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QUINTESSENTIAL PRIMES AND IDEAL TOPOLOGIES OVER A MODULE

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ABSTRACT

Let I be an ideal of a Noetherian ring R, N a finitely generated R-module and let S be a multiplicatively closed subset of R. We define the nth (S)-symbolic power of I w.r.t. N as $S(I^nN) = \bigcup_{s\in S}(I^nN :_N s)$. The purpose of this paper is to show that the topologies defined by $\{I^nN\}_{n\geq 0}$ and $\{S(I^nN)\}_{n\geq 0}$ are equivalent (resp. linearly equivalent) if and only if S is disjoint from the quintessential (resp. essential) primes of I w.r.t. N.

1. INTRODUCTION

Let R be a commutative Noetherian ring, S a multiplicatively closed subset of R, I an ideal of R and N a finitely generated R-module. The nth (S)-symbolic power of I w.r.t. N, denoted by $S(I^nN)$, is defined to be the union of $I^nN :_N s$, where s varies in S. The I-adic filtration $\{I^nN\}_{n\geq 0}$ and the (S)-symbolic filtration $\{S(I^nN)\}_{n\geq 0}$ induce topologies on N which are called I-adic topology and (S)-symbolic topology respectively. These two topologies are said to be equivalent (resp. linearly equivalent), if for every integer $m \geq 0$, there is an integer $n \geq 0$ such that $S(I^nN) \subseteq I^mN$, (resp. there is an integer $h \geq 0$ such that $S(I^{k+h}N) \subseteq I^kN$ for all $k \geq 0$). In the case $S = R\backslash \cup \{\mathfrak{p} \in m\operatorname{Supp}_R(N/IN)\}$, where $m\operatorname{Supp}_R(N/IN)$ is the set of the minimal primes of $\operatorname{Supp}_R(N/IN)$, the n-th (S)-symbolic power of I w.r.t. N is denoted by $(IN)^{(n)}$. Equivalence of I-adic topology and (S)-symbolic topology has been studied, in the case $N = R$, in [4, 5, 8, 10–12, 15–17] and has led to some interesting results.

The purpose of the present paper is to characterize the equivalence (resp. linearly equivalence) between these topologies in terms of quintessential (resp. essential) primes. Using this characterization, we show that the topologies $\{I^mN\}_{m\geq 0}$ and $\{S(I^mN)\}_{m\geq 0}$ are equivalent if and only if, for any prime ideal $\mathfrak{p}$ containing I with $\mathfrak{p} \cap S \neq \phi$, the topologies defined by the filtrations $\{I^mN_{\mathfrak{p}} :_{N_{\mathfrak{p}}} \langle \mathfrak{p} \rangle\}_{m\geq 0}$ and $\{I^mN_{\mathfrak{p}}\}_{m\geq 0}$ are equivalent. We also show that, for a prime ideal $\mathfrak{p} \in \operatorname{Supp}(N/IN)\backslash m\operatorname{Supp}(N/IN)$, if the topology given by the filtration $\{(IN)^{(m)}\}_{m\geq 0}$ is finer than the topology given by the filtration $\{(\mathfrak{p}N)^{(m)}\}_{m\geq 0}$, then $\mathfrak{p} \notin \bar{Q}(I, N)$.

Throughout this paper all rings are commutative Noetherian, with identity, unless otherwise specified. We shall use R to denote such a ring, I an ideal of R, and N a non-zero finitely generated module over R. We denote by $\mathcal{R}$ the Rees ring $R[u, It] := \bigoplus_{n\in\mathbb{Z}} I^nt^n$ of R w.r.t. I, where t is an indeterminate and $u = t^{-1}$. Also, the graded Rees module $N[u, It] := \bigoplus_{n\in\mathbb{Z}} I^nN$ over $\mathcal{R}$ is denoted by $\mathcal{N}$, which is a finitely generated $\mathcal{R}$-module. If $(R, \mathfrak{m})$ is local, then R^* (resp. N^*) denotes the completion of R (resp. N) w.r.t. the $\mathfrak{m}$-adic topology. In particular, for any $\mathfrak{p} \in \operatorname{Spec} R$, we denote $R^*_{\mathfrak{p}}$ and $N^*_{\mathfrak{p}}$ the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. For any submodule M of N, the submodules $\bigcup_{s\in S}(M :_N s)$ and $\bigcup_{n\geq 0}(M :_N I^n)$ of N are denoted by $S(M)$ and $M :_N \langle I \rangle$ respectively, where S is a multiplicatively closed subset of R. In the case $S = R\backslash \bigcup\{\mathfrak{p} \in m\operatorname{Supp}_R N/IN\}$, the submodule $S(I^mN)$ is denoted by $(IN)^{(m)}$. In particular if $\mathfrak{p}$ is a prime ideal of R, then $(\mathfrak{p}N)^{(m)} := \bigcup_{s\in R\backslash\mathfrak{p}}(\mathfrak{p}^mN :_N s)$. For any ideal J of R, the *radical* of J, denoted by $\operatorname{Rad}(J)$, is defined to be the set $\{a \in R : a^n \in J \text{ for some } n \in \mathbb{N}\}$. Finally, for any R-module L, we shall use $m\operatorname{Ass}_R L$ to denote the set of minimal elements of $\operatorname{Ass}_R L$.

In the second section we will study some basic results which will be needed in the Sections 3 and 4. Some of these results has been established, by S. McAdam and L. J. Ratliff, Jr, in [7], in certain case when $N = R$. In this section, among other things, we prove the following theorem.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a complete ring. Then, with the above notations, the following conditions are equivalent:*

(i) *There is a $\mathfrak{p} \in \mathrm{Ass}_R N$ such that $\mathrm{Rad}(I + \mathfrak{p}) = \mathfrak{m}$.*
(ii) *There is an integer $k \geq 1$ such that, for all integers $m \geq 1$, $I^m N :_N \langle \mathfrak{m} \rangle \not\subseteq \mathfrak{m}^k N$.*

In the third section we examine the equivalence of the topologies defined by $\{I^n N\}_{n\geq 0}$, $\{(IN)^{(n)}\}_{n\geq 0}$ and $\{S(I^n N)\}_{n\geq 0}$ by using the quintessential primes of I. Some of these results has been established, by P. Schenzel in [11,12] and S. McAdam in [8], in certain case when $N = R$. A typical result in this direction is the following:

Theorem 1.2. *The following conditions are equivalent:*

(i) $Q(I, N) = m\ \mathrm{Ass}_R(N/IN)$.
(ii) *The topologies $\{(IN)^{(n)}\}_{n\geq 0}$ and $\{I^n N\}_{n\geq 0}$ are equivalent.*

The proof of Theorem 1.2 is given in 3.7. Finally in the fourth section, we study the linearly equivalence of the I-adic and (S)-symbolic topologies in terms of essential primes of I. The main result of this section is:

Theorem 1.3. *Let S be a multiplicatively closed subset of R. Then the I-adic and (S)-symbolic topologies, on N, are linearly equivalent if and only if S is disjoint from the essential primes of I.*

2. SOME BASIC RESULTS

The purpose of this section is to establish some results which will be needed later. The aim goal of this section is Theorem 2.6, which plays a key role in this paper. The following lemma is needed in the proof of that theorem.

Lemma 2.1. (i) (cf. [7, (2.1) LEMMA]). *Let $(R, \mathfrak{m})$ be local and $\mathfrak{p} \in \mathrm{Ass}_R N$. Then there exists a non-zero element y in N such that, for every submodule M of N with $\mathrm{Rad}(M :_R N + \mathfrak{p}) = \mathfrak{m}$, either $y \in M$ or $\mathfrak{m} \in \mathrm{Ass}_R N/M$.*

(ii) (cf. [7, (2.2) COROLLARY]). *Let* $\mathfrak{p} \in \operatorname{Spec} R$ *and* $\mathfrak{q} \in \operatorname{Ass}_R N$ *be such that* $\mathfrak{q} \subseteq \mathfrak{p}$. *Then there is an integer* $k \geq 1$ *such that* $\mathfrak{p} \in \operatorname{Ass}_R N/M$ *for any submodule* M *of* N *with* $M \subseteq (\mathfrak{p}N)^{(k)}$ *and that* $\mathfrak{p} \in m \operatorname{Ass}_R (R/M :_R N + \mathfrak{q})$.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}_R N$, then there is a non-zero element $y \in N$ such that $\mathfrak{p} = 0 :_R y$. Let M be any submodule of N such that $\operatorname{Rad}(M :_R N + \mathfrak{p}) = \mathfrak{m}$ and $\mathfrak{m} \notin \operatorname{Ass}_R N/M$. It is enough to show that $y \in M$. To this end, let $Q_1 \cap \cdots \cap Q_n = M$ be an irredundant primary decomposition of the submodule M, with Q_i is $\mathfrak{p}_i$-primary submodule. Then $\mathfrak{m} \neq \mathfrak{p}_i$, for all $i = 1, 2, \ldots, n$. Since $(M :_R N) \subseteq \mathfrak{p}_i$ with $1 \leq i \leq n$, we have $\mathfrak{p} \not\subseteq \mathfrak{p}_i$ for every $i = 1, 2, \ldots, n$. Therefore, because of $\mathfrak{p}y = 0$, we have $\mathfrak{p}y \subseteq Q_i$ for all $i = 1, 2, \ldots, n$. Thus $y \in Q_1 \cap \cdots \cap Q_n$. Therefore $y \in M$, so that (i) holds.

In order to prove (ii), we may assume that $(R, \mathfrak{p})$ is local. Then $(\mathfrak{p}N)^{(k)} = \mathfrak{p}^k N$. Let y be as in (i). Using Krull's Intersection Theorem, there exists an integer $k \geq 1$ such that $y \notin \mathfrak{p}^k N$. Now, it is straightforward to check that the assertion follows from (i). □

Remark 2.2. Let M be a submodule of N. The increasing sequence of submodules

$$M \subseteq M :_N I \subseteq M :_N I^2 \subseteq \cdots \subseteq M :_N I^n \subseteq \cdots$$

becomes stationary. Denote its ultimate constant value by $M :_N \langle I \rangle$. Note that $M :_N \langle I \rangle = M :_N I^n$ for all large n. Let $M = Q_1 \cap \cdots \cap Q_r\ apQ_{r+1} \cap \cdots \cap Q_s$ be an irredundant primary decomposition of M with $I \subseteq \operatorname{Rad}(Q_i :_R N)$ exclusively for $r + 1 \leq i \leq s$. Then, from the definition, it easily follows that $M :_N \langle I \rangle = Q_1 \cap \cdots \cap Q_r$. Therefore

$$\operatorname{Ass}_R N/M :_N \langle I \rangle = \{\mathfrak{p} \in \operatorname{Ass}_R N/M : I \not\subseteq \mathfrak{p}\} = \operatorname{Ass}_R (N/M) \backslash V(I).$$

In the case $N = R$, the following lemma was proved by Chevalley. The proof given in [9, Theorem 30.1] can be easily carried over to a module, (see, [2, Ch. IV. Section 2.5, Corollary 4]).

Lemma 2.3. *Let* $(R, \mathfrak{m})$ *be a complete local ring and let* $\{N_n : n = 1, 2, \ldots\}$ *be a decreasing sequence of submodules of N such that* $\bigcap_{n \geq 0} N_n = 0$. *Then for all integers* $r \geq 1$ *there exists* $n(r) \geq 1$ *such that* $N_{n(r)} \subseteq \mathfrak{m}^r N$.

Remark 2.4. (see, [9, Theorem 18.1]). Let T be a ring which is a flat R-module and let N_1, N_2 be submodules of N. Then

$$(N_1 :_R N_2) \otimes_R T = (N_1 \otimes_R T :_T N_2 \otimes_R T).$$

In particular, if $(R, \mathfrak{m})$ is local, then $(N_1 :_R N_2)R^* = (N_1^* :_{R^*} N_2^*)$.

The following lemma is almost certainly known, but I could not find a reference for it. So it is explicitly stated and proved here.

Lemma 2.5. *Let y be an element of N. Then there is an integer $r \geq 0$ such that, for all $m \geq r$,*

$$I^m N :_R Ry = I^{m-r}(I^r N :_R Ry) + (0 :_R Ry).$$

Proof. By the Artin-Rees Lemma, there exists an integer $r \geq 0$ such that, for all $m \geq r$,

$$I^m N \cap Ry = I^{m-r}(I^r N \cap Ry).$$

It is easy to see that, $I^m N \cap Ry = (I^m N :_R Ry)y$. Consequently, when $m \geq r$, we have

$$(I^m N :_R Ry)y = I^{m-r}(I^r N :_R Ry)y.$$

Let $a \in I^m N :_R Ry$ and suppose that $m \geq r$. Then ay belongs to $(I^m N :_R Ry)y$ and therefore $ay = a'y$ for some $a' \in I^{m-r}(I^r N :_R Ry)$. It follows that $a \in I^{m-r}(I^r N :_R Ry) + (0 :_R Ry)$. Accordingly $I^m N :_R Ry \subseteq I^{m-r}(I^r N :_R Ry)+$ $(0 :_R Ry)$. As the opposite inclusion is obvious, the result follows. $\square$

Now we are prepared to prove the main theorem of this section, which is a comparison of the topologies defined by certain decreasing families of submodules of a finitely generated module over a commutative Noetherian complete local ring.

Theorem 2.6. (cf. [7, (2.5) LEMMA]). *Let $(R, \mathfrak{m})$ be local. Consider the following conditions*:

(i) $\bigcap_{m=0}^{\infty}(I^m N^* :_{N^*} \langle \mathfrak{m} \rangle) \neq 0$.
(ii) *There is a $k > 0$ such that, for all $m > 0$, $I^m N :_N \langle \mathfrak{m} \rangle \not\subseteq \mathfrak{m}^k N$.*
(iii) *There is a $\mathfrak{p} \in \mathrm{Ass}_R N$ with $\mathrm{Rad}(I + \mathfrak{p}) = \mathfrak{m}$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii); *and these conditions are equivalent, whenever R is complete.*

Proof. The equivalence of conditions (i) and (ii) is proved in [13, Lemma 2.2]. In order to prove the implication (iii) $\Rightarrow$ (i), suppose there is an associated prime ideal $\mathfrak{p}$ of N such that $\mathrm{Rad}(I + \mathfrak{p}) = \mathfrak{m}$. By [6, Theorem 23.2] there exists $\mathfrak{q} \in \mathrm{Ass}_{R^*} N^*$ with $\mathfrak{q} \cap R = \mathfrak{p}$. Now, it is easy to see that $\mathrm{Rad}(IR^* + \mathfrak{q}) = \mathfrak{m}R^*$. If $\mathrm{Rad}(IN^* :_{R^*} N^*) = \mathfrak{m}R^*$, then one easily sees that $\mathrm{Rad}(I^n N^* :_{R^*} N^*) = \mathfrak{m}R^*$, for all $n \geq 1$; and so $I^n N^* :_{N^*} \langle \mathfrak{m} \rangle = N^*$. Therefore $\bigcap_{n \geq 0}(I^n N^* :_{N^*} \langle \mathfrak{m} \rangle) = N^* \neq 0$, as desired. Accordingly, we may assume that $\mathrm{Rad}(IN^* :_{R^*} N^*) \subsetneq \mathfrak{m}R^*$. Then we have

$$I^n R^* \subseteq I^n N^* :_{R^*} N^* \subseteq (I^n N^* :_{N^*} \langle \mathfrak{m} \rangle) :_{R^*} N^* \subseteq \mathfrak{m}R^*,$$

for all $n \geq 1$. So $\mathrm{Rad}((I^nN^* :_{N^*} \langle \mathfrak{m} \rangle) :_{R^*} N^* + \mathfrak{q}) = \mathfrak{m}R^*$. Therefore, because of $\mathfrak{m}R^* \notin \mathrm{Ass}_{R^*} N^*/I^nN^* :_{N^*} \langle \mathfrak{m} \rangle$, (i) follows from Lemma 2.1 (i).

Finally, assume that R is complete and that (i) holds. We show that (iii) is true. To this end, let y be a non-zero element of $\bigcap_{n \geq 0}(I^nN :_N \langle \mathfrak{m} \rangle)$. Then, by Lemma 2.5, there exists an integer $r \geq 0$ such that, for all $m \geq r$,

$$I^mN :_R Ry \subseteq I^{m-r} + (0 :_R Ry) \subseteq I^{m-r} + \mathfrak{p},$$

where $\mathfrak{p}$ is an associated prime ideal of N with $\mathfrak{p} \supseteq (0 :_R Ry)$. Now, for any $m \geq 0$ there is an integer $n \geq 0$ with $\mathfrak{m}^n \subseteq I^mN :_R Ry \subseteq I^{m-r} + \mathfrak{p}$. So $\mathrm{Rad}(I + \mathfrak{p}) = \mathfrak{m}$, as required. □

3. QUINTESSENTIAL PRIMES AND COMPARISON OF TOPOLOGIES

The purpose of this section is to establish a relationship between the topologies defined by the filtrations $\{I^nN\}_{n \geq 0}$, $\{(IN)^{(n)}\}_{n \geq 0}$ and $\{S(I^nN)\}_{n \geq 0}$, by using the quintessential primes of I w.r.t. N. The main results are Theorems 3.5 and 3.6.

Definition. *A prime ideal $\mathfrak{p}$ of R is called a Quintessential prime ideal of I w.r.t. N precisely when there exists $\mathfrak{q} \in \mathrm{Ass}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}}$ such that $\mathrm{Rad}(IR^*_{\mathfrak{p}} + \mathfrak{q}) = \mathfrak{p}R^*_{\mathfrak{p}}$. The set of Quintessential prime ideals of I w.r.t. N is denoted by $Q(I, N)$.*

Lemma 3.1. *Let S be a multiplicatively closed subset of R. Then*

(i) *For any prime ideal $\mathfrak{p}$ of R disjoint from S, $\mathfrak{p} \in Q(I, N)$ if and only if $S^{-1}\mathfrak{p} \in Q(S^{-1}I, S^{-1}N)$.*
(ii) *$m\,\mathrm{Ass}_R N/IN \subseteq Q(I, N)$.*
(iii) *If J is a second ideal of R with $\mathrm{Rad}(J) = \mathrm{Rad}(I)$, then $Q(J, N) = Q(I, N)$.*
(iv) [1, 3.5] *If $\mathfrak{q} \in \mathrm{Ass}_R N$, then $m\mathrm{Ass}_R R/(I + \mathfrak{q}) \subseteq Q(I, N)$.*

Proof. (i), (ii) and (iii) follow immediately from the definition. To prove (iv) use [6, Theorem 23.2] and the fact that $\dim T = \dim T^*$ for any local ring T. □

Remark 3.2. Before bringing the next result we fix a notation, which is employed by P. Schenzel in [14] in the case $N = R$. Let S be a multiplicatively closed subset of R. For a submodule M of N, we use $S(M)$ to denote the submodule $\bigcup_{s \in S}(M :_N s)$. Note that the primary decomposition of $S(M)$ consists of the intersection of all primary components of M whose associated prime ideals do not meet S. In other words

$\mathrm{Ass}_R N/S(M) = \{\mathfrak{p} \in \mathrm{Ass}_R N/M : \mathfrak{p} \cap S = \phi\}$

In particular, if $S = R \setminus \cup\{\mathfrak{p} \in m\mathrm{Ass}_R N/IN\}$ then, for every $n \in \mathbb{N}, S(I^n N)$ is denoted by $(IN)^{(n)}$. The following theorem is originally shown by S. McAdam and L. J. Ratliff, Jr. in [7, (3.2) PROPOSITION] in the case $N = R$. Our approach is shorter than given in [1, 3.12].

Theorem 3.3. *Let $\mathfrak{p}$ be a prime ideal containing I. Then the following conditions are equivalent*:

(i) $\mathfrak{p} \in Q(I, N)$.

(ii) *There is an integer $k \geq 1$ such that $\mathfrak{p} \in \mathrm{Ass}_R N/M$ for any submodule M of N with $\mathrm{Rad}(M :_R N) = \mathrm{Rad}(IN :_R N)$ and $M \subseteq (\mathfrak{p}N)^{(k)}$.*

(iii) *There is an integer $k \geq 0$ such that $I^m N :_N \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}N)^{(k)}$ for all $m \geq 0$.*

Proof. (i)⇒(ii). Let $\mathfrak{p} \in Q(I, N)$. Then there exists $\mathfrak{q} \in \mathrm{Ass}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$ such that $\mathrm{Rad}(IR^*_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}R^*_\mathfrak{p}$. Now, let k be as in Lemma 2.1(ii), applied to $\mathfrak{q} \in \mathrm{Ass}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$, and let M be any submodule of N such that $\mathrm{Rad}(M :_R N) = \mathrm{Rad}(IN :_R N)$ and $M \subseteq (\mathfrak{p}N)^{(k)}$. Then $M^*_\mathfrak{p} \subseteq \mathfrak{p}^k N^*_\mathfrak{p} = (\mathfrak{p}N^*_\mathfrak{p})^{(k)}$ and, in view of Remark 2.4, it is easy to see that $IR^*_\mathfrak{p} \subseteq \mathrm{Rad}(M^*_\mathfrak{p} :_{R^*_\mathfrak{p}} N^*_\mathfrak{p})$. One can use Nakayama's Lemma to show that $M^*_\mathfrak{p} :_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$ is a proper ideal of $R^*_\mathfrak{p}$. Therefore $\mathrm{Rad}(M^*_\mathfrak{p} :_{R^*_\mathfrak{p}} N^*_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}R^*_\mathfrak{p}$. We can now use Lemma 2.1 (ii) and [6, Theorem 23.2] to complete the proof of (ii).

In order to prove the implication (ii) ⇒ (iii), we may assume that $(R, \mathfrak{p})$ is local. Then $(\mathfrak{p}N)^{(k)} = \mathfrak{p}^k N$. Suppose, the contrary, that (iii) is not true. Then, for any integer $k \geq 0$, there is an integer $m \geq 0$ with $I^m N :_N \langle \mathfrak{p} \rangle \subseteq \mathfrak{p}^k N$. Now, by Nakayama's Lemma, $\mathrm{Rad}(IN :_R N) \neq \mathfrak{p}$; and hence $\mathrm{Rad}((IN :_N \langle \mathfrak{p} \rangle) :_R N) = \mathrm{Rad}(IN :_R N)$. Therefore the condition (ii) implies that $\mathfrak{p} \in \mathrm{Ass}_R N/ IN :_N \langle \mathfrak{p} \rangle$ which provides a contradiction (see Remark 2.2).

Finally, in order to complete the proof, we have to show the implication (iii)⇒(i). To do this end, suppose that there is an integer $k \geq 0$ such that for all $m \geq 0$, $I^m N :_N \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}N)^{(k)}$. Then by Remark 2.4 and [9, Corollary 17.10] we have $I^m N^*_\mathfrak{p} :_{N^*_\mathfrak{p}} \langle \mathfrak{p}R^*_\mathfrak{p} \rangle \nsubseteq (\mathfrak{p}N^*_\mathfrak{p})^{(k)}$. By virtue Theorem 2.6 this proves (i). □

The next lemma, which is a consequence of Krull's Intersection Theorem, is of assistance in the proof of the main results 3.5 and 3.6.

Lemma 3.4. *Let $\mathfrak{p} + I$ be a proper ideal of R, for all $\mathfrak{p} \in \mathrm{Ass}_R N$. Then*

$$\bigcap_{n \geq 1} I^n N = 0.$$

Theorem 3.5. (cf. [8, Theorem 1.2]). *Let S be a multiplicatively closed subset of R. Then the following conditions are equivalent.*

(i) $S \subseteq R \backslash \cup \{\mathfrak{p} \in Q(I, N)\}$.
(ii) *The (S)-symbolic topology is finer than the I-adic topology.*
(iii) *The (S)-symbolic topology is finer than the topology defined by the filtration $\{(\mathfrak{p}N)^{(k)}\}_{k \geq 0}$ for all primes $\mathfrak{p}$ containing I.*
(iv) *The (S)-symbolic topology is finer than the $\mathfrak{p}$-adic topology for all prime ideals $\mathfrak{p} \supseteq I$.*

Proof. (i) $\Rightarrow$ (iii). Let $\mathfrak{p} \in \operatorname{Spec} R$ with $I \subseteq \mathfrak{p}$ and let $l \geq 1$. We need to show that there exists an integer $m \geq 1$ such that $S(I^m N) \subseteq (\mathfrak{p}N)^{(l)}$. To this end, let S' be the natural image of S in $R_{\mathfrak{p}}$. Then, in view of assumption (i) and Lemma 3.1, we have $S' \subseteq R_{\mathfrak{p}} \backslash \cup \{\mathfrak{q} \in Q(IR_{\mathfrak{p}}, N_{\mathfrak{p}})\}$. Also, it is easy to see that $S'(I^m N_{\mathfrak{p}}) \subseteq (\mathfrak{p}N_{\mathfrak{p}})^{(l)}$ implies $S(I^m N) \subseteq (\mathfrak{p}N)^{(l)}$. Therefore we may assume that R is local at $\mathfrak{p}$. Now, because $M^* \cap N = M$ for any submodule M of N, we may assume in addition in view of [1, Proposition 3.8] and Remark 2.4, that R is complete. (Note that $(\mathfrak{p}N)^{(l)} = \mathfrak{p}^l N$). Now, for any $q \in \operatorname{Ass}_R N$, Lemma 3.1 (iv) and the assumption (i) show that S is disjoint from $I + q$. Therefore by Lemma 3.4, we have $\bigcap_{n \geq 1} I^n S^{-1} N = 0$. Consequently $\bigcap_{n \geq 1} S(I^n N) = 0$. As R is complete, Chevalleys' theorem (see, Lemma 2.3) implies (iii).

The conclusion (ii) $\Rightarrow$ (iv) is obviously true.

In order to prove that (iii) $\Rightarrow$ (ii), let $k \geq 1$. Then, by considering a primary decomposition for $I^k N$ and using hypothesis for elements of $\operatorname{Ass}_R N/I^k N$, it is easily seen that there is an integer $m \geq 1$ such that $S(I^m N) \subseteq \mathfrak{p}^k N$, as desired. Finally, we prove the implication (iv) $\Rightarrow$ (i). To this end, let $\mathfrak{p} \in Q(I, N)$. We show that $\mathfrak{p} \cap S = \phi$. Suppose the contrary is true and let $s \in S \cap \mathfrak{p}$. By virtue of Theorem 3.3, there is a $k \geq 0$ such that, for all $n \geq 0$, $I^n N :_N \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}N)^{(k)}$. However, the condition (iv) says that for some $m \geq 1$, $S(I^m N) \subseteq \mathfrak{p}^k N$, and so for such m we have $I^m N :_N \langle \mathfrak{p} \rangle \nsubseteq S(I^m N)$. Let $y \in I^m N :_N \langle \mathfrak{p} \rangle \backslash S(I^m N)$. Then, for sufficiently large l, $\mathfrak{p}^l y \subseteq I^m N$. Therefore $s^l y \in I^m N$, and hence $y \in S(I^m N)$ which is a contradiction. So $\mathfrak{p} \cap S = \phi$ and (i) follows. □

Theorem 3.6 (cf. [17, LEMMA 3.1]). *Let $\mathfrak{p} \in \operatorname{Supp}(N/IN) \backslash m \operatorname{Ass}_R(N/IN)$ and consider the following statements.*

(i) $\mathfrak{p} \in Q(I, N)$.
(ii) *There is an integer $k \geq 1$ such that $(IN)^{(m)} \nsubseteq (\mathfrak{p}N)^{(k)}$ for all $m \in \mathbb{N}$.*
(iii) *There is a prime $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{q} \in Q(I, N) \backslash m \operatorname{Ass}_R(N/IN)$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) ⇒ (ii). Suppose that (i) holds and that $k \geq 1$ be as in Theorem 3.3 (ii). Assume the contrary. Then, for such k, there is an integer $m \geq 1$ such that $(IN)^{(m)} \subseteq (\mathfrak{p}N)^{(k)}$. Next, using the definition of $(IN)^{(m)}$, it is easy to see that, $\mathrm{Rad}((IN)^{(m)} :_R N) = \mathrm{Rad}(IN :_R N)$. Therefore by Theorem 3.3, $\mathfrak{p} \in \mathrm{Ass}_R(N/(IN)^{(m)})$. Recall that $\mathrm{Ass}_R(N/(IN)^{(m)}) = \{\mathfrak{q} \in \mathrm{Ass}_R(N/I^m N) \,|\, \mathfrak{q} \cap S = \phi\}$, where $S = R \setminus \cup\{\mathfrak{q} \in m\,\mathrm{Ass}_R N/IN\}$. Accordingly $\mathfrak{p} \subseteq \cup\{\mathfrak{q} \in m\,\mathrm{Ass}_R N/IN\}$; so $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in m\,\mathrm{Ass}_R N/IN$. It follows that $\mathfrak{p} = \mathfrak{q}$ and hence $\mathfrak{p} \in m\,\mathrm{Ass}_R N/IN$, which is the required contradiction.

The implication (ii) ⇒ (iii) follows from (i) ⇒ (ii) of Theorem 3.5 as a special case. □

The following corollary gives a characterization of $Q(I, N)$.

Corollary 3.7 (cf. [8, Remark (i) after Theorem 1.2]). *The following conditions are equivalent*:

(i) $Q(I, N) = m\,\mathrm{Ass}_R N/IN$.
(ii) *The topology defined by* $\{(IN)^{(m)}\}_{m \geq 0}$ *is equivalent to the* I*-adic topology*.

Proof. This follows immediately from Theorem 3.5 and Lemma 3.1 (i), (ii). □

Corollary 3.8. *Let* S *be a multiplicatively closed subset of* R. *Then the following conditions are equivalent:*

(i) $\dim R^*_{\mathfrak{p}}/(IR^*_{\mathfrak{p}} + \mathfrak{q}) > 0$ *for all prime ideals* $\mathfrak{p}$ *containing* I *with* $\mathfrak{p} \cap S \neq \phi$, *and for all* $\mathfrak{q} \in \mathrm{Ass}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}}$.
(ii) *The* I*-adic and* (S)*-symbolic topologies are equivalent*.
(iii) *The topology given by* $\{I^n N_{\mathfrak{p}} :_{N_{\mathfrak{p}}} \langle \mathfrak{p}R_{\mathfrak{p}} \rangle\}_{n \geq 0}$ *is equivalent to the* $IR_{\mathfrak{p}}$*-adic topology, on* $N_{\mathfrak{p}}$, *for all prime ideals* $\mathfrak{p}$ *containing* I *with* $\mathfrak{p} \cap S \neq \phi$.

Proof. The equivalence of (i) and (ii) follows from Theorem 3.5. In order to complete the proof, let us show the equivalence between (ii) and (iii). Firstly, suppose that (ii) holds and let $\mathfrak{p} \supseteq I$ be a prime ideal with $\mathfrak{p} \cap S \neq \phi$. Let S' be the image of S in $R_{\mathfrak{p}}$. Then, for any $n \geq 0$, there exists an integer $m \geq 0$ such that $S'(I^m N_{\mathfrak{p}}) \subseteq I^n N_{\mathfrak{p}}$. Therefore, since $\mathfrak{p} \cap S \neq \phi$, we have $I^m N_{\mathfrak{p}} :_{N_{\mathfrak{p}}} \langle \mathfrak{p}R_{\mathfrak{p}} \rangle \subseteq I^n N_{\mathfrak{p}}$ and (iii) follows.

To prove the implication (iii) ⇒ (ii), it is enough, in view of Theorem 3.5, to show that $S \subseteq R \setminus \cup\{\mathfrak{q} \in Q(I, N)\}$. To achieve this, suppose the contrary is true. Then there is an element $\mathfrak{p} \in Q(I, N)$ such that $\mathfrak{p} \cap S \neq \phi$. Now, because of $\mathfrak{p}R_{\mathfrak{p}} \in Q(IR_{\mathfrak{p}}, N_{\mathfrak{p}})$, Theorem 3.3 provides a contradic tion. □

4. ESSENTIAL PRIMES AND LINEAR EQUIVALENCE

In this section we study the relationship between the linear equivalence of certain topologies and essential primes of I. The I-adic and (S)-symbolic topologies on N are said to be *linearly equivalent* if there exists an integer $k \geq 0$ such that $S(I^{n+k}N) \subseteq I^nN$ for all integers $n \geq 0$.

Before we state the main result of this section, let us give a definition.

Definition. *A prime ideal $\mathfrak{p}$ of R is an essential prime of I w.r.t.N, if $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in Q(u\mathcal{R}, \mathcal{N})$. The set of essential primes of I w.r.t.N will be denoted by $E(I, N)$.*

Theorem 4.1 (cf. [8, Corollary 1.3]). *Let S be a multiplicatively closed subset of R. Then the following conditions are equivalent.*

(i) $S \subseteq R \backslash \cup \{\mathfrak{p} \in E(I, N)\}$.
(ii) *The I-adic and (S)-symbolic topologies are linearly equivalent.*

Proof. (i) $\Rightarrow$ (ii). It follows from the hypothesis that $S \subseteq \mathcal{R} \backslash \cup \{\mathfrak{q} \in Q(u\mathcal{R}, \mathcal{N})\}$. Hence, in view of Theorem 3.5, there exists an integer $m \geq 0$ such that $S(u^m\mathcal{N}) \subseteq u\mathcal{N}$. Now, one easily sees that, for all $k \geq 1$, $S(u^{m+k-1}\mathcal{N}) \subseteq u^k\mathcal{N}$. Therefore by intersecting with N, we have $S(I^{m+k-1}N) \subseteq I^kN$, so (ii) holds for $h = m - 1$.

(ii) $\Rightarrow$ (i). Using Theorem 3.5 and the hypothesis, we see that S is disjoint from all the primes in $Q(u\mathcal{R}, \mathcal{N})$. So that, by definition of $E(I, N)$, we have $S \subseteq R \backslash \cup \{\mathfrak{q} \in E(I, N)\}$, as required. □

Following [3], we shall use $A^*(I, N)$ to denote the ultimately constant values of $\mathrm{Ass}_R N/I^nN$ for large n. Next, as an application of Theorems 3.5 and 4.1, we provide the following nice alternate proofs for the main results of [1].

Consequence 4.2 (cf. [1, Theorem 3.17] and [8, Lemma 2.1]). *$Q(I, N) \subseteq E(I, N) \subseteq A^*(I, N)$.*

Proof. For the proof of $E(I, N) \subseteq A^*(I, N)$, let $\mathfrak{p} \in E(I, N)$. Since both $E(I, N)$ and $A^*(I, N)$ behave well under localization, without loss of generality, we may assume that $(R, \mathfrak{p})$ is local. Let $A^*(I, N) = \mathrm{Ass}_R N/I^nN$ for large n, and set $S = R \backslash \cup \{\mathfrak{q} \in A^*(I, N)\}$. Then, for all $k \geq 0$, by using a normal primary decomposition for $I^{n+k}N$ and the fact that $A^*(I, N) = \mathrm{Ass}_R N/I^{n+k}N$, it is easy to see that $S(I^{n+k}N) = I^{n+k}N \subseteq I^kN$. Therefore, by Theorem 4.1, we have $S \subseteq R \backslash \cup \{\mathfrak{p}' \in E(I, N)\}$. Consequently, $\cup\{\mathfrak{p}' \in E(I, N)\} \subseteq \cup\{\mathfrak{q} \in A^*(I, N)\}$. Since $A^*(I, N)$ is finite, there is a $\mathfrak{q} \in A^*(I, N)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$; so $\mathfrak{p} = \mathfrak{q} \in A^*(I, N)$, as required.

To show that $Q(I,N) \subseteq E(I,N)$, let $\mathfrak{p} \in Q(I,N)$. We may assume that $(R,\mathfrak{p})$ is local. Now, let $S = R \backslash \cup \{\mathfrak{q} \in E(I,N)\}$. Then Theorem 4.1 shows that Theorem 3.5(ii) is satisfied. Hence by Theorem 3.5(i), we have $S \subseteq R \backslash \cup \{\mathfrak{p}' \in Q(I,N)\}$. Therefore $\cup\{\mathfrak{p}' \in Q(I,N)\} \subseteq \cup\{\mathfrak{q} \in E(I,N)\}$. As $E(I,N)$ is finite, we have $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in E(I,N)$. Hence $\mathfrak{p} = \mathfrak{q} \in E(I,N)$, as desired. □

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