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INTEGRAL CLOSURE AND IDEAL TOPOLOGIES IN MODULES

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1. INTRODUCTION

The notion of integral closure of an ideal I of a commutative Noetherian ring R has proved useful in many situations both in commutative algebra and algebraic geometry. This concept is extended to modules by D. Rees [1]. Similar to the theory of integral closure of ideals, it is natural to expect many interesting applications for notion of integral closure of modules. In fact, a lot of articles are concerned with this notion and its applications (see *e.g*. [12, 13, 10, 5, 4 and 6]). In this paper we investigate further applications of this concept. In particular we extend some of the main results of McAdam [9] to finitely generated modules.

Let I denote an ideal of the commutative Noetherian ring R. The set $\bar{Q}^*(I)$, quintasymptotic prime ideals of I, was systematically studied by S. McAdam in [9] and S.H. Ahn in [1] extended this notation to finitely

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generated R-modules. For a finitely generated R-module N, the set of quintasymptotic prime ideals of I with respect to N is defined as

$$\bar{Q}^*(I,N) = \{\mathfrak{p} \in V(I) | \text{ there is a } z \in \mathrm{Min}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}} \text{ with } \mathrm{Rad}(IR^*_{\mathfrak{p}} + z) = \mathfrak{p}R^*_{\mathfrak{p}}\}.$$

Ahn's paper contains no notion of integral closure on an R-module, so that the techniques related to integral closure of an ideal are not available on modules. Therefore the counterparts of [9, Theorem 1.5 and Proposition 3.5] for finitely generated modules are not proved in [1].

We prove the counterparts of these results for finitely generated modules in certain cases .

Let R be a Noetherian domain and K be its field of fractions. Let N be a finitely generated torsion-free R-module. The (Rees) integral closure of N is defined as $\bar{N} = \cap NV$, where V ranges over all DVR's of K which contain R. For a submodule M of N, we define the integral closure of M in N as $M_a = \bar{M} \cap N$. We denote the submodule $\bigcup_{n\geq 0}(M :_N I^n)$ of N by $M :_M < I >$. In section two, we examine the behaviour of integral closure of modules with respect to localization and completion. The main result of this section asserts that if, R is a normal domain, N a finitely generated torsion-free R-module and M is an integrally closed submodule of N with $\mathrm{rank}_R M = \mathrm{rank}_R N$, then M has a primary decomposition each primary component of which is integrally closed.

In the third section, we focus on locally analytically normal domains. Let R be a locally analytically normal domain. For a finitely generated torsion-free R-module N, it follows that $N^*_{\mathfrak{p}}$ is a torsion-free $R^*_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}\, R$, and hence $\bar{Q}^*(I,N)$ coincides with $\mathrm{Min}_R(R/I)$. Thus Theorems 3.3 and 3.4 may be considered as generalizations of the above mentioned results of McAdam. Let N be an R-module and S a multiplicatively closed subset of R. For a submodule M of N, we put $S(M) = \bigcup_{s\in S}(M :_N s)$. In Theorem 3.4, we provide a criterion for comparison of the topologies defined by the filtrations $\{S(I^nN)\}_{n\geq 0}$ and $\{(I^nN)_a\}_{n\geq 0}$. Namely, we show that over a locally analytically domain R, the S-symbolic topology $\{S(I^nN)\}_{n\geq 0}$ is finer than the topology given by the integral closure filtration $\{(I^nN)_a\}_{n\geq 0}$ if and only if S is disjoint of each minimal prime ideal of I.

In the end of this section, we give some applications to local cohomology.

All rings considered in this paper are assumed to be commutative with non-zero identities. For a module N over a commutative Noetherian ring R, we denote by $\mathrm{Min}_R N$, the set of minimal elements of $\mathrm{Ass}_R N$. If $(R, \mathfrak{m})$ is a local ring, the $\mathfrak{m}$-adic completion of an R-module N is denoted by N^*.

By a locally analytically normal domain, we mean a Noetherian domain R such that $R_{\mathfrak{p}}^*$ is a normal domain for each prime ideal $\mathfrak{p}$ of R. For a domain R, we let K_R denote its field of fractions.

2. INTEGRAL CLOSURE IN MODULES

This section is devoted to examining of integral closure of modules as developed by Rees in [12]. The theory of reductions and integral closure of modules can be developed for torsion-free modules over an arbitrary Noetherian domain without assumptions of locality or regularity. So, we let R be a Noetherian domain with the field of fractions K_R, and let N be a finitely-generated torsion-free R-module. By N_K we denote the finite-dimensional K_R-vector space $N \otimes_R K_R$. If M is a submodule of N, then M_K is naturally identified with a subspace of N_K. For any birational overring S of R, we use the notation NS to be the S-submodule of N_K generated by N. In this section, the birational overring of R that we will focus on are the discrete valuation rings of K_R containing R.

Definition. *With the previous notation, an element $v \in N_K$ is said to be integral over N if $v \in NV$ for every discrete valuation ring (DVR for short) V of K_R containing R. The* Rees integral closure *of N is the set of all elements of N_K that are integral over N, and it is denoted by $\bar{N}$. Moreover, for a submodule M of N, the* integral closure *of M in N is the submodule $M_a := \bar{M} \cap N$. We say that M is* integrally closed *if $M = M_a$.*

Example 2.1. (i) Let R be a DVR and N a finitely generated torsion-free R-module. Then any submodule of N is integrally closed.

(ii) Let R be a normal domain (*i.e.* integrally closed in the field of fractions K_R) and let N be a finitely generated torsion-free R-module. Then, by using [6, Corollary 3.5], it follows that every projective submodule P of N is integrally closed.

Let R be a Noetherian domain. Let N be a finitely generated torsion-free R-module of rank r, and let F be a free R-submodule of N_K containing N with $\operatorname{rank}_R(F) = r$. Then the symmetric algebra $Sym_R(F) = S(F)$ is equal to $R[T_1, \ldots, T_r] = R[T]$, a polynomial ring over R with r variables. Suppose N is generated by $b_1, \ldots, b_t \in F$, with $b_i = (b_{i1}, \ldots, b_{ir})$. We denote by I_N, the ideal of $S(F)$ generated by N, that is $I_N = (b_1T, \ldots, b_tT)S(F)$, where $b_iT = b_{i1}T_1 + \cdots + b_{ir}T_r$.

The following result will be served as the spring board for our investigations into the properties of the integral closure in modules. We shall use it in Lemmas 2.3 and 2.4 and Theorem 2.6.

Lemma 2.2. *With the previous notation, let R be a normal domain. Then* $\bar{N} = (I_N)_a \cap F$, *where* $(I_N)_a$ *denotes the integral closure of the ideal* I_N *in* $S(F)$.

Proof. See [6, Proposition 3.6]. □

Lemma 2.3. *Let R a normal domain. Let N be a finitely generated torsion-free R-module and M a submodule of N. If U is a multiplicatively closed subset of R, then* $U^{-1}M_a = (U^{-1}M)_a$.

Proof. By the definition $M_a = \bar{M} \cap N$, where $\bar{M}$ denotes the Rees integral closure of M. Hence $U^{-1}M_a = U^{-1}\bar{M} \cap U^{-1}N$. Therefore it is enough to show that $U^{-1}\bar{M} = \overline{U^{-1}M}$. To this end, first note that by Lemma 2.2, $\bar{M} = (I_M)_a \cap F$, where F is a free R-submodule of M_K containing M with $\operatorname{rank}_R F = \operatorname{rank}_R M$, and I_M denotes the ideal of $S(F)$ generated by M. It is easy to see that $U^{-1}(S(F)) \cong S(U^{-1}F)$ naturally and that the image of $U^{-1}I_M$ under this isomorphism is $I_{U^{-1}M}$. By [15, Lemma 2.3], $(U^{-1}I_M)_a = U^{-1}(I_M)_a$ and hence

$$\begin{aligned} U^{-1}\bar{M} &= U^{-1}(I_M)_a \cap U^{-1}F = (U^{-1}I_M)_a \cap U^{-1}F \\ &= (I_{U^{-1}M})_a \cap U^{-1}F. \end{aligned}$$

Now, the field K_R is also the field of fractions of $U^{-1}R$. Moreover $U^{-1}F$ is a free $U^{-1}R$-submodule of $(U^{-1}M)_{K_R}$ containing $U^{-1}M$ with $\operatorname{rank}_{U^{-1}R}(U^{-1}F) = \operatorname{rank}_{U^{-1}R}(U^{-1}M)$. Therefore $U^{-1}\bar{M} = \overline{U^{-1}M}$. □

Lemma 2.4. *Let R be a locally analytically normal domain and let N be a finitely generated torsion-free R-module. For any submodule M of N with* $\operatorname{rank}_R M = \operatorname{rank}_R N$,

$$(M^*_{\mathfrak{p}})_a \cap N_{\mathfrak{p}} = (M_{\mathfrak{p}})_a$$

for all $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. First of all, it is easy to see that over a domain T, a T-module C is torsion-free if and only if $\operatorname{Ass}_T C = \{0\}$. Having kept this in mind and using [7, Theorem 23.2], we deduce that $N^*_{\mathfrak{p}}$ is a torsion-free $R^*_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$. Hence, it is enough to show that over a local normal domain R, we have $(M^*)_a \cap N = M_a$. Since N is a finitely generated torsion-free R-module, there is a free R-submodule F of N_K containing N such that $\operatorname{rank}_R N = \operatorname{rank}_R F$. Let K^* denote the field of fractions of R^*. Because $M \subseteq M^* \subseteq F^*$, it follows that F^* is a free R^*-submodule of $M^* \otimes_{R^*} K^*$ containing M^* and that $\operatorname{rank}_{R^*} M^* = \operatorname{rank}_{R^*} F^*$. Note that the R^*-module $M_K \otimes_R R^*$ can be naturally embedded in $M^* \otimes_{R^*} K^*$. It is easy to establish the following facts:

(i) $S(F^*) \cong S(F) \otimes_R R^*$,

(ii) the natural homomorphism $S(F) \longrightarrow S(F) \otimes_R R^* = S(F^*)$ is faithfully flat (see [7, P. 53, Exercise 7.1]), and,

(iii) $I_M S(F^*) = I_{M^*}$.

Now, we have $\bar{M}^* \cap N = (I_{M^*})_a \cap F^* \cap N$, by Lemma 2.2. Since the natural homomorphism $S(F) \longrightarrow S(F^*)$ is faithfully flat, it follows from [8, Lemma 3.15] that $(I_M S(F^*))_a \cap S(F) = (I_M)_a$, and hence

$$\begin{aligned}\bar{M}^* \cap N &= (I_{M^*})_a \cap N = (I_M S(F^*))_a \cap N \\ &= (I_M S(F^*))_a \cap S(F) \cap N = (I_M)_a \cap N \\ &= (I_M)_a \cap F \cap N = \bar{M} \cap N.\end{aligned}$$

This completes the proof. □

Although, the part (i) of the following result (which may be considered as an extended form of Krull's intersection Theorem) may be deduced from [2, p.288, Ex.15], we bring it here just for comparison with the part (ii) of this result.

Proposition 2.5. *Let I denote a proper ideal of the Noetherian ring R and let N be a finitely generated R-module. Suppose that $0 = \bigcap_{i=1}^{k} Q_i$ is an irredundant primary decomposition of the zero submodule of N with Q_i belongs to $\mathfrak{p}_i$. Then*

(i) $\bigcap_{n\geq 1} I^n N = \bigcap\{Q_i \mid \mathfrak{p}_i + I \subsetneqq R\}$. In particular if $\mathfrak{p}_i + I \subsetneqq R$ for each $\mathfrak{p}_i \in \mathrm{Ass}_R N$, then $\bigcap_{n\geq 1} I^n N = 0$.

(ii) If R is domain and N is torsion-free, then $\bigcap_{n\geq 1} (I^n N)_a = 0$.

Proof. (ii) It is known that $I_a = (\cap IV) \cap R$, where the intersection is taken over all DVR's V of the field of fractions of R containing R. Since I is a proper ideal and $\mathrm{Rad}(I) = \mathrm{Rad}(I_a)$, it follows that there exists a such DVR, W such that $IW \neq W$. Then we have $\bigcap_{n=1}^{\infty} (I^n N)_a = \bigcap_{n=1}^{\infty} (\overline{I^n N} \cap N) = \bigcap_{n=1}^{\infty} [\bigcap_{R\subseteq V} (I^n V) NV \cap N]$, the second intersection is taken over all DVR's V of the field of fractions of R containing R. Thus

$$\begin{aligned}\bigcap_{n=1}^{\infty} (I^n N)_a &= \left[\bigcap_{R\subseteq V}\left(\bigcap_{n=1}^{\infty} (I^n V) NV\right)\right] \cap N \\ &\subseteq \left(\bigcap_{n=1}^{\infty} (IW)^n (NW)\right) \cap N = 0.\end{aligned}$$ □

Theorem 2.6. *Let R be a Noetherian normal domain and let M be an integrally closed proper submodule of the finitely generated torsion-free R-module*

N with $\operatorname{rank}_R M = \operatorname{rank}_R N$. Then M has a primary decomposition each primary component of which is integrally closed.

Proof. Since N is torsion-free, there is a free R-submodule F of $N_K = N \otimes_R K$ containing N such that $\operatorname{rank}_R F = \operatorname{rank}_R N$. Because $\operatorname{rank}_R M = \operatorname{rank}_R N$ and $M = M_a$ it follows that $M = (I_M)_a \cap F \cap N = (I_M)_a \cap N$, by Lemma 2.2. Now, since $(I_M)_a$ is an integrally closed ideal of the Noetherian ring $S(F)$, by [9, Lemma 1.4], $(I_M)_a$ has a primary decomposition $q_1 \cap \cdots \cap q_s$ each primary component of which is integrally closed. Hence

$$\begin{aligned} M &= (I_M)_a \cap N = (q_1 \cap q_2 \cap \cdots \cap q_s) \cap N \\ &= (q_1 \cap N) \cap (q_2 \cap N) \cap \cdots \cap (q_s \cap N). \end{aligned} \tag{$*$}$$

For a given element a of R, consider the following commutative diagram in which vertical maps are the natural monomorphisms.

$$\begin{array}{ccc} N/(q_i \cap N) & \xrightarrow{a} & N/(q_i \cap N) \\ \downarrow & & \downarrow \\ S(F)/q_i & \xrightarrow{a} & S(F)/q_i \end{array}$$

It follows from this diagram that $q_i \cap N$ is a primary submodule of N, whenever it is proper. Now, we show that $q_i \cap N$ is integrally closed submodule in N for each $1 \leq i \leq s$. To this end, first note that for a DVR, W with $S(F) \subseteq W \subseteq K_{S(F)}$, by [11, 33.7], $W \cap K_R$ is a DVR of R and clearly $R \subseteq W \cap K_R \subseteq K_R$. Thus

$$\begin{aligned} (q_i \cap N)_a = \overline{(q_i \cap N)} \cap N &= \left[\bigcap_{R \subseteq V \subseteq K_R} (q_i \cap N)V \right] \cap N \\ &\subseteq \left[\bigcap_{S(F) \subseteq W \subseteq K_{S(F)}} q_i(W \cap K_R) \right] \cap N \\ &\subseteq \left[\bigcap_{S(F) \subseteq W \subseteq K_{S(F)}} q_i W \right] \cap N = q_i \cap N. \end{aligned}$$

Hence $(q_i \cap N)_a = q_i \cap N$. Therefore deleting unneeded components in (*) leaves the desired primary decomposition of M in N. □

3. IDEAL TOPOLOGIES IN MODULES

In this section, we compare certain ideal topologies on finitely generated torsion-free modules over an analytically normal domain. In particular, for a multiplicatively closed subset S of R the topologies defined by the filtrations $\{S(I^nN)\}_{n\geq 0}$ and $\{(I^nN)_a\}_{n\geq 0}$ are compared.

In the case $N = R$, the following lemma was proved by Chevalley. The proof given in [11, Theorem 30.1], can be easily carried over to a module. (see, [2, Ch. IV, Section 2.5, Corollary 4]).

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, N a finitely generated R-module and $(N_n)_{n\geq 0}$ a decreasing sequence of submodules of N such that $\bigcap_{n\geq 0} N_n = 0$. Then, for all $r > 0$, there exists $n(r) > 0$ such that $N_{n(r)} \subseteq \mathfrak{m}^r N$.*

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a locally analytically normal domain, which is complete in the $\mathfrak{m}$-adic topology. Let N be a non-zero finitely generated torsion-free R-module. Then for any proper ideal I of R, the following are equivalent:*

(i) $\bigcap_{n=1}^{\infty}((I^nN)_a :_N < \mathfrak{m} >) \neq 0$.

(ii) *There is a $k \geq 0$ such that for all $m \geq 0$, $(I^mN)_a :_N < \mathfrak{m} > \not\subseteq (\mathfrak{m}^kN)_a$.*

(iii) *I is $\mathfrak{m}$-primary.*

Proof. In view of Proposition 2.5 and Lemma 3.1, it is clear that (i) and (ii) are equivalent. Now, we show the implication (i)$\Longrightarrow$(iii). By [13, Theorem 5.11], the set $\bigcup_{n=1}^{\infty} \mathrm{Ass}_R(N/(I^nN)_a)$ is finite. If $\mathfrak{m}$ does not belong to this union, then $(I^nN)_a :_N < \mathfrak{m} >= (I^nN)_a$, for each $n \in \mathbb{N}$, which contradicts (i), by Proposition 2.5. Thus we can choose $s \in \mathfrak{m}$ such that it does not belong to any associated prime to $N/(I^tN)_a, t \in \mathbb{N}$, which is not equal to the maximal ideal. Let S be the set of all non-negative powers of s. By considering a minimal primary decomposition of the submodule $(I^nN)_a$, it easily follows that $(I^nN)_a :_N < \mathfrak{m} >= S^{-1}(I^nN)_a \cap N$ for all $n \in \mathbb{N}$. But, by Lemma 2.3, $S^{-1}(I^nN)_a = ((S^{-1}I^n)(S^{-1}N))_a$. Thus, in view of Proposition 2.5, (i) implies that $S^{-1}I = S^{-1}R$. That is $s \in \sqrt{I}$. To prove (iii), we should show that $\mathfrak{m}$ is a minimal prime of I. By the choice of s, it suffices to show that if $\mathfrak{p}$ is a minimal prime ideal of I, then $\mathfrak{p} \in \mathrm{Ass}_R(N/(I^mN)_a)$, for some $m \in \mathbb{N}$. We may and do assume that R is local at $\mathfrak{p}$ by Lemma 2.3. Since $\bigcap_{n\geq 1}(I^nN)_a = 0$, there exists an integer m such that $(I^mN)_a \subsetneqq N$. Now, we have $\mathfrak{p} = \mathrm{Rad}(I) \subseteq \mathrm{Rad}((I^mN)_a :_R N) \subseteq \mathfrak{p}$. Thus $\mathrm{Rad}((I^mN)_a :_R N) = \mathfrak{p}$ and this implies that $\mathfrak{p} \in \mathrm{Ass}_R(N/(I^mN)_a)$.

(iii) $\Longrightarrow$ (i). Since $\mathrm{Rad}(I) = \mathfrak{m}$, for any integer n there is an integer $l(n)$ such that $\mathfrak{m}^{l(n)} \subseteq I^n$. Hence

$$(I^n N)_a :_N < \mathfrak{m} > \supseteq I^n N :_N \mathfrak{m}^{l(n)} \supseteq I^n N :_N I^n = N$$

Thus $\bigcap_{n=1}^{\infty}((I^n N)_a :_N < \mathfrak{m} >) = N \neq 0$. □

Let I denote a non-zero ideal of the Noetherian domain R and N a finitely generated torsion-free R-module. Then one can check easily that $\mathrm{rank}_R(IN) = \mathrm{rank}_R(N)$. We shall use this fact in the remainder of this section without further comment.

Theorem 3.3. *Let R be a locally analytically normal (Noetherian) domain and I an ideal of R. Let N be a non-zero finitely generated torsion-free R-module. For any prime ideal $\mathfrak{p} \supseteq I$ of R, the following are equivalent:*

(i) $\mathfrak{p} \in \mathrm{Min}_R R/I$.

(ii) There exists an integer $k \geq 1$ such that $(I^m N :_N < \mathfrak{p} >) \nsubseteq (\mathfrak{p}^k N_{\mathfrak{p}})_a \cap N$, for all $m \in \mathbb{N}$.

(iii) There exists an integer $k \geq 1$ such that $(I^m :_R < \mathfrak{p} >) \nsubseteq (\mathfrak{p}^k R_{\mathfrak{p}})_a \cap R$, for all $m \in \mathbb{N}$.

Proof. The equivalence of conditions (i) and (iii) is proved in [9, Proposition 3.5]. We have $I^m N :_N < \mathfrak{p} > \subseteq (\mathfrak{p}^k N_{\mathfrak{p}})_a \cap N$ if and only if $I^m N_{\mathfrak{p}} :_{N_{\mathfrak{p}}} < \mathfrak{p} R_{\mathfrak{p}} > \subseteq (\mathfrak{p}^k N_{\mathfrak{p}})_a$, and $\mathfrak{p} \in \mathrm{Min}_R R/I$ if and only if $\mathfrak{p} R_{\mathfrak{p}} \in \mathrm{Min}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}})$. Hence we assume that R is local at $\mathfrak{p}$ and write $(\mathfrak{p}^k N)_a$ instead of $(\mathfrak{p}^k N_{\mathfrak{p}})_a \cap N$.

(i)$\Longrightarrow$(ii). Suppose that $\mathfrak{p} \in \mathrm{Min}_R R/I$. Then I is $\mathfrak{p}$-primary. By Proposition 2.5, $\bigcap_{n=1}^{\infty}(\mathfrak{p}^n N)_a = 0$. Since $N \neq 0$, there exists $k \in \mathbb{N}$ such that $(\mathfrak{p}^k N)_a \subsetneqq N$. We show that

$$I^m N :_N < \mathfrak{p} > \nsubseteq (\mathfrak{p}^k N)_a$$

for all $m \in \mathbb{N}$. Assume there exists $m \in \mathbb{N}$ such that $I^m N :_N < \mathfrak{p} > \subseteq (\mathfrak{p}^k N)_a$. Then

$$I \subseteq (I^m N :_N < \mathfrak{p} >) :_R N \subseteq \mathrm{Rad}((\mathfrak{p}^k N)_a :_R N) \subseteq \mathfrak{p}.$$

Hence $\mathrm{Rad}((I^m N :_N < \mathfrak{p} >) :_R N) = \mathfrak{p}$ and so $\mathfrak{p} \in \mathrm{Ass}_R(N/I^m N :_N < \mathfrak{p} >)$ which is a contradiction as one can see easily.

(ii)$\Longrightarrow$(i). In view of Lemma 2.4, the condition (ii) implies that $(I^m N^*)_a :_{N^*} < \mathfrak{p} R^* > \nsubseteq (\mathfrak{p}^k N^*)_a$ for all $m \in \mathbb{N}$. Hence by Proposition 3.2, the ideal IR^* is $\mathfrak{p} R^*$-primary and so $\mathfrak{p} \in \mathrm{Min}_R R/I$. □

Before bringing the next result we fix some notation. Let S be a multiplicatively closed subset of R. For a submodule M of the Noetherian

R-module N, we use $S(M)$ to denote $\bigcup_{s\in S}(M :_N s)$. Also, for an integer $k \geq 1$, and a prime ideal $\mathfrak{p}$ of R, we denote $\bigcup_{s\in R\setminus\mathfrak{p}}((\mathfrak{p}^k N)_a :_N s)$ by $(\mathfrak{p}N)^{<k>}$.

Now, we are ready to prove the main result of this section.

Theorem 3.4. *Let R be a locally analytically normal (Noetherian) domain, I an ideal of R and S a multiplicatively closed subset of R. Then the following are equivalent:*

(i) $S \subseteq R\setminus \bigcup\{\mathfrak{p} \in \mathrm{Min}_R R/I\}$.

(ii) *Let N be a finitely generated torsion-free R-module. For all primes $\mathfrak{q} \supseteq I$ and $k \geq 0$, there is a $m \geq 0$ with $S((I^m N)_a) \subseteq (\mathfrak{q}N)^{<k>}$.*

(iii) *The topology $\{S(I^n)\}_{n\geq 1}$ is finer than the topology defined by the integral closure filtration $\{(I^n)_a\}_{n\geq 1}$.*

(iv) *The topology $\{S(I^n N)\}_{n\geq 1}$ is finer than the topology defined by the integral closure filtration $\{(I^n N)_a\}_{n\geq 1}$, for any finitely generated torsion-free R-module N.*

Proof. (i)$\Longrightarrow$(ii). Suppose that (i) holds, and let $\mathfrak{q}$ and k be as in (ii). First, we localize at $\mathfrak{q}$. If S' is the image of S in $R_\mathfrak{q}$, under the natural monomorphism $R \longrightarrow R_\mathfrak{q}$, then (i) implies that $S' \subseteq R_\mathfrak{q}\setminus \bigcup\{\mathfrak{q}' \in \mathrm{Min}_{R_\mathfrak{q}}(R_\mathfrak{q}/IR_\mathfrak{q})\}$. Also it is easy, from Lemma 2.3, to see that if $S'(I_\mathfrak{q}^m N_\mathfrak{q})_a \subseteq (\mathfrak{q}N_\mathfrak{q})^{<k>}$, then $S(I^m N)_a \subseteq (\mathfrak{q}N)^{<k>}$. Therefore we may assume that R is local at $\mathfrak{q}$ (and write $(\mathfrak{q}^k N)_a$ instead of $(\mathfrak{q}N)^{<k>}$). Next we see that we may assume that R is complete. To this end, first note that condition (i) holds for S and the ideal IR^* of R^*. On the other hand Lemma 2.4 implies that, if $S(I^m N^*)_a \subseteq (\mathfrak{q}^k N^*)_a$, then $S(I^m N)_a \subseteq (\mathfrak{q}^k N)_a$. Hence we may assume in addition that R is complete. Since I is disjoint from S, by Proposition 2.5, $\bigcap_{n=1}^{\infty}[(I^n S^{-1}N)_a \cap N] = 0$. Hence by Chevalley's theorem (see, Lemma 3.1), there is an $m \geq 0$ with $(I^m S^{-1}N)_a \cap N \subseteq \mathfrak{q}^k N$. Thus by Lemma 2.3, $S^{-1}(I^m N)_a \cap N \subseteq \mathfrak{q}^k N$. That is $S(I^m N)_a \subseteq \mathfrak{q}^k N \subseteq (\mathfrak{q}^k N)_a$. So that (ii) holds.

(ii)$\Longrightarrow$(i). Let (ii) holds, and let $\mathfrak{p} \in \mathrm{Min}_R R/I$. By Theorem 3.3, there is $k \geq 1$ such that $I^n N :_N < \mathfrak{p} > \not\subseteq (\mathfrak{p}N)^{<k>}$ for all $n \geq 1$. The condition (ii) implies that, there is a $m \geq 1$ with $S(I^m N)_a \subseteq (\mathfrak{p}N)^{<k>}$. For this m, we have $I^m N :_N < \mathfrak{p} > \not\subseteq S(I^m N)_a$. Let $x \in I^m N :_N < \mathfrak{p} >$ but $x \notin S(I^m N)_a$, in particular $x \notin S(I^m N)$. Now, there exists a $n \geq 1$ such that $\mathfrak{p}^n x \subseteq I^m N$. If $\mathfrak{p} \cap S \neq \emptyset$, then there is a $r \in \mathfrak{p}^n \cap S$. Hence $rx \in I^m N$ or $x \in I^m N :_S r \subseteq S(I^m N)$. Thus $x \in S(I^m N)$, which is a contradiction. Therefore $\mathfrak{p} \cap S = \emptyset$, so that (i) holds.

(ii)$\Longrightarrow$(iv). Let (ii) hold and let $k \geq 0$. Let N be a finitely generated torsion-free R-module. We consider a primary decomposition $Q_1 \cap \cdots \cap Q_n$ of $(I^k N)_a$, with Q_i primary submodule to $\mathfrak{p}_i$ and $(Q_i)_a = Q_i$. Note that such primary decomposition exists by Theorem 2.6. For some k_i, we have

$\mathfrak{p}_i^{k_i}N \subseteq Q_i$ for $i = 1, 2, \ldots, n$, and furthermore by (ii) for some m_i, $S(I^{m_i}N) \subseteq (\mathfrak{p}_i N)^{<k_i>}$, (note that for all $1 \leq i \leq n$, $I \subseteq \mathfrak{p}_i$). Let $m = \max\{m_1, \ldots, m_n\}$, we see that

$$S(I^m N) \subseteq (\mathfrak{p}_i N)^{<k_i>}$$

for all $1 \leq i \leq n$. But we have

$$(\mathfrak{p}_i N)^{<k_i>} = \bigcup_{s \in R \setminus \mathfrak{p}_i} [(\mathfrak{p}_i^{k_i} N)_a :_N s] \subseteq \bigcup_{s \in R \setminus \mathfrak{p}_i} (Q_i :_N s) = Q_i.$$

Therefore $S(I^m N) \subseteq \bigcap_{i=1}^n Q_i = (I^k N)_a$, and so (iv) holds.

(iv)$\Longrightarrow$(i). Let $\mathfrak{p} \in \mathrm{Min}_R R/I$. By Theorem 3.3, there exists $k \geq 1$ such that $I^n N :_N < \mathfrak{p} > \not\subseteq (\mathfrak{p}N)^{<k>}$ for all $n \geq 1$. Since $(I^k N)_a \subseteq (\mathfrak{p}^k N)_a \subseteq (\mathfrak{p}N)^{<k>}$, it follows from condition (iv), that there is $m \geq 1$ with $S(I^m N) \subseteq (\mathfrak{p}N)^{<k>}$. For this m we have $I^m N :_N < \mathfrak{p} > \not\subseteq S(I^m N)$. Choose $x \in (I^m N :_N < \mathfrak{p} >) \setminus S(I^m N)$. Now, we can process similar to the proof of the implication (ii)$\Longrightarrow$(i) to deduce that $\mathfrak{p} \cap S = \emptyset$. Hence (i) holds.

Finally, (i)$\Longleftrightarrow$(iii) is trivial using [9, Theorem 1.5]. □

Corollary 3.5. *Let R be a locally analytically normal (Noetherian) domain, I an ideal of R and S a multiplicatively closed subset of R. Suppose that N is a finitely generated torsion-free R-module. Then the following are equivalent:*

(i) The topology $\{S(I^n N)\}_{n \geq 0}$ *is finer than the topology* $\{(I^n N)_a\}_{n \geq 0}$.
(ii) $H^{\mathrm{ht}\,\mathfrak{q}}_{IR_\mathfrak{q}}(N_\mathfrak{q}) = 0$ *for all* $\mathfrak{q} \in V(I)$ *with* $\mathfrak{q} \cap S \neq \emptyset$.

Proof. (i)$\Longrightarrow$(ii). By Theorem 3.4, $S \subseteq R \setminus \cup \{\mathfrak{p} \in \mathrm{Min}_R R/I\}$. Let $\mathfrak{q} \in V(I)$, with $\mathfrak{q} \cap S \neq \emptyset$, it follows that $\mathfrak{q} \notin \mathrm{Min}_R R/I$. Thus we have $\dim R^*_\mathfrak{q}/IR^*_\mathfrak{q} > 0$. So from the Lichtenbaum-Hartshorne vanishing Theorem [14, Theorem 1.1 (c)], it follows that $H^{\mathrm{ht}\,q}_{IR_\mathfrak{q}}(N_\mathfrak{q}) = 0$ and so (ii) holds.

(ii)$\Longrightarrow$(i). Let (ii) hold, and $\mathfrak{p}$ be any prime of $\mathrm{Min}_R R/I$. Then $IR_\mathfrak{p}$ is$\mathfrak{p}R_\mathfrak{p}$-primary and so by Grothendieck's non-vanishing Theorem [3, Theorem 6.1.2], $H^{\mathrm{ht}\,\mathfrak{p}}_{IR_\mathfrak{p}}(N_\mathfrak{p}) \neq 0$. Hence by assumption $S \cap \mathfrak{p} = \emptyset$. Thus $S \subseteq R \setminus \cup \{\mathfrak{p} \in \mathrm{Min}_R R/I\}$, which completes the proof of the corollary by Theorem 3.4. □

Corollary 3.6. *Let* $(R, \mathfrak{m})$ *be a locally analytically Noetherian normal domain, and let I be an unmixed ideal of R which is not* $\mathfrak{m}$*-primary. Suppose that N is a finitely generated torsion-free R-module. Then*

$$\mathrm{Supp}_R\, H^i_I(N) \subseteq \{\mathfrak{q} \in V(I) | \mathrm{ht}\,\mathfrak{q} \geq i + 1\}$$

for all i with $\mathrm{ht}\, I < i \leq d$, *where* $d = \dim R$.

Proof. It is clear that $\operatorname{Supp}_R H_I^i(N) \subseteq V(I)$, by [3, Exercise 6.2.6]. Furthermore if $\mathfrak{p} \in V(I)$ is such that $\operatorname{ht}\mathfrak{p} < i$, then by [3, Theorems 4.3.2 and 6.1.2] we have $(H_I^i(N))_\mathfrak{p} = 0$, since $\dim N_\mathfrak{p} = \operatorname{ht}\mathfrak{p} < i$. Thus

$$\operatorname{Supp}_R H_I^i(N) \subseteq \{\mathfrak{q} \in V(I) | \operatorname{ht}\mathfrak{q} \geq i\}.$$

Now, let $S = R \setminus \cup \{\mathfrak{p} \in \operatorname{Min}_R R/I\}$ then by Theorem 3.4 and Corollary 3.5, $(H_I^{\operatorname{ht}\mathfrak{q}}(N))_\mathfrak{q} = 0$ for all $\mathfrak{q} \in V(I)$ with $\mathfrak{q} \cap S \neq \emptyset$. Hence for all i with $\operatorname{ht} I < i \leq d$, $\operatorname{Supp}_R H_I^i(N) \subseteq \{\mathfrak{q} \in V(I) | \operatorname{ht}\mathfrak{q} = i \text{ and } \mathfrak{q} \cap S = \emptyset\} \cup \{\mathfrak{q} \in V(I) | \operatorname{ht}\mathfrak{q} \geq i+1\}$. Now, if $\mathfrak{q} \in V(I)$ and $\mathfrak{q} \cap S = \emptyset$, then $\mathfrak{q} \in \operatorname{Min}_R R/I$ and since I is unmixed, it follows that $\operatorname{ht}\mathfrak{q} = \operatorname{ht} I$. Thus $\{\mathfrak{q} \in V(I) | \operatorname{ht}\mathfrak{q} = i$ and $\mathfrak{q} \cap S = \emptyset\} = \emptyset$, for all i with $\operatorname{ht} < i \leq d$. Therefore

$$\operatorname{Supp}_R H_I^i(N) \subseteq \{\mathfrak{q} \in V(I) | \operatorname{ht}\mathfrak{q} \geq i+1\}$$

for all $\operatorname{ht} I < i \leq d$ as desired. □

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