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Locally Unmixed Modules and Ideal Topologies

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Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R-module. In this paper, the main result asserts that N is locally unmixed if and only if, for any N-proper ideal $\mathfrak{a}$ of R generated by $\mathrm{ht}_N\,\mathfrak{a}$ elements, the topology defined by $(\mathfrak{a}N)^{(n)}$, $n \geq 0$, is equivalent to the $\mathfrak{a}$-adic topology.

1. INTRODUCTION

Throughout this paper, all rings considered will be commutative and Noetherian and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by $\mathfrak{a}$. Let N be a non-zero finitely generated module over R. We denote by $\mathscr{R}$ the Rees ring $R[u, \mathfrak{a}t] := \oplus_{n \in \mathbb{Z}} \mathfrak{a}^n t^n$ of R w.r.t. $\mathfrak{a}$, where t is an indeterminate and $u = t^{-1}$. Also, the graded Rees module $N[u, \mathfrak{a}t] := \oplus_{n \in \mathbb{Z}} \mathfrak{a}^n N$ over $\mathscr{R}$ is denoted by $\mathscr{N}$, which is a finitely generated $\mathscr{R}$-module. If $(R, \mathfrak{m})$ is local, then R^* (resp. N^*) denotes the completion of R (resp. N) w.r.t. the $\mathfrak{m}$-adic topology. In particular, for any $\mathfrak{p} \in \operatorname{Spec} R$, we denote $R_{\mathfrak{p}}^*$ and $N_{\mathfrak{p}}^*$ the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. For any multiplicatively closed subset S of R, the nth (S)-*symbolic power of* $\mathfrak{a}$ w.r.t. N, denoted by $S(\mathfrak{a}^n N)$, is defined to be the union of $\mathfrak{a}^n N :_N s$ where s varies in S. The $\mathfrak{a}$-adic filtration $\{\mathfrak{a}^n N\}_{n \geq 0}$ and the (S)-symbolic filtration $\{S(\mathfrak{a}^n N)\}_{n \geq 0}$ induce topologies on N which are called the $\mathfrak{a}$-*adic topology* and the (S)-*symbolic topology*, respectively. These two topologies are said to be equivalent if, for every integer $m \geq 0$, there is an integer $n \geq 0$ such

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that $S(\mathfrak{a}^n N) \subseteq \mathfrak{a}^m N$. In particular, if $S = R \setminus \cup \{\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a}N\}$, where $m \operatorname{Ass}_R N/\mathfrak{a}N$ denotes the set of minimal prime ideals of $\operatorname{Ass}_R N/\mathfrak{a}N$, the nth (S)-symbolic power of $\mathfrak{a}$ w.r.t. N is denoted by $(\mathfrak{a}N)^{(n)}$, and the topology defined by the filtration $\{(\mathfrak{a}N)^{(n)}\}$ is called the *symbolic topology*. The purpose of the present paper is to show that N is locally unmixed if and only if, for each N-proper ideal $\mathfrak{a}$ that is generated by $\operatorname{ht}_N \mathfrak{a}$ elements, the $\mathfrak{a}$-adic and the symbolic topologies are equivalent. Schenzel has characterized unmixed local rings [8, Theorem 7] in terms of comparison of the topologies defined by certain filtrations. Also, Katz [4, Theorem 3.5] and Verma [11, Theorem 5.2] have proved a characterization of locally unmixed rings in terms of s-ideals. Equivalence of $\mathfrak{a}$-adic topology and (S)-symbolic topology has been studied, in the case $N = R$, in [4, 6–10], and has led to some interesting results.

Let $\mathfrak{p} \in \operatorname{Supp}(N)$. Then the *N-height* of $\mathfrak{p}$, denoted by $\operatorname{ht}_N \mathfrak{p}$, is defined to be the supremum of lengths of chains of prime ideals of $\operatorname{Supp}(N)$ terminating with $\mathfrak{p}$. We have $\operatorname{ht}_N \mathfrak{p} = \dim_{R_\mathfrak{p}} N_\mathfrak{p}$. We shall say that an ideal $\mathfrak{a}$ of R is *N-proper* if $N/\mathfrak{a}N \neq 0$, and, when this is the case, we define the *N-height of* $\mathfrak{a}$ (written $\operatorname{ht}_N \mathfrak{a}$) to be

$$\inf\{\operatorname{ht}_N \mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}(N) \cap V(\mathfrak{a})\}$$

$$\left(= \inf\{\operatorname{ht}_N \mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R(N/\mathfrak{a}N)\}\right).$$

For any N-proper ideal $\mathfrak{a}$ of R, denote by $\operatorname{grade}(\mathfrak{a}, N)$ the maximum length of all N-sequences contained in $\mathfrak{a}$. Suppose for the moment that $(R, \mathfrak{m})$ is local. It follows from Nakayama's lemma that every proper ideal of R is N-proper. N is said to be a *Cohen–Macaulay module* (abbreviated as *CM module*) if and only if $\operatorname{grade}(\mathfrak{m}, N) = \operatorname{ht}_N \mathfrak{m}$.

More generally, if R is not necessarily local and N is non-zero and finitely generated, N is said to be CM if and only if $N_\mathfrak{m}$ is a CM $R_\mathfrak{m}$*-module* in the above sense for each maximal ideal $\mathfrak{m} \in \operatorname{Supp}(N)$. In the following we refer to [2, 3, 5] for the basic results about CM modules. For any ideal $\mathfrak{a}$ of R, *the radical of* $\mathfrak{a}$, denoted by $\operatorname{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. For any R-module L, we denote by $m \operatorname{Ass}_R L$ the set of minimal prime ideals of $\operatorname{Ass}_R L$.

In the second section, we characterize the CM property of a non-zero finitely generated R-module N in terms of the equalities $\mathfrak{a}^n N$ and $(\mathfrak{a}N)^{(n)}$ for certain N-proper ideals $\mathfrak{a}$ of R. More precisely we prove the following result:

THEOREM 1.1. *The following conditions are equivalent*:

(i) *N is* CM.

(ii) *For any N-proper ideal* $\mathfrak{a}$ *of R that is generated by* $\operatorname{ht}_N \mathfrak{a}$ *elements*, $\mathfrak{a}^n N = (\mathfrak{a}N)^{(n)}$ *for all* $n \in \mathbb{N}$.

The result of 1.1 is proved in 2.3. Let $(R, \mathfrak{m})$ be local and let N be a non-zero finitely generated R-module. N is said to be an *unmixed module* if for any $\mathfrak{p} \in \operatorname{Ass}_{R^*} N^*$, $\dim R^*/\mathfrak{p} = \dim N$.

More generally, if R is not necessarily local and N is non-zero finitely generated, N is a *locally unmixed module* if for any $\mathfrak{p} \in \operatorname{Supp}(N)$, $N_\mathfrak{p}$ is *an unmixed* $R_\mathfrak{p}$*-module*. As the main result of Section 3 we characterize the *locally unmixed* property of a non-zero finitely generated R-module N in terms of the equivalence of the topologies defined by $\{\mathfrak{a}^n N\}_{n \geq 0}$ and $\{(\mathfrak{a}N)^{(n)}\}_{n \geq 0}$, for certain N-proper ideals $\mathfrak{a}$ of R. More precisely we shall show that:

THEOREM 1.2. *The following conditions are equivalent*:

(i) *N is locally unmixed.*

(ii) *For each N-proper ideal $\mathfrak{a}$ of R that is generated by $\operatorname{ht}_N \mathfrak{a}$ elements, the topology given by $\{(\mathfrak{a}N)^{(n)}\}_{n \geq 0}$ is equivalent to the $\mathfrak{a}$-adic topology on N.*

The proof of Theorem 1.2 is given in 3.12.

2. COHEN–MACAULAY MODULES AND SYMBOLIC POWERS

Let N be a non-zero and finitely generated R-module and let $\mathfrak{a}$ be an N-proper ideal of R. The following theorem is well known when $N = R$. The proof in [2, 5] can be easily carried over to a module, so we omit the proof.

THEOREM 2.1. *Let N and $\mathfrak{a}$ be as above. Then the following hold*:

(i) $\operatorname{ht}_N \mathfrak{a} = \operatorname{ht}(\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N)$.

(ii) (*Krull's principle ideal theorem*) *If $\mathfrak{x} = x_1, \ldots, x_n$ is a sequence of elements of R, then $\operatorname{ht}_N \mathfrak{p} \leq n$ for all $\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{x}N$. Furthermore, if $\mathfrak{x}$ is an N-sequence, then any minimal element of $\operatorname{Ass}_R N/\mathfrak{x}N$ has N-height n. In particular, $\operatorname{ht}_N \mathfrak{x} = n$.*

(iii) *If $\operatorname{ht}_N \mathfrak{a} = n$, then there exist $x_1, \ldots, x_n$ in $\mathfrak{a}$ such that $\operatorname{ht}_N(x_1, \ldots, x_i) = i$ for $i = 1, 2, \ldots, n$.*

(iv) *N is* CM *if and only if $\operatorname{ht}_N \mathfrak{b} = \operatorname{grade}(\mathfrak{b}, N)$ for any N-proper ideal $\mathfrak{b}$ of R.*

PROPOSITION 2.2. *Let N be a non-zero finitely generated R-module. Then the following conditions are equivalent*:

(i) *N is* CM.

(ii) *For every N-proper ideal $\mathfrak{a}$ of R generated by $\operatorname{ht}_N \mathfrak{a}$ elements, $m \operatorname{Ass}_R(N/\mathfrak{a}N) = \operatorname{Ass}_R(N/\mathfrak{a}N)$.*

Proof. (i) ⇒ (ii) Let $\operatorname{ht}_N \mathfrak{a} := n$ and let $\mathfrak{a} = (x_1, \ldots, x_n)$. Suppose that $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass}_R N/\mathfrak{a}N$, and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{q} \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. It is enough to show that $\mathfrak{q} = \mathfrak{p}$. To this end, we have

$$\mathfrak{p} R_m, \mathfrak{q} R_m \in \operatorname{Ass}_{R_{\mathfrak{m}}}(N_{\mathfrak{m}}/\mathfrak{a}N_{\mathfrak{m}}).$$

$\operatorname{ht}_N \mathfrak{a} = n$ and $N_{\mathfrak{m}}$ is CM [2, Theorem 2.1.2] imply that $x_1, \ldots, x_n$ is an $N_{\mathfrak{m}}$-sequence. Therefore by [2, Theorem 2.1.3] $N_{\mathfrak{m}}/(x_1, \ldots, x_n)N_{\mathfrak{m}}$ is a CM $R_{\mathfrak{m}}$-module. When $\mathfrak{p} R_{\mathfrak{m}} = \mathfrak{q} R_{\mathfrak{m}}$ and so $\mathfrak{p} = \mathfrak{q}$, as required.

In order to prove the implication (ii) ⇒ (i) assume that $\mathfrak{b}$ is an arbitrary N-proper ideal of R. In view of Theorem 2.1(iv), it is enough to show that $\operatorname{ht}_N \mathfrak{b} = \operatorname{grade}(\mathfrak{b}, N)$. To achieve this, suppose $x_1, \ldots, x_n$ in $\mathfrak{b}$ are such that $\operatorname{ht}_N(x_1, \ldots, x_i) = i$ for all $1 \leq i \leq n$, where $n = \operatorname{ht}_N \mathfrak{b}$. Thus we have

$$x_i \notin \bigcup\{\mathfrak{p} \in m \operatorname{Ass}_R N/(x_1, \ldots, x_{i-1})N\}, \qquad \text{for all } 1 \leq i \leq n.$$

So the condition (ii) implies that $x_i \notin \bigcup\{\mathfrak{p} \in \operatorname{Ass}_R N/(x_i, \ldots, x_{i-1})N\}$ for all $1 \leq i \leq n$. That is, $x_1, \ldots, x_n$ in $\mathfrak{b}$ is an N-sequence. Now the assertion follows. ∎

We are now ready to state and prove the main theorem of this section.

THEOREM 2.3. *Let N be a non-zero finitely generated R-module. Then the following conditions are equivalent*:

(i) *N is* CM.

(ii) *For any N-proper ideal* $\mathfrak{a}$ *of R generated by* $\operatorname{ht}_N \mathfrak{a}$ *elements*, $(\mathfrak{a}N)^{(n)} = \mathfrak{a}^n N$ *for all* $n \in \mathbb{N}$.

Proof. (i) ⇒ (ii) Suppose N is CM and let $\mathfrak{a}$ be an arbitrary N-proper ideal of R generated by $\operatorname{ht}_N \mathfrak{a}$ elements. Then by Proposition 2.2, we have $\operatorname{Ass}_R N/\mathfrak{a}N = m \operatorname{Ass}_R N/\mathfrak{a}N$. Thus, by assumption (i) and [3, Theorem 125 and Exercise 13, p. 103], it follows that $\operatorname{Ass}_R N/\mathfrak{a}^n N = m \operatorname{Ass}_R N/\mathfrak{a}^n N$ for any $n \in \mathbb{N}$. (Note that $m \operatorname{Ass}_R N/\mathfrak{a}^n N = m \operatorname{Ass}_R N/\mathfrak{a}N$.) Now, by considering a normal primary decomposition for $\mathfrak{a}^n N$ and using the definition of $(\mathfrak{a}N)^{(n)}$, it is straightforward to check that $(\mathfrak{a}N)^{(n)} = \mathfrak{a}^n N$ for all $n \in \mathbb{N}$, as required.

(ii) ⇒ (i) Let $\mathfrak{a}$ be an ideal of R generated by $\operatorname{ht}_N \mathfrak{a}$ elements. In view of Proposition 2.2, it is enough to show that the associated prime ideals of $N/\mathfrak{a}N$ are minimal. Suppose this is not the case and let $\mathfrak{q}$ be an element of $\operatorname{Ass}_R N/\mathfrak{a}N$ which does not belong to $m \operatorname{Ass}_R N/\mathfrak{a}N$. Then $\mathfrak{q} = \mathfrak{a}N :_R x$ for some $x \in N \setminus \mathfrak{a}N$ and $q \nsubseteq \bigcup\{\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a}N\}$. Let $s \in \mathfrak{q}$ be such that $s \notin \bigcup\{\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a}N\}$. Then $sx \in \mathfrak{a}N$ and, so $x \in \mathfrak{a}N :_N s$. Now, the condition (ii) provides a contradiction. ∎

3. LOCALLY UNMIXED MODULES AND COMPARISON OF TOPOLOGIES

The purpose of this section is to prove that a non-zero finitely generated module over a Noetherian ring R is locally unmixed if and only if, for any N-proper ideal $\mathfrak{a}$ of R that can be generated by $\operatorname{ht}_N \mathfrak{a}$ elements, the topologies $\mathfrak{a}$-adic and symbolic, on N, are equivalent. We begin with

DEFINITION 3.1. Let N be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. A prime ideal $\mathfrak{p}$ of R is called a *quintessential prime ideal* of $\mathfrak{a}$ w.r.t. N precisely when there exists $\mathfrak{q} \in \operatorname{Ass}_{R_{\mathfrak{p}}^*} N_{\mathfrak{p}}^*$ such that $\operatorname{Rad}(\mathfrak{a} R_{\mathfrak{p}}^* + q) = \mathfrak{p} R_{\mathfrak{p}}^*$. The set of *quintessential primes of* $\mathfrak{a}$ w.r.t. N is denoted by $Q(\mathfrak{a}, N)$. It is easy to see that $Q(\mathfrak{a}, N) = Q(\mathfrak{a} + \operatorname{Ann}_R N, N)$.

LEMMA 3.2. *Let N be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. Suppose $\mathfrak{p} \in \operatorname{Supp}(N/\mathfrak{a} N) \setminus m \operatorname{Ass}_R(N/\mathfrak{a} N)$ is such that the topology defined by $(\mathfrak{a} N)^{(n)}$, $n \geq 1$, is finer than the topology defined by $(\mathfrak{p} N)^{(n)}$, $n \geq 1$. Then $\mathfrak{p} \notin Q(\mathfrak{a}, N)$.*

Proof. Suppose the contrary; i.e., $\mathfrak{p} \in Q(\mathfrak{a}, N)$. Let $k \geq 1$ be as [1, Proposition 3.12]. Then, for such k, there exists an integer $n \geq 1$ such that $(\mathfrak{a} N)^{(n)} \subseteq (\mathfrak{p} N)^{(k)}$. Again, from [1, Proposition 3.12], it follows that $\mathfrak{p} \in \operatorname{Ass}_R(N/(\mathfrak{a} N)^{(n)})$. Since $\operatorname{Ass}_R N/(\mathfrak{a} N)^{(n)} = \{q \in \operatorname{Ass}_R N/\mathfrak{a}^n N : \mathfrak{q} \cap S = \phi\}$, where $S = R \setminus \cup \{\mathfrak{q} \in m \operatorname{Ass}_R N/\mathfrak{a} N\}$, it yields $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in m \operatorname{Ass}_R N/\mathfrak{a} N$. Consequently, $\mathfrak{p} = \mathfrak{q}$, and so $\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a} N$, which is a contradiction. ∎

PROPOSITION 3.3. *Let N and $\mathfrak{a}$ be as above. Then the following conditions are equivalent*:

(i) $Q(\mathfrak{a}, N) = m \operatorname{Ass}_R N/\mathfrak{a} N$.

(ii) *The symbolic topology is equivalent to the $\mathfrak{a}$-adic topology.*

Proof. (i) ⇒ (ii) Let $n \geq 1$, and let

$$\mathfrak{a}^n N = (\mathfrak{a} N)^{(n)} \cap Q_2 \cap \cdots \cap Q_r \qquad \text{where } Q_i \text{ is } \mathfrak{p}_i\text{-primary in } N \quad (2 \leq i \leq r)$$

be a normal primary decomposition of $\mathfrak{a}^n N$. Then, $\mathfrak{p}_i \in \operatorname{Supp}(N/\mathfrak{a} N) \setminus m \operatorname{Ass}_R N/\mathfrak{a} N$ for all $2 \leq i \leq r$. It is easy to see that $(\mathfrak{p}_i N)^{(k_i)} \subseteq Q_i$ for sufficiently large k_i $(2 \leq i \leq r)$. Furthermore, by assumption (i) and Lemma 3.2, it follows that $(\mathfrak{a} N)^{(m_i)} \subseteq (\mathfrak{p}_i N)^{(k_i)}$ for sufficiently large m_i with $2 \leq i \leq n$. Letting m be the maximum of $m_2, \ldots, m_r$, we see that $(\mathfrak{a} N)^{(m)} \subseteq (\mathfrak{p}_i N)^{(k_i)}$ $(2 \leq i \leq r)$. Thus $(\mathfrak{a} N)^{(mn)} \subseteq \mathfrak{a}^n N$, as required.

In order to prove that (ii) $\Rightarrow$ (i), suppose that for any $k \geq 1$ there is an integer $m \geq 1$ such that $(\mathfrak{a}N)^{(m)} \subseteq \mathfrak{a}^k N$. Let $\mathfrak{p} \in Q(\mathfrak{a}, N)$. Then $(\mathfrak{a}N)^{(m)} \subseteq (\mathfrak{p}N)^{(k)}$. By virtue of Lemma 3.2, $\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a}N$. Hence, [1, Lemma 3.5] yields the claim. ∎

LEMMA 3.4. *Let N be a non-zero finitely generated R-module. Suppose that $\mathfrak{p} \in \operatorname{Supp}(N)$ and let $\mathfrak{b}$ be an $N_{\mathfrak{p}}$-proper ideal of $R_{\mathfrak{p}}$ generated by $\operatorname{ht}_{N_{\mathfrak{p}}} \mathfrak{b}$ elements. Then there is an N-proper ideal $\mathfrak{c}$ of R generated by $\operatorname{ht}_N \mathfrak{c}$ elements such that*

$$\mathfrak{b} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \mathfrak{c}R_{\mathfrak{p}} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}.$$

Proof. Since $\operatorname{ht}_{N_{\mathfrak{p}}} \mathfrak{b} = \operatorname{ht}((\mathfrak{b} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})$, by Theorem 2.1 (i), it follows that the ideal $(\mathfrak{b} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ of $R_{\mathfrak{p}}/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ is generated by $\operatorname{ht}((\mathfrak{b} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})$ elements. According to Verma [11, Lemma 5.1], there is an ideal $\mathfrak{a}/\operatorname{Ann}_R N$ of $R/\operatorname{Ann}_R N$ that is a generated by $\operatorname{ht}(\mathfrak{a}/\operatorname{Ann}_R N) = \operatorname{ht}_N \mathfrak{a}$ elements, and

$$\left(\mathfrak{b} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}\right)/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = (\mathfrak{a}/\operatorname{Ann}_R N) R_{\mathfrak{p}}/\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}.$$

The result now follows. ∎

PROPOSITION 3.5. *Let $(R, \mathfrak{m})$ be local and let N be a non-zero finitely generated R-module. Suppose that $\mathfrak{q} \in \operatorname{Ass}_{R^*} N^*$ with $\dim R^*/\mathfrak{q} > 0$. Then there exists an N-proper ideal $\mathfrak{a}$ of R generated by $\operatorname{ht}_N \mathfrak{a}$ elements, such that*

$$\operatorname{Rad}(\mathfrak{a}R^* + \mathfrak{q}) = \mathfrak{m}R^* \qquad \text{and} \qquad \operatorname{ht}_N \mathfrak{a} = \dim R^*/\mathfrak{q}.$$

Proof. Let $\mathfrak{q} \in \operatorname{Ass}_{R^*} N^*$ with $\dim R^*/\mathfrak{q} := n > 0$. Using Krull's principal ideal theorem and prime avoidance arguments one constructs elements $x_1, \dots, x_n \in \mathfrak{m}$ such that

$$x_i \notin \left(\bigcup_{\mathfrak{p} \in m \operatorname{Ass}_R N/(x_1, \dots, x_{i-1})N} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p}^* \in m \operatorname{Ass}_{R^*} N^*/(\mathfrak{q} + (x_1, \dots, x_{i-1})R^*)N^*} \mathfrak{p}^* \cap R \right),$$

for all $1 \leq i \leq n$. Select $\mathfrak{a} = (x_1, \dots, x_n)$. We need to show that $\operatorname{ht}_N \mathfrak{a} = n$ and $\mathfrak{a}R^* + \mathfrak{q}$ is $\mathfrak{m}R^*$-primary. First we show $\operatorname{ht}_N \mathfrak{a} = n$. We prove this by induction on n. The case $n = 1$ follows from the principal ideal theorem, together with $x_1 \notin \bigcup_{\mathfrak{p} \in m \operatorname{Ass}_R N} \mathfrak{p}$. So let $n > 1$ and suppose that the result is true for $n - 1$. Let $\mathfrak{p} \in m \operatorname{Ass}_R N/\mathfrak{a}N$. Because of $x_n \in \mathfrak{p}$, we have $\mathfrak{p} \notin m \operatorname{Ass}_R N/(x_1, \dots, x_{n-1})N$. The result now follows from Krull's principal ideal theorem and inductive hypothesis. Now we prove $\operatorname{Rad}(\mathfrak{a}R^* + \mathfrak{q}) = \mathfrak{m}R^*$. To this end let $\mathfrak{q}_n^*$ be a minimal prime over $\mathfrak{a}R^* + \mathfrak{q}$. Then from $x_n \in \mathfrak{q}_n^* \cap R$, follows that $\mathfrak{q}_n^* \notin m\operatorname{Ass}_{R^*} N^*/(\mathfrak{q} + (x_1, \ldots, x_{n-1})R^*)N^*$. Therefore there is a $\mathfrak{q}_{n-1}^* \in m\operatorname{Ass}_{R^*} N^*/(\mathfrak{q} + (x_1, \ldots, x_{n-1})R^*)N^*$ such that $\mathfrak{q}_{n-1}^* \subsetneqq \mathfrak{q}_n^*$, and so on. Thus we have a saturated chain of primes from $\mathfrak{q}$ to $\mathfrak{q}_n^*$, as $\mathfrak{q} = \mathfrak{q}_0^* \subsetneqq \cdots \subsetneqq \mathfrak{q}_{n-1}^* \subsetneqq \mathfrak{q}_n^*$, with $1 \leq i \leq n$, $\mathfrak{q}_i^* \in m\operatorname{Ass}_{R^*} N^*/(\mathfrak{q} + (x_1, \ldots, x_i)R^*)N^*$. Because $\dim R^*/\mathfrak{q} = n$, it follows that $\mathfrak{q}_n^* = \mathfrak{m}R^*$, so that $\operatorname{Rad}(\mathfrak{a}R^* + \mathfrak{q}) = \mathfrak{m}R^*$, as desired. ∎

DEFINITION 3.6. Let N be a non-zero finitely generated R-module. A prime ideal $\mathfrak{p}$ of R is an *essential prime of* $\mathfrak{a}$ w.r.t. N, if $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in Q(u\mathscr{R}, \mathscr{N})$. The set of *essential primes of* $\mathfrak{a}$ w.r.t. N will be denoted by $E(\mathfrak{a}, N)$.

DEFINITION 3.7. A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of elements of R is called an *essential sequence on* N if the following conditions are satisfied:

(i) For all $1 \leq i \leq n$, $x_i \notin \bigcup\{\mathfrak{p} \in E((x_1, \ldots, x_{i-1}), N)\}$.

(ii) $N/\boldsymbol{x}N \neq 0$.

An essential sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of elements of R (contained in an ideal $\mathfrak{a}$) on N is *maximal* (in $\mathfrak{a}$), if $x_1, \ldots, x_n, x_{n+1}$ is not an essential sequence on N for any $x_{n+1} \in R (x_{n+1} \in \mathfrak{a})$. It is shown that (see [1, Theorem 4.16]) all maximal essential sequences on N in an ideal $\mathfrak{a}$ have the same length. This allows us to introduce the fundamental notion of *essential grade* (see [1, Definition 4.17]).

LEMMA 3.8. *Let N be a non-zero finitely generated R-module and let $\boldsymbol{x} = x_1, \ldots, x_n$ be elements of R which form an essential sequence on N. Then the following hold*:

(i) $\operatorname{ht}_N(x_1, \ldots, x_i) = i$ *for all* $1 \leq i \leq n$.

(ii) *If N is locally unmixed, then* $E(\boldsymbol{x}, N) = m\operatorname{Ass}_R N/\boldsymbol{x}N$.

Proof. In order to prove (i), it is sufficient to show that if $\mathfrak{p} \in m\operatorname{Ass}_R(N/(x_1, \ldots, x_i)N)$, then $\operatorname{ht}_N \mathfrak{p} = i$. To this end recall that $m\operatorname{Ass}_R N/\mathfrak{a}N \subseteq E(\mathfrak{a}, N)$, for any ideal $\mathfrak{a}$ of R, and $x_1, \ldots, x_i$ is an essential sequence on N. Putting this together the proof of (i) follows by induction. For the proof of (ii), it is clearly sufficient to prove that $E(\boldsymbol{x}, N) \subseteq m\operatorname{Ass}_R N/\boldsymbol{x}N$. Let $\mathfrak{p} \in E(\boldsymbol{x}, N)$. Then by virtue of (i), $\operatorname{ht}_N \mathfrak{p} \geq n$. Thus we need only to show that $\operatorname{ht}_N \mathfrak{p} \leq n$, which implies that $\mathfrak{p} \in m\operatorname{Ass}_R N/\boldsymbol{x}N$. To do this, let $\operatorname{ht}_N \mathfrak{p} = k$ and let $\mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_k = \mathfrak{p}$ be a saturated chain of primes of $\operatorname{Supp}(N)$ with $\mathfrak{p}_0 \in m\operatorname{Ass}_R N$. Then, by

[1, Lemmas 4.9, 3.2, 4.11], $\operatorname{ht}(\mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}_0R_{\mathfrak{p}}) = n$ and so $n \geq k$. This completes the assertion. ∎

PROPOSITION 3.9. *Let $(R, \mathfrak{m})$ be local and let N be a non-zero finitely generated R-module. Then* $e\operatorname{grade}(\mathfrak{a}, N) = \operatorname{Min}\{\operatorname{ht}(\mathfrak{a}R^* + \mathfrak{q}/\mathfrak{q}) : \mathfrak{q} \in$ *$\mathit{Ass}_{R^*}\, N^*\}$.*

Proof. Let $e\operatorname{grade}(\mathfrak{a}, N) = n$, and let $\mathbf{x} = x_1, \ldots, x_n$ be a maximal essential sequence on N in $\mathfrak{a}$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in E(\mathbf{x}, N)$, by [1, Theorem 3.17]. By virtue of [1, Proposition 3.8], there exists a prime $\mathfrak{q}^*$ in R^* such that $\mathfrak{q}^* \cap R = \mathfrak{p}$ and $\mathfrak{q}^* \in E(\mathbf{x}R^*, N^*)$. Furthermore, by [1, Proposition 3.6], there is a $\mathfrak{q}_0^* \in \operatorname{Ass}_{R^*} N^*$ such that $\mathfrak{q}_0^* \subseteq \mathfrak{q}^*$ and $\mathfrak{q}^*/\mathfrak{q}_0^* \in E(\mathbf{x}R^* + \mathfrak{q}_0^*/\mathfrak{q}_0^*, R^*/\mathfrak{q}_0^*)$. Now, by [1, Lemma 4.9], $x_1 + \mathfrak{q}_0^*, \ldots, x_n + \mathfrak{q}_0^*$ is an essential sequence in the complete domain $R^*/\mathfrak{q}_0$. Therefore, Lemma 3.8 shows that $\operatorname{ht}(\mathfrak{q}^*/\mathfrak{q}_0^*) = n$. As $\mathfrak{a}R^* + \mathfrak{q}_0^* \subseteq \mathfrak{q}^*$, we have $\operatorname{ht}(\mathfrak{a}R^* + \mathfrak{q}_0^*/\mathfrak{q}_0^*) \leq n$. Now it is easy to see that the assertion follows from [1, Theorem 4.16]. ∎

THEOREM 3.10. *Let N be a non-zero finitely generated R-module and let $\mathbf{x} = x_1, \ldots, x_n$ be an essential sequence on N. Then $E(\mathbf{x}, N) = Q(\mathbf{x}, N)$.*

Proof. In view of [1, Theorem 3.17], it is sufficient to show that $E(\mathbf{x}, N) \subseteq Q(\mathbf{x}, N)$. Let $\mathfrak{p} \in E(\mathbf{x}, N)$. By [1, Lemma 3.2], $\mathfrak{p}R_{\mathfrak{p}} \in E(\mathbf{x}R_{\mathfrak{p}}, N_{\mathfrak{p}})$, and so $e\operatorname{grade}(\mathfrak{p}R_{\mathfrak{p}}, N_{\mathfrak{p}}) = n$. Thus, by virtue of Proposition 3.9, there is a $\mathfrak{q} \in \operatorname{Ass}_{R_{\mathfrak{p}}^*} N_{\mathfrak{p}}^*$ such that $\dim R_{\mathfrak{p}}^*/\mathfrak{q} = n$. Furthermore, by [1, Lemma 4.9], $x_1 + \mathfrak{q}, \ldots, x_n + \mathfrak{q}$ is an essential sequence on the complete local domain $R_{\mathfrak{p}}^*/\mathfrak{q}$. So, Lemma 3.8 implies that $\operatorname{ht}(x_1 + \mathfrak{q}, \ldots, x_n + \mathfrak{q}) = n$. That is, $\operatorname{ht}(\mathbf{x}R_{\mathfrak{p}}^* + \mathfrak{q}/\mathfrak{q}) = n$. Hence $\mathfrak{p}R_{\mathfrak{p}}^*/\mathfrak{q}$ is minimal over $\mathbf{x}R_{\mathfrak{p}}^* + \mathfrak{q}/\mathfrak{q}$. Consequently, $\operatorname{Rad}(\mathbf{x}R_{\mathfrak{p}}^* + \mathfrak{q}) = \mathfrak{p}R_{\mathfrak{p}}^*$, as required. ∎

COROLLARY 3.11. *Let N be a non-zero finitely generated R-module and let $x_1, \ldots, x_n$ in R be such that $(x_1, \ldots, x_n)N \neq N$. Suppose N is locally unmixed and* $\operatorname{ht}_N(x_1, \ldots, x_i) = i$ *for $1 \leq i \leq n$. Then $x_1, \ldots, x_n$ is an essential sequence on N.*

Proof. First assume that $n = 1$. Since $E(0_R, N) = \operatorname{Ass}_R N$, we need to show that $x_1 \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. Since N is locally unmixed, $\operatorname{Ass}_R N = \{\mathfrak{p} \in \operatorname{Supp}(N) : \operatorname{ht}_N \mathfrak{p} = 0\}$. Thus, since $\operatorname{ht}_N(x_1) = 1$, x_1 is not a zero-divisor on N. Let $n > 1$, and suppose that the result is true for $n - 1$. Let $\mathfrak{p} \in E((x_1, \ldots, x_{n-1}), N)$. Then, by the inductive hypothesis and Lemma 3.8, we have $x_n \notin \mathfrak{p}$. Therefore, $x_1, \ldots, x_n$ is an essential sequence on N, as desired. ∎

We are now ready to state and prove the main theorem of this section, which is a characterization of locally unmixed modules in terms of comparison of the topologies defined by certain decreasing families of submodules of a finitely generated module over a commutative Noetherian ring.

THEOREM 3.12. *Let N be a non-zero finitely generated R-module. Then the following conditions are equivalent*:

(i) *N is locally unmixed.*

(ii) *For every N-proper ideal $\mathfrak{a}$ of R generated by $\mathrm{ht}_N\, \mathfrak{a}$ elements, the $\mathfrak{a}$-adic topology is equivalent to the symbolic topology.*

Proof. First we show (i) ⇒ (ii). Let $\mathfrak{a}$ be an N-proper ideal of R which is generated by $\mathrm{ht}_N\, \mathfrak{a}$ elements. Suppose that $\mathfrak{p} \in Q(\mathfrak{a}, N)$. We show that $\mathfrak{p} \in m\,\mathrm{Ass}_R\, N/\mathfrak{a}N$. To do this, it is straightforward to check that the ideal $\mathfrak{a}R_\mathfrak{p}$ of $R_\mathfrak{p}$ can be generated by $\mathrm{ht}_{N_\mathfrak{p}}\, \mathfrak{a}R_\mathfrak{p}$ elements, by Theorem 2.1(ii). Therefore, by [1, Lemma 3.2], we may assume that $(R, \mathfrak{p})$ is local. Let $\mathrm{ht}_N\, \mathfrak{a} = n$. By Theorem 2.1, there exist $x_1, \ldots, x_n$ in $\mathfrak{a}$ with $\mathrm{ht}_N(x_1, \ldots, x_i) = i$ for all $1 \leq i \leq n$. As shown in Corollary 3.11, $x_1, \ldots, x_n$ is an essential sequence on N, so $e\,\mathrm{grade}(\mathfrak{a}, N) = n$ by Proposition 3.9. Now, analogous to the proof of [3, Theorem 125], it is easy to see that $\mathfrak{a}$ can be generated by an essential sequence of length n. Therefore by Theorem 3.10 and Lemma 3.8(ii), we have $\mathfrak{p} \in m\,\mathrm{Ass}_R\, N/\mathfrak{a}N$. We can now use Proposition 3.3 and the fact that $m\,\mathrm{Ass}_R\, N/\mathfrak{a}N \subseteq Q(\mathfrak{a}, N)$ to complete the proof of (ii).

In order to prove (ii) ⇒ (i), suppose that $\mathfrak{p} \in \mathrm{Supp}(N)$. By Lemma 3.4 and the fact that $Q(\mathfrak{b}, L) = Q(\mathfrak{b} + \mathrm{Ann}_R\, L, L)$ for any ideal $\mathfrak{b}$ of R and any R-module L, and [1, Lemma 3.2], we may assume without loss of generality that $(R, \mathfrak{p})$ is local. If $\mathrm{grade}(\mathfrak{p}, N) = 0$, then $\mathfrak{p} \in \mathrm{Ass}_R\, N$. Thus $\mathfrak{p} \in Q(0_R, N)$. Therefore by condition (ii) and Proposition 3.3, $\mathfrak{p} \in m\,\mathrm{Ass}_R\, N$. So $\mathrm{ht}_N\, \mathfrak{p} = 0$; that is, $\dim N = 0$. Whence N is a CM module, and so N is unmixed. We may assume that $\mathrm{grade}(\mathfrak{p}, N) > 0$. Let $\mathfrak{q} \in \mathrm{Ass}_{R^*}\, N^*$ be such that $\dim R^*/\mathfrak{q} = n$. We need to show that $\dim N = n$. Because $\mathrm{grade}(\mathfrak{p}, N) > 0$ we have $n > 0$. Then, by virtue of Proposition 3.5 there exits an N-proper ideal $\mathfrak{a}$ of R generated by $\mathrm{ht}_N\, \mathfrak{a} = n$ elements such that $\mathrm{Rad}(\mathfrak{a}R^* + \mathfrak{q}) = \mathfrak{p}R^*$. Consequently, by Proposition 3.3, $\mathfrak{p} \in m\,\mathrm{Ass}_R\, N/\mathfrak{a}N$. Hence $\mathrm{ht}_N\, \mathfrak{p} = n$ and the claim is true. ∎

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