follow from the one-dimensionality of V_1 that $V_2\subset V_1$. Then we would have $V_1=(V_2)_M$, whence $YK(W,X,Z)[Y]_{(Y)}=M_1=M(V_2)_M=M$ and $1=YY^{-1}\in MV=M$, a contradiction. Thus V_1 and V_2 are incomparable. Then $T:=V_1\cap V_2$ is a three dimensional Prüfer domain with $\mathfrak{m}_1:=M_1\cap T$ and $\mathfrak{m}_2:=M_1\cap T$ as maximal ideals such that $T_{\mathfrak{m}_1}=V_1$ and $T_{\mathfrak{m}_2}=V_2$ by [19, Theorem 11.11]. With $\varphi:T\to T/\mathfrak{m}_1(\cong V_1/M_1\cong K(W,X,Z))$ denoting the canonical surjection, consider the pullback $R_3:=\varphi^{-1}(K[W,X])$. Notice that T is a DW-domain since it is a Prüfer domain, $d:=\operatorname{tr.deg.}(K(W,X,Z)/K(W,X))=1$, and that $K[W,X]$ is a Noetherian Krull domain. In particular $K[W,X]$ is a Jaffard domain (of dimension

2) and w-Jaffard domain (of w-dimension 1). Thus using Theorem 2.12 we have:

$$\begin{aligned} w\text{-}\dim(R_3) &= \max\{w\text{-}\dim(T), w\text{-}\dim(K[W,X]) + \dim(T_{\mathfrak{m}_1})\} \\ &= \max\{3, 1+1\} = 3, \ \textit{and} \\ w\text{-}\dim_v(R_3) &= \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(K[W,X]) + \dim_v(T_{\mathfrak{m}_1}) + d\} \\ &= \max\{3, 1+1+1\} = 3. \end{aligned}$$

This means that R_3 is a w-Jaffard domain of w-dimension 3. But by [1, Theorem 2.11] *we have*

$$\begin{aligned} \dim(R_3) &= \max\{\dim(T), \dim(K[W,X]) + \dim(T_{\mathfrak{m}_1})\} \\ &= \max\{3, 2+1\} = 3, \ \textit{and} \\ \dim_v(R_3) &= \max\{\dim_v(T), \dim_v(K[W,X]) + \dim_v(T_{\mathfrak{m}_1}) + d\} \\ &= \max\{3, 2+1+1\} = 4. \end{aligned}$$

Therefore R_3 is not a Jaffard domain.

Now set $F := qf(R_3)$. Suppose that $V := F + M$ is a rank 1 valuation domain with maximal ideal M. Set $R_4 := R_3 + M$. It is easy to see that R_4 is w-Jaffard of $w\text{-}\dim(R_4) = 4$, $\dim(R_4) = 4$, *and* $\dim_v(R_4) = 5$. *Iterating in the same way we obtain R_n with desired properties.*

Example 3.2. *For each $n \geq 2$ there is an integral domain R_n which is Jaffard of dimension n but not a w-Jaffard domain.*
Let K be a field and let X, Y, Z be indeterminates over K. Let $C := K[X,Y,Z]$ and set $P := (X)$ and $Q := (Y,Z)$. Let $T := C_S$ where $S := C \backslash (P \cup Q)$, which is a multiplicatively closed subset of C. Then $\mathrm{Max}(T) = \{PT, QT\}$, $\dim(T_{PT}) = 1$ *and* $\dim(T) = \dim(T_{QT}) = 2$. *Next notice that we have a surjective ring homomorphism $\psi : C_P \to K(Y,Z)$ sending $f/g \mapsto f(0,Y,Z)/g(0,Y,Z)$, with* $\mathrm{Ker}(\psi) = PC_P$. *Thus we have $T/PT \cong C_P/PC_P \cong K(Y,Z)$. With $\varphi : T \to T/PT$ denoting the canonical surjection, consider the pullback $R_2 := \varphi^{-1}(K(Y))$. Note that T is a Noetherian Krull domain. Thus T is a 2 dimensional Jaffard domain and a w-Jaffard domain of w-dimension 1. Also notice that* $d := \mathrm{tr.deg.}(K(Y,Z)/K(Y)) = 1$. *Thus by* [1, Theorem 2.11] *we have*

$$\begin{aligned} \dim(R_2) &= \max\{\dim(T), \dim(K(Y)) + \dim(T_{PT})\} \\ &= \max\{2, 0+1\} = 2, \ \textit{and} \\ \dim_v(R_2) &= \max\{\dim_v(T), \dim_v(K(Y)) + \dim_v(T_{PT}) + d\} \\ &= \max\{2, 0+1+1\} = 2. \end{aligned}$$

Therefore R_2 is a Jaffard domain of dimension 2. On the other hand using Theorem 2.12 we have:

$$\begin{aligned} w\text{-}\dim(R_2) &= \max\{w\text{-}\dim(T), w\text{-}\dim(K(Y)) + \dim(T_{PT})\} \\ &= \max\{1, 0+1\} = 1, \ \textit{and} \\ w\text{-}\dim_v(R_2) &= \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(K(Y)) + \dim_v(T_{PT}) + d\} \\ &= \max\{1, 0+1+1\} = 2. \end{aligned}$$

This means that R_2 is not a w-Jaffard domain.

Now set $F := qf(R_2)$. Suppose that $V := F + M$ is a rank 1 valuation domain with maximal ideal M. Set $R_3 := R_2 + M$. It is easy to see that R_3 is Jaffard of $\dim(R_3) = 3$, *w-*$\dim(R_3) = 2$, *and w-*$\dim_v(R_3) = 3$. *Iterating in the same way we obtain R_n with desired properties.*

Here we give our promised example of a w-Jaffard domain which is not a strong Mori nor a UMt domain.

Example 3.3. *Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \cdots)$ be an algebraic extension field of $\mathbb{Q}$ such that $[\mathbb{K} : \mathbb{Q}] = \infty$. Let X and Y be indeterminates over $\mathbb{K}$ and set $R := \mathbb{Q} + (X,Y)\mathbb{K}[X,Y]$. Then R is a w-Jaffard domain of w-dimension 2 by Theorem 2.12, but it is not a strong Mori domain by* [20, Theorem 3.11]*. Next we claim that R is not a UMt domain. In fact if R is a UMt domain,* [8, Corollary 3.2] *implies that (X,Y) is a t-prime ideal of $\mathbb{K}[X,Y]$, which is absurd since $\mathbb{K}[X,Y]$ is a Krull domain and (X,Y) has height 2. Note that in this case R is a 2 dimensional Jaffard domain.*

Recall that an integral domain is called a *Mori domain* if it satisfies the ascending chain condition on divisorial ideals. Every strong Mori domain is a Mori domain. The following example is designed to show that a Mori domain *need not* be a w-Jaffard domain.

Example 3.4. *Let K be a field and let X, Y be two indeterminates over K and set $R := K + YK(X)[Y]$. Then R is not a w-Jaffard domain by Corollary 2.14, but it is a Mori domain by* [11, Theorem 4.18].

The following example shows that a w-Jaffard domain *need not* be w-locally Jaffard.

Example 3.5. *Let K be a field and X_1, X_2 indeterminates over K. It is proved in* [1, Example 3.2(a)] *that there are two incomparable valuation domains (V_1, M_1) and (V_2, M_2) of dimension 1 and 2 respectively. Set $T := V_1 \cap V_2$ which is a two-dimensional Prüfer domain with exactly two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ so that $T_{\mathfrak{m}_1} = V_1$ and $T_{\mathfrak{m}_2} = V_2$. Denoting $\varphi : T \to T/\mathfrak{m}_1 (\cong V_1/M_1 \cong K(X_1, X_2))$ consider the pullback $R := \varphi^{-1}(K(X_1))$. Since $K(X_1)$ and T are DW-domains ,* [21, Theorem 3.1(3)] *implies that R is also a DW-domain. In particular w-*$\dim(R) = \dim(R)$ *and w-*$\dim_v(R) = \dim_v(R)$. *It follows that w-*$\dim(R) = \max\{2, 0+1\} = 2$, *$w$-*$\dim_v(R) = \max\{2, 0+1+1\} = 2$. *Thus R is a w-Jaffard (=Jaffard) domain. It is observed in* [1, Example 3.2(a)] *that for the prime ideal $\mathfrak{n}_1 := \mathfrak{m}_1 \cap R$ of R,* $\dim(R_{\mathfrak{n}_1}) = 1$ *and* $\dim_v(R_{\mathfrak{n}_1}) = 2$. *This shows that R is not w-locally Jaffard.*

By [13, Exercise 17(1), Page 372], for each positive integer n, there exists a finite-dimensional (non-Jaffard) domain R such that $\dim_v(R) - \dim(R) = n$ (see also [1, Example 3.1(a)]).

Example 3.6. *For each positive integer n, there exists a finite w-dimensional domain R such that $w\text{-}\dim_v(R) - w\text{-}\dim(R) = n$.*
Indeed let D be a Krull domain. Let K be the quotient field of D and $\{X_1, \cdots, X_n, Y\}$ be a set of $n+1$ indeterminates over K. Let L denote the field $K(X_1, \cdots, X_n)$. Also define the valuation domain $V := L[Y]_{(Y)} = L + M$ (with $M = YV$) and the ring $R := D + M$. Applying Proposition 2.12 to the pullback description of R, we have $w\text{-}\dim(R) = w\text{-}\dim(D) + 1$ and $w\text{-}\dim_v(R) = w\text{-}\dim_v(D) + 1 + n$. Since D is a Krull domain we have $w\text{-}\dim(D) = w\text{-}\dim_v(D) = 1$. So that $w\text{-}\dim_v(R) - w\text{-}\dim(R) = n$. In particular R is not a w-Jaffard domain. Note that $\dim(R) = \dim(D) + 1$ *by* [7, Proposition 2.1(5)] *and* $\dim_v(R) = \dim_v(D) + 1 + n$ *by* [1, Theorem 2.6(a)] *while $w\text{-}\dim(R) = 2$ and $w\text{-}\dim_v(R) = 2 + n$. In particular if* $\dim(D) \geq 2$ *then* $\dim(R) \neq w\text{-}\dim(R)$ *and* $\dim_v(R) \neq w\text{-}\dim_v(R)$.

We remark that in the above example if D is a Dedekind domain then R is a DW-domain by [21, Theorem 3.1(2)]. This means that $\dim(R) = w\text{-}\dim(R)$ and $\dim_v(R) = w\text{-}\dim_v(R)$. Therefore R is a non-Jaffard domain with $\dim_v(R) - \dim(R) = n$.

4. An application

Recall that in [27] Seidenberg proved that if n, m are positive integers such that $n+1 \leq m \leq 2n+1$, there is an integrally closed domain R such that $\dim(R) = n$ and $\dim(R[X]) = m$. More recently in [29, Theorem 2.10] Wang showed that for any pair of positive integers n, m with $1 \leq n \leq m \leq 2n$, there is an integrally closed domain R such that w-$\dim(R) = n$ and w-$\dim(R[X]) = m$. By Proposition 1.1 we have for an integral domain R if $n = w$-$\dim(R)$ then

$$n + 1 \leq w[X]\text{-}\dim(R[X]) \leq 2n + 1.$$

Now we show that these bounds are the best possible. We say that an integral domain R is of *w_x-type* (n, m) if w-$\dim(R) = n$ and $w[X]$-$\dim(R[X]) = m$.

Theorem 4.1. *Let D be an integral domain of w_x-type (n, m) with quotient field K. Let L be a purely transcendental field extension of K. Then:*

(a) *If $V_1 = K + M_1$ is a DVR and $R_1 = D + M_1$, then R_1 is of w_x-type $(n+1, m+1)$.*
(b) *If $V_2 = L + M_2$ is a DVR and $R_2 = D + M_2$, then R_2 is of w_x-type $(n+1, m+2)$.*

Proof. (a) Using Proposition 2.2 we have

$$w\text{-}\dim(R_1) = w\text{-}\dim(D) + \dim(V_1) = n + 1.$$

Since R_1 is a pullback of a diagram of type $(\square^*)$, Proposition 2.4 yields that

$$w[X]\text{-}\dim(R_1[X]) = w[X]\text{-}\dim(D[X]) + \dim(V_1[X]) - 1 = m + 2 - 1 = m + 1.$$

(b) By the same way as (a) we have $w\text{-}\dim(R_2) = n + 1$. Now we compute $w[X]\text{-}\dim(R_2[X])$. If $Q_1 \subset \cdots \subset Q_m$ is a chain of $w[X]$-prime ideals of $D[X]$ of length m, then

$$M_2[X] \subset Q_1 + M_2[X] \subset \cdots \subset Q_m + M_2[X]$$

is a chain of prime ideals of $R_2[X]$ of length $m+1$. Notice that $(Q_i + M_2[X]) \cap R_2 = Q_i \cap D + M_2$ for $i = 1, \cdots, m$. Since Q_i is a $w[X]$-prime ideal of D then $Q_i \cap D$ is a w-prime ideal of D (or equal to zero) by [22, Remark 2.3]. Therefore by [29, Lemma

2.3] we see that $Q_i \cap D + M_2 = (Q_i + M_2[X]) \cap R_2$ is a w-prime ideal of R_2. Thus using [22, Remark 2.3] we obtain that $Q_i + M_2[X]$ is a $w[X]$-prime ideal of $R_2[X]$. On the other hand $(R_2)_{M_2} = K + M_2$, and therefore it is not a valuation domain. Thus [13, Theorem 19.15(2)] yields that $M_2[X]$ is not minimal in $R_2[X]$. Therefore $w[X]\text{-}\dim(R_2[X]) \geq m+2$. We consider a chain $P_1 \subset \cdots \subset P_s$ of $w[X]$-prime ideals of $R_2[X]$ of maximal length. Since P_2 is not minimal in $R_2[X]$, $P_2 \cap R_2 \neq (0)$. By [13, Part (3) of Exercise 12 Page 202] M_2 is the unique minimal prime ideal of R_2. Therefore $M_2 \subseteq P_2 \cap R_2$ and $M_2[X] \subseteq P_2$. Each $P_j \cap R_2$ is a w-prime ideal of R_2 by [22, Remark 2.3] for $j = 1, \cdots, s$. Since $(P_j/M_2[X]) \cap D = (P_j/M_2[X]) \cap R_2/M_2 = (P_j \cap R_2)/M_2$ we claim that $(P_j \cap R_2)/M_2 = (P_j/M_2[X]) \cap D$ is a w-prime ideal of D by Lemma 2.1. Therefore $P_j/M_2[X]$ is a $w[X]$-prime ideal of $D[X]$ by [22, Remark 2.3]. So that $P_3/M_2[X] \subset \cdots \subset P_s/M_2[X]$ is a chain of $w[X]$-prime ideals of $D[X]$, and thus $s - 2 \leq m$. It follows that $w[X]\text{-}\dim(R_2[X]) = m+2$ completing the proof. □

Remark 4.2. *Let D^c and R_i^c denote the integral closures of D and R_i in their quotient fields, respectively ($i = 1, 2$). Then $R_i^c = D^c + M_i$ by* [29, Lemma 2.6(1)]. *Therefore, R_i is integrally closed if and only if D is integrally closed.*

Following Seidenberg, we say that a domain R is an *F-ring* if $\dim(R) = 1$ and $\dim(R[X]) = 3$. By [16, Corollary 3.6] and [13, Proposition 30.14], a one-dimensional domain R is an F-ring if and only if R is not a UMt-domain. For an F-ring, w-$\dim(R) = 1$ and $w[X]$-$\dim(R[X]) = 3$ by [22, Corollary 3.6]. Thus a F-ring is a domain of w_x-type $(1, 3)$.

Corollary 4.3. *For any pair of positive integers (n, m) with $n + 1 \leq m \leq 2n + 1$, there is an integrally closed integral domain R of w_x-type (n, m).*

Proof. A PID is an integrally closed integral domain of w_x-type $(1, 2)$. By [27, Theorem 3] there is an integrally closed F-ring. Thus by the comments before the corollary it is of w_x-type $(1, 3)$. So if $n = 1$ the result is true. Using Theorem 4.1 and by an induction argument similar to the proof of [27, Theorem 3], the proof is complete. □

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