Int. J. Appl. Comput. Math (2024) 10:1
https://doi.org/10.1007/s40819-023-01573-9

ORIGINAL PAPER

Numerical Methods for Solving a Riesz Space Partial Fractional Differential Equation: Applied to Fractional Kinetic Equations

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Accepted: 19 August 2023

Abstract
In this paper, we design new numerical methods for solving the partial fractional differential equations. The fractional kinetics equation has been considered in this paper. The Riesz fractional derivative has been approximated by linear interpolation polynomial and numerical methods have been designed by using the Euler and the Crank–Nicolson methods. Furthermore, the error bounds of proposed methods are obtained. We also show that our numerical methods are stable and convergent with the accuracy of $O(\kappa + h)$ and $O(\kappa^2 + h)$ respectively. Finally, some numerical examples are constructed to demonstrate the efficacy and usefulness of the numerical methods.

Keywords Riesz space partial fractional differential equation · The fractional kinetics equation · Stability and convergent

Mathematics Subject Classification 35R11 · 65L20 · 65N06 · 65N22

Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, control theory, fractal phenomena, biosciences, bioengineering, fluid mechanics, nonlinear control, control of power electronics, chaotic dynamics, thermodynamics and signal and image processing etc [1–3]. Fractional order differential equations (FDEs) are generalizations of classical integer order differential equations. Recently, works in this field have shown that sometimes physical

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systems can be modeled more accurately using fractional derivative formulations. For example, ultrasonic wave propagation in human cancellous bone [4], modeling of speech signals [5], modeling the cardiac tissue electrode interface [6], the sound waves propagation in rigid porous materials [7], lateral and longitudinal control of autonomous vehicles [8], the theory of viscoelasticity [9], fractional differentiation for edge detection [10], wave propagation in viscoelastic horns using a fractional calculus rheology model [11], fluid mechanics [12], etc.

There are different methods to solve FDEs analytically. The most common analytical methods used to solve very special FDEs are the Laplace transform method, fractional Green's function, power series method, method of orthogonal polynomials, etc [13]. Hence, developing efficient and reliable numerical methods for solving general FDEs is of particular usefulness in the application. Several numerical methods have been proposed, such as, PI method, Multistep Methods, Krylov projection methods, etc [14–17].

Partial fractional differential equations (PFDEs) arise in the unification of diffusion and wave propagation phenomenon. The solutions of these equations are very important in the physical sciences. Some analytical techniques are presented in literature for solving PFDEs, such as, method of separating variables [18], decomposition method [19], variational iteration method [20], homotopy-perturbation method [21], etc. For details on numerical methods for partial fractional differential equations, for instance, see [22–31].

The fractional kinetic equation was introduced to describe chaotic Hamiltonian dynamics in typical low dimensional systems [32]. A fractal set of islands in the stochastic sea makes the phase space non-uniform and imposes a specific, sticky behavior near the boundaries of islands. In its turn, the boundary layers of the islands can be considered as domains where the intermittent dynamics reveals its fractal properties both in space and time. Such a space-time complexity can be considered as a motivation to use fractal derivatives in space and time to describe the sticky trajectories on a fractal support. A number of simulations confirms the correctness of the approach to the anomalous transport problem based on the fractional kinetics [33]. During the last decade, different limit cases of the fractional kinetic equation have been discussed intensively. One case is related to the so-called Lévy process and Lévy flights [34], another to the problem of fractal time and fractal Brownian motion [35]. These cases can be matched to the Montroll-Weiss equation and to the continuous time random walk [36]. All of these cases can be united by the fractional kinetic equation. Some different special cases of the fractional kinetics were considered in applications that are not related to typical problems of the theory of dynamical systems with chaos, such as theory of turbulence [34], diffusion in porous media [37] and kinetics in viscoelastic media [38]. A derivation of the fractional kinetic equation (FKE) consists of two steps. First, it is based on the existence of singular zones in phase space that creates a set of sticky domains. The dynamics considering only parts of trajectories in sticky domains and neglecting the parts of trajectories of their transition from one sticky domain to another one can be plotted. Second, the new support based on a set of sticky domains is a fractal structure in time and in phase space simultaneously [39].

To approximate the Riesz fractional derivative and use to numerical solving FDEs and FPDEs, there have existed several analytical and numerical methods. In the following methods α belongs to (1,2). Zhang and Liu [40] obtained the analytical solutions of space Riesz and time Caputo fractional partial differential equations. Chen et al. [41] studied the analytical solution for the space Riesz fractional reaction-dispersion equation by using the Laplace and Fourier transform. Yang et al. [42] studied the space Riesz fractional diffusion and advection-dispersion equations and solving these problems on a finite domain by using the L1, L2 methods. Later on, they [43] presented two numerical methods for solving the Riesz fractional advection-dispersion equation by using the Galerkin finite element and a backward difference methods. Shen et al. [44] used a second-order numerical scheme for the space Riesz fractional advection-dispersion equation based on fractional central difference operator. Yang et al. [42] presented three numerical methods for the space Riesz fractional diffusion and advection-dispersion equations on a finite domain. Celik and Duman [45] obtained the numerical solutions of the fractional diffusion equation with the space Riesz fractional derivative on a finite domain based on the fractional central difference operator. Recently, Ding et al. [46] constructed two different fourth-order numerical schemes for the Riesz derivative and applied to the spatial Riesz fractional diffusion equation. Later on, they [47] developed two high-order approximate schemes (sixth- and eighth-order) for the Riesz fractional derivative and use to solve the following space Riesz fractional reaction-dispersion equation. Very recently, Ding and Li [48] constructed high-order (from 2nd-order to 6th-order) numerical schemes to approximate the Riesz derivatives with $\alpha \in (0, 1)$, and develop three kinds of difference schemes for the Riesz space fractional turbulent diffusion equation. Also, Ding and Li [49] derived even-order fractional-compact numerical differential formulas for Riesz derivatives $\alpha \in (1, 2)$, and the fourth-order formula applied to solving the Riesz spatial fractional telegraph equation.

At present, the numerical schemes to approximate the Riesz derivatives with $\alpha \in (1, 2)$ are usually obtained by weighting the shifted and non-shifted Grunwald–Letnikov or Lubich difference operators. In the present paper, our main goal is to construct a numerical method for approximate the Riesz derivatives with $\alpha \in (0, 1)$ by using the piecewise linear interpolation polynomial. The novelty of the paper is firstly to propose a formula for the Riesz derivatives based on the piecewise linear interpolation polynomial. Then, in this work, we approximate the Riesz derivatives with order $\alpha \in (0, 1)$. Until now the Riesz derivatives approximate for $\alpha \in (1, 2)$. The main advantage of the method is the one can easily get unconditionally stable finite difference scheme.

In this paper, we consider special cases of the fractional kinetics and design a new numerical method for the Riesz space partial fractional differential equation. Next, we proposed, two numerical methods based on the forward Euler, Crank–Nicolson methods and linear interpolation polynomial for time and space, respectively. Also, we prove that our numerical methods are unconditionally stable and convergent with the accuracy of $\mathrm{O}(\kappa + h)$ and $\mathrm{O}(\kappa^2 + h)$ respectively. Finally, to show the efficacy of the numerical methods, some numerical examples are given.

The rest of this paper is organized as follows. Section “Preliminaries” contains some definitions. Equation of fractional kinetics is introduced in Section “Equation of Fractional Kinetics”. In Section “Numerical Method”, numerical method for our problem is presented. In Section “Test Examples”, examples illustrating the performance of the new numerical schemes are presented. In the last section, conclusions are given.

Preliminaries

In this section, we recall some definitions related to our work. Let $\mathbb{C}(\mathbb{J}, \mathbb{R})$ denote the Banach space of all continuous functions from $\mathbb{J} = [0, T]$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_\infty = \sup\{|u(t)| : t \in \mathbb{J}\}, \quad T > 0 \tag{2.1}$$

and $\mathbb{C}^n(\mathbb{J}, \mathbb{R})$ denotes the class of all real valued functions possessing derivatives up to order n on $\mathbb{J} = [0, T]$, $T > 0$.

Definition 1 [13] The left and right fractional integrals of order $\alpha > 0$ of $f \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ are defined as

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \tag{2.2} $$

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds. \tag{2.3} $$

Definition 2 [13] The left and right Riemann-Liouville fractional derivatives of order $n-1 \leq \alpha < n, \ n \in \mathbb{N}$ of $f \in \mathbb{C}^n(\mathbb{J}, \mathbb{R})$ are defined as

$$ {}^{RL}{}_0D_t{}^\alpha f(t) = D^n{}_0I_t^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d^n}{dt^n}) \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.4} $$

$$ {}^{RL}{}_tD_b{}^\alpha f(t) = (-1)^n D^n{}_tI_b^{n-\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} (\frac{d^n}{dt^n}) \int_t^b \frac{f(s)}{(s-t)^{\alpha-n+1}} ds. \tag{2.5} $$

Definition 3 [13] The left and right Caputo fractional derivatives of order $n-1 < \alpha < n, \ n \in \mathbb{N}$ of $f \in \mathbb{C}^n(\mathbb{J}, \mathbb{R})$ are defined as

$$ {}^C{}_0D_t{}^\alpha f(t) = {}_0I_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.6} $$

$$ {}^C{}_tD_b{}^\alpha f(t) = {}_tI_b^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{(-1)^n f^{(n)}(s)}{(s-t)^{\alpha-n+1}} ds. \tag{2.7} $$

Definition 4 [42] The Riesz fractional derivative of order $0 < \alpha < 2$ of the function $f \in \mathbb{C}^2(\mathbb{J}, \mathbb{R})$ on a finite domain [0, b] is defined as

$$ \frac{\partial^\alpha f(t)}{\partial |t|^\alpha} = -(-\Delta)^{\frac{\alpha}{2}} f(t) = -c_\alpha [{}^{RL}{}_0D_t^\alpha f(t) + {}^{RL}{}_tD_b^\alpha f(t)], \tag{2.8} $$

where

$$ c_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}, \quad 0 < \alpha < 2, \ \alpha \neq 1, \tag{2.9} $$

also, ${}^{RL}{}_0D_t^\alpha$ and ${}^{RL}{}_tD_b^\alpha$ are the left and right Riemann-Liouville fractional derivative operators respectively.

Equation of Fractional Kinetics

A generalized fractional kinetic equation can be presented as the following form

$$ \frac{\partial^\beta u(x,t)}{\partial t^\beta} = L(x)u(x,t) + f(x,t), \tag{3.1} $$

where f is the source function, $0 < \beta \leq 1$ and $L(x)$ is fractional differential operator as

$$L(x) = a\frac{\partial}{\partial x} + b^{+}\frac{\partial^{\alpha^{+}}}{\partial x^{\alpha^{+}}} + b^{-}\frac{\partial^{\alpha^{-}}}{\partial x^{\alpha^{-}}}, \tag{3.2}$$

where a, b^{+} and b^{-} are costants and $0 < \alpha^{+}, \alpha^{-} \leq 2$ (for more details see [50]).

In the definition of $L(x)$, we have to introduce two kinds of fractional derivatives along $x < 0$ and $x > 0$ of the corresponding orders, since the wandering process can be performed in both directions of x. Also, The first term in the L corresponds to a regular (non-fractional) flux. The next two terms correspond to asymmetric wandering along fractional support in positive and negative directions of x. The case of symmetric fractional kinetics has been observed in simulations for the web-map and standard map [51].

If we consider $a \equiv 0,\ \ b^{+} = b^{-} = 1,\ \ \alpha^{+} = \alpha^{-} = \alpha$, thus, the corresponding form of the fractional kinetic equation will be as

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), \tag{3.3}$$

where $0 < \beta \leq 1,\ \ 0 < \alpha \leq 2$ and function $u(x,t)$ is a probability density [50].

In the case $\beta = 1, \alpha = 2$, it is normal diffusion equation. For $0 < \beta < 1, \alpha = 2$, is called the equation of fractional Brownian motion. For $\beta = 1, 1 < \alpha < 2$, the FKE corresponds to the Lévy process [33, 51]. To study dynamical behaviors of fractional kinetics equation, introduction numerical methods for solving this problem, are very important. In this work, we take $\beta = 1,\ \ 0 < \alpha < 1$. This paper focuses on designing a new numerical method for finding a solution $u(x,t)$ that satisfies the fractional kinetics equation with a source term $f(x,t)$

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), \quad t > 0,\ \ x \in (0, L), \\ u(x,0) = g(x), \\ u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t). \end{cases} \tag{3.4}$$

Where $0 < \alpha < 1$, and $\dfrac{\partial^{\alpha}}{\partial |x|^{\alpha}}$ is the Riesz fractional derivative, also $\mu_1(t)$, $\mu_2(t)$, $g(x)$ and $f(x,t)$ are sufficiently smooth functions.

Numerical Method

The purpose of this section, is to present a new numerical method by using the piecewise linear interpolation polynomial for solving the fractional kinetics equation (3.4). We begin by introducing a relation between the Riemann-Liouville and the Caputo fractional derivatives [52].

$$\begin{aligned} {}_{a}^{C}D_{x}{}^{\alpha} f(x) &= {}^{RL}{}_{a}D_{x}{}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}, \\ {}_{x}^{C}D_{b}{}^{\alpha} f(x) &= {}^{RL}{}_{x}D_{b}{}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}, \quad n-1 < \alpha < n. \end{aligned} \tag{4.1}$$

Let $0 < \alpha < 1$, then

$$\begin{aligned} {}_{a}^{RL}D_x{}^{\alpha} f(x) &= {}^{C}{}_{a}D_t{}^{\alpha} f(x) + \frac{f(a)(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \\ {}_{x}^{RL}D_b{}^{\alpha} f(x) &= {}^{C}{}_{t}D_b{}^{\alpha} f(x) + \frac{f(b)(b-x)^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{4.2}$$

therefore, for $t > 0$ and $0 \le x \le L$, we can write

$$\begin{aligned} \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} &= -c_{\alpha}\left[{}^{RL}{}_{0}D_x{}^{\alpha} u(x,t) + {}^{RL}{}_{x}D_L{}^{\alpha} u(x,t)\right] \\ &= -c_{\alpha}\left[{}^{C}{}_{0}D_x{}^{\alpha} u(x,t) + \frac{u(0,t)x^{-\alpha}}{\Gamma(1-\alpha)} + {}^{C}{}_{x}D_L{}^{\alpha} u(x,t) + \frac{u(L,t)(L-x)^{-\alpha}}{\Gamma(1-\alpha)}\right] \\ &= -c_{\alpha}\left[{}^{C}{}_{0}D_x{}^{\alpha} u(x,t) + {}^{C}{}_{x}D_L{}^{\alpha} u(x,t)\right] \\ &\quad -c_{\alpha}\left[\frac{u(0,t)x^{-\alpha}}{\Gamma(1-\alpha)} + \frac{u(L,t)(L-x)^{-\alpha}}{\Gamma(1-\alpha)}\right]. \end{aligned} \tag{4.3}$$

We consider Eq (3.4), by using (2.6), (2.7) and (4.3), we can write

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t) \\ &= \frac{-c_{\alpha}}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} \frac{\partial u(\tau,t)}{\partial \tau} d\tau - \frac{-c_{\alpha}}{\Gamma(1-\alpha)} \int_x^L (\tau - x)^{-\alpha} \frac{\partial u(\tau,t)}{\partial \tau} d\tau \\ &\quad + f(x,t) - c_{\alpha} g(x,t), \end{aligned} \tag{4.4}$$

where $g(x,t) = \dfrac{u(0,t)x^{-\alpha}}{\Gamma(1-\alpha)} + \dfrac{u(L,t)(L-x)^{-\alpha}}{\Gamma(1-\alpha)}$.

In this section, for approximate the time derivative, we consider two cases. In case 1, we approximate the time derivative with the backward Euler method. In case 2, we approximate the time derivative with the Crank–Nicolson scheme. In case 1 and 2, the Riesz space fractional derivative approximate by using the piecewise linear interpolation polynomial. We partition $[0, L]$ into a uniform mesh with the space step size $h = L/M$ and the time step size $\kappa = T/N$, where M, N are two positive integers. Also we have, $x_n = nh$ and $t_j = j\kappa$ for $n = 1, \ldots, M$ and $j = 1, \ldots, N$.

Case 1. We apply numerical method to Eq(3.4) as follows. Let $u(x_n, t_j) = u_n^j$, $f(x_n, t_j) = f_n^j$ and $g(x_n, t_j) = g_n^j$. Then

$$\begin{aligned} \frac{u_{n+1}^j - u_{n+1}^{j-1}}{\kappa} &= \frac{-c_{\alpha}}{\Gamma(1-\alpha)} \int_0^{x_{n+1}} (x_{n+1} - \tau)^{-\alpha} \frac{\partial u(\tau, t_j)}{\partial \tau} d\tau \\ &\quad - \frac{-c_{\alpha}}{\Gamma(1-\alpha)} \int_{x_{n+1}}^L (\tau - x_{n+1})^{-\alpha} \frac{\partial u(\tau, t_j)}{\partial \tau} d\tau + f_{n+1}^j - c_{\alpha} g_{n+1}^j, \end{aligned} \tag{4.5}$$

we use the piecewise linear interpolation polynomial for $u(\tau, t)$ at the nodes x_i, x_{i+1} to approximate two integrals on the right hand of (4.5), these integrals for $\tau \in [x_i, x_{i+1}]$, can be written as

$$
\begin{aligned}
&\int_{0}^{x_{n+1}} (x_{n+1}-\tau)^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau = \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau\\
&\approx \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha}\frac{\partial}{\partial\tau}\left[\frac{(\tau-x_{i+1})}{(x_i-x_{i+1})}u_i^j+\frac{(\tau-x_i)}{(x_{i+1}-x_i)}u_{i+1}^j\right]d\tau\\
&= \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha}\left[\frac{1}{(x_i-x_{i+1})}u_i^j+\frac{1}{(x_{i+1}-x_i)}u_{i+1}^j\right]d\tau\\
&= \sum_{i=0}^{n}\left[\frac{1}{(x_i-x_{i+1})}u_i^j+\frac{1}{(x_{i+1}-x_i)}u_{i+1}^j\right]\int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha}d\tau\\
&= \sum_{i=0}^{n}\left[u_{i+1}^j-u_i^j\right]\frac{(x_{n+1}-x_i)^{1-\alpha}-(x_{n+1}-x_{i+1})^{1-\alpha}}{(1-\alpha)(x_{i+1}-x_i)},
\end{aligned}
\tag{4.6}
$$

also, we have

$$
\begin{aligned}
&\int_{x_{n+1}}^{L} (\tau-x_{n+1})^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau = \sum_{i=n+1}^{M-1}\int_{x_i}^{x_{i+1}} (\tau-x_{n+1})^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau\\
&\approx \sum_{i=n+1}^{M-1}\int_{x_i}^{x_{i+1}} (\tau-x_{n+1})^{-\alpha}\frac{\partial}{\partial\tau}\left[\frac{(\tau-x_{i+1})}{(x_i-x_{i+1})}u_i^j+\frac{(\tau-x_i)}{(x_{i+1}-x_i)}u_{i+1}^j\right]d\tau\\
&= \sum_{i=n+1}^{M-1}\int_{x_i}^{x_{i+1}} (\tau-x_{n+1})^{-\alpha}\left[\frac{1}{(x_i-x_{i+1})}u_i^j+\frac{1}{(x_{i+1}-x_i)}u_{i+1}^j\right]d\tau\\
&= \sum_{i=n+1}^{M-1}\left[\frac{1}{(x_i-x_{i+1})}u_i^j+\frac{1}{(x_{i+1}-x_i)}u_{i+1}^j\right]\int_{x_i}^{x_{i+1}} (\tau-x_{n+1})^{-\alpha}d\tau\\
&= \sum_{i=n+1}^{M-1}\left[u_{i+1}^j-u_i^j\right]\frac{(x_{i+1}-x_{n+1})^{1-\alpha}-(x_i-x_{n+1})^{1-\alpha}}{(1-\alpha)(x_{i+1}-x_i)}.
\end{aligned}
\tag{4.7}
$$

So, Eq. (3.4) by using (4.6) and (4.7), will be as

$$
\begin{aligned}
&u_{n+1}^j+\frac{\kappa c_\alpha}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\frac{(x_{n+1}-x_i)^{1-\alpha}-(x_{n+1}-x_{i+1})^{1-\alpha}}{(x_{i+1}-x_i)}\left[u_{i+1}^j-u_i^j\right]\\
&\quad+\frac{\kappa c_\alpha}{\Gamma(2-\alpha)}\sum_{i=n+1}^{M-1}\frac{(x_i-x_{n+1})^{1-\alpha}-(x_{i+1}-x_{n+1})^{1-\alpha}}{(x_{i+1}-x_i)}\left[u_{i+1}^j-u_i^j\right]\\
&\quad= u_{n+1}^{j-1}+\kappa f_{n+1}^j-\kappa c_\alpha g_{n+1}^j,
\end{aligned}
\tag{4.8}
$$

finally, we can write

$$
\begin{aligned}
&u_{n+1}^j+\sum_{i=0}^{n}\omega(\alpha,i,n)\left[u_{i+1}^j-u_i^j\right]+\sum_{i=n+1}^{M-1}\nu(\alpha,i,n)\left[u_{i+1}^j-u_i^j\right]\\
&\quad= u_{n+1}^{j-1}+\kappa f_{n+1}^j-\kappa c_\alpha g_{n+1}^j,
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
\omega(\alpha,i,n) &= \frac{\kappa h^{-\alpha}c_\alpha}{\Gamma(2-\alpha)}\left[(n+1-i)^{1-\alpha}-(n-i)^{1-\alpha}\right],\ i=0,1,2,\ldots,n,\\
\nu(\alpha,i,n) &= \frac{\kappa h^{-\alpha}c_\alpha}{\Gamma(2-\alpha)}\left[(i-n-1)^{1-\alpha}-(i-n)^{1-\alpha}\right],\ i=n+1,n+2,\ldots,M-1.
\end{aligned}
\tag{4.10}
$$

Lemma 1 *Let $\omega(\alpha, j, i)$ and $\nu(\alpha, j, i)$ are introduced at (4.10). The following relation for $j \leq i-1$ with $i = 2, 3, \ldots M-1$ holds:*

$$\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) = \nu(\alpha, i, j-1) - \nu(\alpha, i-1, j-1).$$

Proof. *From Eq (4.10), it follows that*

$$\begin{aligned}\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) &= \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \\ &\times \left[2(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha} - (i-j+1)^{1-\alpha}\right],\end{aligned} \tag{4.11}$$

also, we have

$$\begin{aligned}\nu(\alpha, i, j-1) - \nu(\alpha, i-1, j-1) &= \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \\ &\times \left[2(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha} - (i-j+1)^{1-\alpha}\right].\end{aligned} \tag{4.12}$$

Eq (4.11) and (4.12), conclude the proof.

By using Lemma 1, introducing

$$A = \begin{pmatrix} \omega(\alpha,0,0) & 0 & 0 & \cdots 0 \\ \beta(1,2) & \omega(\alpha,1,1) & 0 & \cdots 0 \\ \beta(1,3) & \beta(2,3) & \omega(\alpha,2,2) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \beta(1,M-2) & \beta(2,M-2) & \beta(3,M-2) & 0 \\ \beta(1,M-1) & \beta(2,M-1) & \beta(3,M-1) & \omega(\alpha,M-2,M-2) \end{pmatrix}, \tag{4.13}$$

where

$$\beta(j,i) = -\left[\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1)\right]$$

and

$$D = A + A^t, \quad U^j = [u_1^j, u_2^j, \ldots, u_{M-1}^j]^t, \tag{4.14}$$

where A^t denotes the transpose of a matrix A. Equation (4.9) takes the form:

$$(I + D)U^j = IU^{j-1} + F^j, \tag{4.15}$$

where

$$F^j = \begin{bmatrix} \kappa f_1^j - \kappa c_\alpha g_1^j + \omega(\alpha,0,0)u_0^j - \nu(\alpha,M-1,0)u_M^j \\ \kappa f_2^j - \kappa c_\alpha g_2^j + \omega(\alpha,0,1)u_0^j - \nu(\alpha,M-1,1)u_M^j \\ \kappa f_3^j - \kappa c_\alpha g_3^j + \omega(\alpha,0,2)u_0^j - \nu(\alpha,M-1,2)u_M^j \\ \vdots \\ \kappa f_{M-2}^j - \kappa c_\alpha g_{M-2}^j + \omega(\alpha,0,M-3)u_0^j - \nu(\alpha,M-1,M-3)u_M^j \\ \kappa f_{M-1}^j - \kappa c_\alpha g_{M-1}^j + \omega(\alpha,0,M-2)u_0^j - \nu(\alpha,M-1,M-2)u_M^j \end{bmatrix}.$$

Case 2. In this case, we approximate solution by using the Crank–Nicolson scheme for the equation (3.4). So we have

$$\begin{aligned}\frac{u_{n+1}^{j}-u_{n+1}^{j-1}}{\kappa}=&\frac{1}{2}\left[\frac{-c_\alpha}{\Gamma(1-\alpha)}\int_{0}^{x_{n+1}}(x_{n+1}-\tau)^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau\right.\\&\left.-\frac{-c_\alpha}{\Gamma(1-\alpha)}\int_{x_{n+1}}^{L}(\tau-x_{n+1})^{-\alpha}\frac{\partial u(\tau,t_j)}{\partial\tau}d\tau+f_{n+1}^{j}-c_\alpha g_{n+1}^{j}\right]\\&+\frac{1}{2}\left[\frac{-c_\alpha}{\Gamma(1-\alpha)}\int_{0}^{x_{n+1}}(x_{n+1}-\tau)^{-\alpha}\frac{\partial u(\tau,t_{j-1})}{\partial\tau}d\tau\right.\\&\left.-\frac{-c_\alpha}{\Gamma(1-\alpha)}\int_{x_{n+1}}^{L}(\tau-x_{n+1})^{-\alpha}\frac{\partial u(\tau,t_{j-1})}{\partial\tau}d\tau+f_{n+1}^{j-1}-c_\alpha g_{n+1}^{j-1}\right],\end{aligned}\tag{4.16}$$

by using the piecewise linear interpolation polynomial for $u(\tau,t)$ at the nodes $x_i,\ \ x_{i+1}$, and for $\tau\in[x_i,\ \ x_{i+1}]$, we can write

$$\begin{aligned}&u_{n+1}^{j}+\sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[u_{i+1}^{j}-u_{i}^{j}\right]+\sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[u_{i+1}^{j}-u_{i}^{j}\right]\\&=u_{n+1}^{j-1}-\sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[u_{i+1}^{j-1}-u_{i}^{j-1}\right]-\sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[u_{i+1}^{j-1}-u_{i}^{j-1}\right]\\&+\tfrac{\kappa}{2}\left[f_{n+1}^{j}+f_{n+1}^{j-1}-c_\alpha(g_{n+1}^{j}+g_{n+1}^{j-1})\right],\end{aligned}\tag{4.17}$$

where

$$\bar{\omega}(\alpha,i,n)=\frac{1}{2}\omega(\alpha,i,n),\ \ \bar{\nu}(\alpha,i,n)=\frac{1}{2}\,\nu(\alpha,i,n).\tag{4.18}$$

Eq. (4.17) takes the form:

$$(I+\bar{D})U^{j}=(I-\bar{D})U^{j-1}+G^{j},\tag{4.19}$$

where

$$\bar{D}=\frac{1}{2}D,\qquad U^{j}=[u_1^{j},u_2^{j},\ \ldots,u_{M-1}^{j}]^{t}\tag{4.20}$$

and

$$G^{j}=[\Phi(1),\Phi(2),\ldots,\Phi(M-1)]^{t},$$

where

$$\begin{aligned}\Phi(i)=&\frac{\kappa}{2}\left[f_i^{j}+f_i^{j-1}-c_\alpha(g_i^{j}+g_i^{j-1})\right]+\bar{\omega}(\alpha,0,i-1)\left[u_0^{j}+u_0^{j-1}\right]\\&-\bar{\nu}(\alpha,M-1,i-1)\left[u_M^{j}+u_M^{j-1}\right],\quad i=1,2,\ldots,M-1.\end{aligned}$$

Stability of Methods

In this subsection, in order to study stability of methods, we present some Lemmas.

Lemma 2 *Let $\omega(\alpha, i, n)$ is introduced at (4.10). The following relation for $j \leq i-1$ with $i = 2, 3, \ldots M-1$ holds:*

$$\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) > 0.$$

Proof From Eq (4.10), we have $i \leq n$. Since $j \leq i-1$, thus $j-1 \leq i-1$. Therefore, we can write

$$\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) = \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \left(\left[(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}\right] \right. \\ \left. - \left[(i-j+1)^{1-\alpha} - (i-j)^{1-\alpha}\right] \right),$$

we take $a = i - j$. Since $a \geq 1$, we have

$$\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) = \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \left(\left[a^{1-\alpha} - (a-1)^{1-\alpha}\right] \right. \\ \left. - \left[(a+1)^{1-\alpha} - a^{1-\alpha}\right] \right).$$

Suppose $H(a) = a^{1-\alpha} - (a-1)^{1-\alpha}$, and by means of the mean value theorem for H, there is a ξ that we can write

$$H(a+1) - H(a) = H'(\xi)(a+1-a) = H'(\xi),$$

therefore, we have

$$H'(\xi) = (1-\alpha)\left[a^{-\alpha} - (a-1)^{-\alpha}\right],$$

also, since $a > a-1$ and $0 < \alpha < 1$, then $a^{-\alpha} < (a-1)^{-\alpha}$, So, $H'(\xi) < 0$. Finally, we can write

$$H(a) - H(a+1) = \left[a^{1-\alpha} - (a-1)^{1-\alpha}\right] - \left[(a+1)^{1-\alpha} - a^{1-\alpha}\right] > 0,$$

by this relation, the proof is complete.

Theorem 1 *The matrix D is a strictly diagonally dominant.*

Proof We have

$$D_{i,j} = \begin{cases} -\left[\omega(\alpha, i, j-1) - \omega(\alpha, i-1, j-1)\right], & j \geq i+1, \\ 2\omega(\alpha, i, i), & j = i, \\ -\left[\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1)\right] & j \leq i-1, \end{cases} \tag{4.21}$$

where $\omega(\alpha, i, n)$ is introduced at (4.10). If $i = n$, since $c_\alpha > 0$ for $0 < \alpha < 1$. We can write $\omega(\alpha, i, i) = \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} > 0$ and according to Lemma 2, for $j \leq i-1$, we have $\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1) > 0$, which leads to

$$-\left[\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1)\right] < 0$$

and for $j \geq i+1$, we have

$$-\left[\omega(\alpha, i, j-1) - \omega(\alpha, i-1, j-1)\right] < 0.$$

For a given i, we can write

$$
\begin{aligned}
\sum_{j=1,\, j\neq i}^{M-1} \left|D_{i,j}\right| &= \sum_{j=1}^{i-1} |-[\omega(\alpha, j, i-1) - \omega(\alpha, j-1, i-1)]| \\
&\quad + \sum_{j=i+1}^{M-1} |-[\omega(\alpha, i, j-1) - \omega(\alpha, i-1, j-1)]| \\
&= [\omega(\alpha, 1, i-1) - \omega(\alpha, 0, i-1) + \omega(\alpha, 2, i-1) - \omega(\alpha, 1, i-1) \\
&\quad + \omega(\alpha, 3, i-1) - \omega(\alpha, 2, i-1) + \cdots + \omega(\alpha, i-2, i-1) \\
&\quad - \omega(\alpha, i-3, i-1) + \omega(\alpha, i-1, i-1) - \omega(\alpha, i-2, i-1)] \\
&\quad + [\omega(\alpha, i, i) - \omega(\alpha, i-1, i) + \omega(\alpha, i, i+1) - \omega(\alpha, i-1, i+1) \\
&\quad + \omega(\alpha, i, i+2) - \omega(\alpha, i-1, i+2) + \cdots + \omega(\alpha, i, M-3) \\
&\quad - \omega(\alpha, i-1, M-3) + \omega(\alpha, i, M-2) - \omega(\alpha, i-1, M-2)] \\
&= [\omega(\alpha, i-1, i-1) - \omega(\alpha, 0, i-1)] \\
&\quad + \frac{\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \left[1 - (2^{1-\alpha} - 1) + (2^{1-\alpha} - 1) - (3^{1-\alpha} - 2^{1-\alpha})\right. \\
&\quad + (3^{1-\alpha} - 2^{1-\alpha}) - (4^{1-\alpha} - 3^{1-\alpha}) + \cdots + ((M-1)^{1-\alpha} - (M-2)^{1-\alpha}) \\
&\quad \left. - ((M)^{1-\alpha} - (M-1)^{1-\alpha})\right] \\
&= \omega(\alpha, i-1, i-1) - \omega(\alpha, 0, i-1) + \omega(\alpha, i, i) - \omega(\alpha, i-1, M-2) \\
&= 2\omega(\alpha, i, i) - [\omega(\alpha, 0, i-1) + \omega(\alpha, i-1, M-2)] \\
&< 2\omega(\alpha, i, i) = \left|D_{i,i}\right|.
\end{aligned}
$$

Lemma 3 *The matrix D is symmetric and positive definite.*

Proof Equation (4.21) shows, the matrix D is clearly symmetric. Let λ_0 be an eigenvalue of the matrix D. Then it follows by the Geršgorin circles Theorem [53] that

$$
\left|\lambda_0 - D_{i,i}\right| \leq \sum_{j=1,\, j\neq i}^{M-1} \left|D_{i,j}\right|,
$$

or

$$
D_{i,i} - \sum_{j=1,\, j\neq i}^{M-1} \left|D_{i,j}\right| \leq \lambda_0 \leq D_{i,i} + \sum_{j=1,\, j\neq i}^{M-1} \left|D_{i,j}\right|.
$$

Then, by Theorem 1, we have

$$
\lambda_0 \geq D_{i,i} - \sum_{j=1,\, j\neq i}^{M-1} \left|D_{i,j}\right| \geq 0,
$$

which shows that D is positive definite.

Lemma 4 *The matrix $\bar{D}$ is a strictly diagonally dominant matrix, symmetric and positive definite.*

Proof We omit the proof as it is similar to that of Lemmas 2 and 3.

Lemma 5 ([54]) *Let B be a positive definite matrix of order $m-1$. Then, for any parameter $\lambda \geq 0$, the following inequalities hold:*

$$\left\|(I+\lambda B)^{-1}\right\|_{\infty} \leq 1, \quad \left\|(I+\lambda B)^{-1}(I-\lambda B)\right\|_{\infty} \leq 1.$$

Theorem 2 *The first numerical method (4.9) is unconditionally stable.*

Proof Let U^j and u^j be the numerical solution and an exact solution, respectively. Since (I + D) is invertible, we can write

$$U^j - u^j = (I+D)^{-1}(U^{j-1} - u^{j-1}),$$

elementary calculation shows that

$$U^j - u^j = \left[(I+D)^{-1}\right]^{j-1}(U^0 - u^0).$$

By Lemma 5, we have

$$\begin{aligned}\left\|U^j - u^j\right\|_{\infty} &= \left\|\left[(I+D)^{-1}\right]^{j-1}(U^0 - u^0)\right\|_{\infty} \\ &\leq \left\|(I+D)^{-1}\right\|_{\infty}^{j-1}\left\|U^0 - u^0\right\|_{\infty} \leq \left\|U^0 - u^0\right\|_{\infty}.\end{aligned}$$

Thus the numerical method (4.9) is unconditionally stable.

Theorem 3 *The second numerical method (4.17) is unconditionally stable.*

Proof Let U^j and u^j be the numerical solution and an exact solution, respectively. Since (I + $\bar{D}$) is invertible, we can write

$$U^j - u^j = (I+\bar{D})^{-1}(I-\bar{D})(U^{j-1} - u^{j-1}),$$

by elementary calculation, we can write

$$U^j - u^j = \left[(I+\bar{D})^{-1}(I-\bar{D})\right]^{j-1}(U^0 - u^0).$$

By Lemma 5, we have

$$\begin{aligned}\left\|U^j - u^j\right\|_{\infty} &= \left\|\left[(I+\bar{D})^{-1}(I-\bar{D})\right]^{j-1}(U^0 - u^0)\right\|_{\infty} \\ &\leq \left\|(I+\bar{D})^{-1}(I-\bar{D})\right\|_{\infty}^{j-1}\left\|U^0 - u^0\right\|_{\infty} \leq \left\|U^0 - u^0\right\|_{\infty}.\end{aligned}$$

Thus the numerical method (4.17) is unconditionally stable.

Convergence of Methods

In this subsection, in order to study convergence of methods, we present a Lemma.

Lemma 6 *Let $u \in C^2[0, L]$ and $0 < \alpha < 1$, then*

$$\left|\int_0^{x_{n+1}} (x_{n+1}-\tau)^{-\alpha}\frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right]d\tau - \int_{x_{n+1}}^{L}(\tau-x_{n+1})^{-\alpha}\frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right]d\tau\right| < Ch,$$

where $\hat{u}$ is the piecewise linear interpolation polynomial for u.

Proof By using the Taylor theorem, for $\tau \in [x_i, x_{i+1}]$. Then, there exist $\xi_i \in [x_i, x_{i+1}]$. Therefore

$$\begin{aligned}
&\left| \int_0^{x_{n+1}} (x_{n+1}-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau - \int_{x_{n+1}}^{L} (\tau - x_{n+1})^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| \\
&\le \left| \int_0^{x_{n+1}} (x_{n+1}-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| + \left| \int_{x_{n+1}}^{L} (\tau - x_{n+1})^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| \\
&= \left| \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| \\
&\quad + \left| \sum_{i=n+1}^{M-1} \int_{x_i}^{x_{+1}} (\tau - x_{n+1})^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| \\
&\approx \left| \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left[(\tau - x_i)(\tau - x_{i+1}) \left.\frac{\partial^2 u}{2\partial \tau^2}\right|_{\tau=\xi_i} \right] d\tau \right| \\
&\quad + \left| \sum_{i=n+1}^{M-1} \int_{x_i}^{x_{i+1}} (\tau - x_{n+1})^{-\alpha} \frac{\partial}{\partial \tau}\left[\left[(\tau - x_i)(\tau - x_{i+1}) \left.\frac{\partial^2 u}{2\partial \tau^2}\right|_{\tau=\xi_i} \right]\right] d\tau \right| \\
&\le \frac{M_2}{2} \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha} \left[2\tau - x_i - x_{i+1}\right] d\tau \\
&\quad + \frac{M_2}{2} \sum_{i=n+1}^{M-1} \int_{x_i}^{x_{i+1}} (\tau - x_{n+1})^{-\alpha} \left[2\tau - x_i - x_{i+1}\right] d\tau,
\end{aligned}$$

since, $\tau \in [x_i, \ x_{i+1}]$ and $2\tau \le 2x_{i+1}$, we have $2\tau - x_i - x_{i+1} \le x_{i+1} - x_i$. Therefore, we can write

$$\begin{aligned}
&\left| \int_0^{x_{n+1}} (x_{n+1}-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right. \\
&\quad \left. - \int_{x_{n+1}}^{L} (\tau - x_{n+1})^{-\alpha} \frac{\partial}{\partial \tau}\left[u(\tau,t_j)-\hat{u}(\tau,t_j)\right] d\tau \right| \\
&\le \frac{M_2}{2} \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha} \left[x_{i+1} - x_i\right] d\tau \\
&\quad + \frac{M_2}{2} \sum_{i=n+1}^{M-1} \int_{x_i}^{x_{i+1}} (\tau - x_{n+1})^{-\alpha} \left[x_{i+1} - x_i\right] d\tau \\
&= \frac{M_2 h}{2} \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (x_{n+1}-\tau)^{-\alpha} d\tau + \frac{M_2 h}{2} \sum_{i=n+1}^{M-1} \int_{x_i}^{x_{i+1}} (\tau - x_{n+1})^{-\alpha} d\tau \\
&= \frac{M_2 h}{2} \left[\sum_{i=0}^{n} \frac{(x_{n+1}-x_i)^{1-\alpha} - (x_{n+1}-x_{i+1})^{1-\alpha}}{(1-\alpha)} \right. \\
&\quad \left. + \sum_{i=n+1}^{M-1} \frac{(x_{i+1}-x_{n+1})^{1-\alpha} - (x_i - x_{n+1})^{1-\alpha}}{(1-\alpha)} \right] \\
&= \frac{M_2 h}{2} \left[\frac{(x_{n+1})^{1-\alpha}}{(1-\alpha)} + \frac{(L - x_{n+1})^{1-\alpha}}{(1-\alpha)} \right] \le Ch,
\end{aligned}$$

where $M_2 = \sup\limits_{z \in [0,L]} \left| \left. \frac{\partial^2 u(\tau,t)}{\partial \tau^2} \right|_{\tau=z} \right|$ and $\hat{u}(\tau,t)$ is the piecewise linear interpolation polynomial at the nodes x_i, x_{i+1}.

By using Lemma 6, we study convergence of methods. So, for the first method (4.9), we can write

$$\begin{aligned} \frac{\partial u_i^j}{\partial t} &= \frac{u_i^j - u_i^{j-1}}{\kappa} + \mathrm{O}(\kappa), \\ \frac{\partial^\alpha u_i^j}{\partial |x|^\alpha} &= \frac{-\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \sum_{i=0}^{n} \left[(n+1-i)^{1-\alpha} - (n-i)^{1-\alpha}\right]\left[u_{i+1}^j - u_i^j\right] \\ &\quad + \frac{-\kappa h^{-\alpha} c_\alpha}{\Gamma(2-\alpha)} \sum_{i=n+1}^{M-1} \left[(i-n-1)^{1-\alpha} - (i-n)^{1-\alpha}\right]\left[u_{i+1}^j - u_i^j\right] + \mathrm{O}(h). \end{aligned} \tag{4.22}$$

The local truncation error of (4.9) can be written as:

$$T_{i,j} = \mathrm{O}(\kappa^2 + \kappa h).$$

Theorem 4 *Let U^j be the numerical solution and u^j be an exact solution of (4.9). Then, we have*

$$\left\| U^j - u^j \right\|_\infty \le C\,O(\kappa + h), \tag{4.23}$$

where C is a positive constant.

Proof We can write for U^j and u^j as

$$\begin{aligned} U_{n+1}^j &+ \sum_{i=0}^{n} \omega(\alpha, i, n)\left[U_{i+1}^j - U_i^j\right] + \sum_{i=n+1}^{M-1} \nu(\alpha, i, n)\left[U_{i+1}^j - U_i^j\right] \\ &= U_{n+1}^{j-1} + \kappa f_{n+1}^j - \kappa c_\alpha g_{n+1}^j \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} u_{n+1}^j &+ \sum_{i=0}^{n} \omega(\alpha, i, n)\left[u_{i+1}^j - u_i^j\right] + \sum_{i=n+1}^{M-1} \nu(\alpha, i, n)\left[u_{i+1}^j - u_i^j\right] \\ &= u_{n+1}^{j-1} + \kappa f_{n+1}^j - \kappa c_\alpha g_{n+1}^j. \end{aligned} \tag{4.25}$$

Letting $e_i^j = U_i^j - u_i^j$ and using (4.24) and (4.25), we obtain

$$\begin{aligned} e_{n+1}^j &+ \sum_{i=0}^{n} \omega(\alpha, i, n)\left[e_{i+1}^j - e_i^j\right] + \sum_{i=n+1}^{M-1} \nu(\alpha, i, n)\left[e_{i+1}^j - e_i^j\right] \\ &= e_{n+1}^{j-1}. \end{aligned} \tag{4.26}$$

Eq. (4.26), can be written in matrix–vector form as

$$(I + D)E^j = I E^{j-1} + \mathrm{O}(\kappa^2 + \kappa h)\chi,$$

where

$$E^j = [e_1^j, e_2^j, \ldots, e_{M-1}^j]^t,\ \chi = [1, 1, \ldots, 1]^t,\ D = A + A^t.$$

We take

$$\digamma = (I + D)^{-1},\ \Theta = \mathrm{O}(\kappa^2 + \kappa h)(I + D)^{-1},$$

we get

$$E^j = \digamma E^{j-1} + \Theta,$$

which, on iterating and using the given initial condition, yields

$$E^j = (F^{j-1} + F^{j-2} + \cdots + I)\Theta.$$

By Lemma 5, we can write

$$\begin{aligned}\left\|E^j\right\|_\infty &\le (\left\|F^{j-1}\right\|_\infty + \left\|F^{j-2}\right\|_\infty + \cdots + \|I\|_\infty)\|\Theta\|_\infty\\ &= (\|F\|_\infty^{j-1} + \|F\|_\infty^{j-2} + \cdots + \|I\|_\infty)\|\Theta\|_\infty\\ &\le (1+1+\cdots+1)\|\Theta\|_\infty\\ &\le j\mathrm{O}(\kappa^2 + \kappa h) = T\mathrm{O}(\kappa + h).\end{aligned}$$

Thus, we can write

$$\left\|E^j\right\|_\infty \le C\mathrm{O}(\kappa + h).$$

For the second method (4.17), by using Lemma 6, we can write

$$\begin{aligned}\frac{u_i^j - u_i^{j-1}}{\kappa} &= \frac{1}{2}(\frac{\partial^\alpha u_i^j}{\partial|x|^\alpha} + \frac{\partial^\alpha u_i^{j-1}}{\partial|x|^\alpha}) + \mathrm{O}(\kappa^2)\\ \frac{\partial^\alpha u_i^j}{\partial|x|^\alpha} &= \frac{-\kappa h^{-\alpha}c_\alpha}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\left[(n+1-i)^{1-\alpha} - (n-i)^{1-\alpha}\right]\left[u_{i+1}^j - u_i^j\right]\\ &\quad + \frac{-\kappa h^{-\alpha}c_\alpha}{\Gamma(2-\alpha)}\sum_{i=n+1}^{M-1}\left[(i-n-1)^{1-\alpha} - (i-n)^{1-\alpha}\right]\left[u_{i+1}^j - u_i^j\right] + \mathrm{O}(h).\end{aligned} \tag{4.27}$$

Thus, the local truncation error of (4.17) can be written as:

$$T_{i,j} = \mathrm{O}(\kappa^3 + \kappa h).$$

Theorem 5 *Let U^j be the numerical solution and u^j be an exact solution of (4.17). Then, we have*

$$\left\|U^j - u^j\right\|_\infty \le CO(\kappa^2 + h), \tag{4.28}$$

where C is a positive constant.

Proof We can write

$$\begin{aligned}&U_{n+1}^j + \sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[U_{i+1}^j - U_i^j\right] + \sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[U_{i+1}^j - U_i^j\right]\\ &= U_{n+1}^{j-1} - \sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[U_{i+1}^{j-1} - U_i^{j-1}\right] - \sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[U_{i+1}^{j-1} - U_i^{j-1}\right]\\ &\quad + \frac{\kappa}{2}\left[f_{n+1}^j + f_{n+1}^{j-1} - c_\alpha(g_{n+1}^j + g_{n+1}^{j-1})\right]\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}&u_{n+1}^j + \sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[u_{i+1}^j - u_i^j\right] + \sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[u_{i+1}^j - u_i^j\right]\\ &\quad = u_{n+1}^{j-1} - \sum_{i=0}^{n}\bar{\omega}(\alpha,i,n)\left[u_{i+1}^{j-1} - u_i^{j-1}\right] - \sum_{i=n+1}^{M-1}\bar{\nu}(\alpha,i,n)\left[u_{i+1}^{j-1} - u_i^{j-1}\right]\\ &\quad + \tfrac{\kappa}{2}\left[f_{n+1}^j + f_{n+1}^{j-1} - c_\alpha(g_{n+1}^j + g_{n+1}^{j-1})\right].\end{aligned} \tag{4.30}$$

Let us set $e_i^j = U_i^j - u_i^j$ and by using (4.29) and (4.30), we can write

$$\begin{aligned} & e_{n+1}^j + \sum_{i=0}^{n} \bar{\omega}(\alpha, i, n) \left[e_{i+1}^j - e_i^j \right] + \sum_{i=n+1}^{M-1} \bar{\nu}(\alpha, i, n) \left[e_{i+1}^j - e_i^j \right] \\ & \quad = e_{n+1}^{j-1} - \sum_{i=0}^{n} \bar{\omega}(\alpha, i, n) \left[e_{i+1}^{j-1} - e_i^{j-1} \right] - \sum_{i=n+1}^{M-1} \bar{\nu}(\alpha, i, n) \left[e_{i+1}^{j-1} - e_i^{j-1} \right], \end{aligned} \tag{4.31}$$

thus, (4.31) can be expressed in matrix–vector form as

$$(I + \bar{D})E^j = (I - \bar{D})E^{j-1} + \mathrm{O}(\kappa^3 + \kappa h)\chi,$$

where

$$E^j = [e_1^j, e_2^j, \ldots, e_{M-1}^j]^t, \; \chi = [1, 1, \ldots, 1]^t, \quad \bar{D} = \frac{1}{2}(A + A^t).$$

Let us take

$$\mathrm{Z} = (I + \bar{D})^{-1}(I - \bar{D}), \qquad R = \mathrm{O}(\kappa^3 + \kappa h)(I + \bar{D})^{-1},$$

thus

$$E^j = \mathrm{ZE^{j-1}} + \mathrm{R},$$

by iterating and using the given initial condition, we have

$$E^j = (\mathrm{Z^{j-1}} + \mathrm{Z^{j-2}} + \cdots + I)R.$$

By Lemma 5, we can write

$$\begin{aligned} \left\| E^j \right\|_\infty & \le (\left\| \mathrm{Z}^{j-1} \right\|_\infty + \left\| \mathrm{Z}^{j-2} \right\|_\infty + \cdots + \|I\|_\infty)\|R\|_\infty \\ & = (\|\mathrm{Z}\|_\infty^{j-1} + \|\mathrm{Z}\|_\infty^{j-2} + \cdots + \|I\|_\infty)\|R\|_\infty \\ & \le (1 + 1 + \cdots + 1)\|R\|_\infty \\ & \le j\mathrm{O}(\kappa^3 + \kappa h) = T\mathrm{O}(\kappa^2 + h). \end{aligned}$$

Finally, we have

$$\left\| E^j \right\|_\infty \le C\mathrm{O}(\kappa^2 + h).$$

Test Examples

In this section, let u_i^j and $u(x_i, t_j)$ be the numerical and exact solutions at (x_i, t_j). The experimental orders of convergence is measured by the following ratios

$$Order = \log_2 \left(\frac{error(h)}{error(h/2)} \right),$$

where $error(h)$ is the absolute error $\left| u(x_i, t_j) - u_i^j \right|$ with step size $h = \kappa$.

Example 1 We consider the following fractional kinetics equation with a source term $f(x, t)$, as

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + f(x, t); \tag{5.1}$$

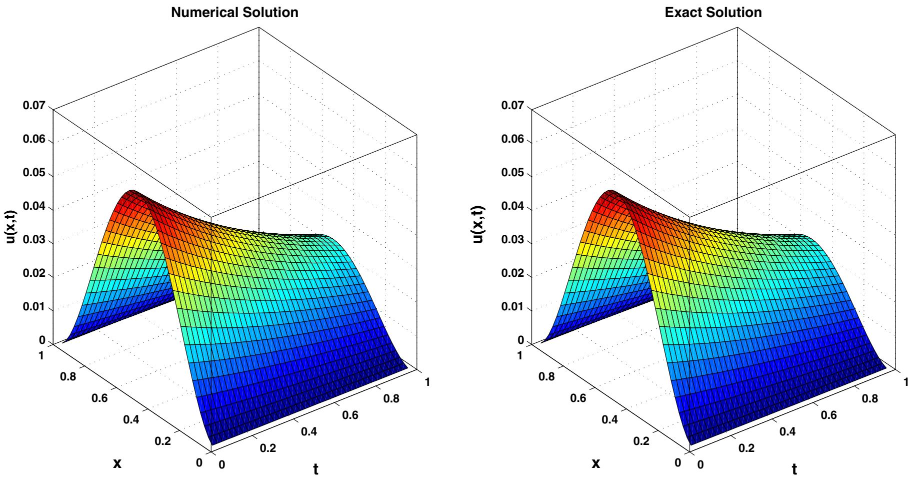

Fig. 1 The numerical and exact solutions by (4.17) for Example 1 (5.1–5.2), at $T = 1$ and $\alpha = 0.2$

subject to the initial condition:

$$\begin{aligned} u(x,0) &= x^2(1-x)^2, \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \quad 0 \le t \le T, \end{aligned} \tag{5.2}$$

where $0 < \alpha < 1$ and

$$\begin{aligned} f(x,t) = -x^2(1-x)^2e^{-t} + \frac{e^{-t}}{2\cos(\frac{\alpha\pi}{2})}\Bigg[\frac{\Gamma(5)\left[(1-x)^{4-\alpha}+x^{4-\alpha}\right]}{\Gamma(5-\alpha)} \\ -\frac{2\Gamma(4)\left[(1-x)^{3-\alpha}+x^{3-\alpha}\right]}{\Gamma(4-\alpha)} + \frac{\Gamma(3)\left[(1-x)^{2-\alpha}+x^{2-\alpha}\right]}{\Gamma(3-\alpha)}\Bigg]. \end{aligned}$$

The exact solution of the Eqs. (5.1–5.2) is $u(x,t) = x^2(1-x)^2e^{-t}$.

The outcomes of Example 1 are presented as:

Figure 1, shows the numerical and exact solutions by (4.17) for Example 1 (5.1–5.2), at $T = 1$ and $\alpha = 0.2$. Tables 1 and 2 show the absolute errors and the convergence orders of the first method (4.9) and the second method (4.17) respectively for Example 1 (5.1–5.2). It can be observed from these figure and tables that, approximate solutions by the proposed scheme are in good agreement with the exact solutions (Fig. 2). Figure 3 shows the numerical solution by (4.17) for Example 1 (5.1–5.2), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$.

Example 2 Consider the fractional kinetics equation with a source term $g(x,t)$ as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + g(x,t); \tag{5.3}$$

subject to the initial condition:

$$\begin{aligned} u(x,0) &= x^2(1-x)^2, \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \quad 0 \le t \le T, \end{aligned} \tag{5.4}$$

Table 1 The absolute errors and the convergence orders of the first method (4.9) for Example 1 (5.1–5.2)

$h=\kappa$	$\alpha=0.1$		$\alpha=0.2$		$\alpha=0.3$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	9.3492e−004	–	6.9978e−004	–	4.5440e−004	–
1/20	4.9973e−004	0.90	4.0841e−004	0.78	2.7179e−004	1.03
1/40	2.5988e−004	0.94	2.2506e−004	0.86	1.7265e−004	0.65
1/80	1.3301e−004	0.97	1.1942e−004	0.91	9.9304e−005	0.80
$h=\kappa$	$\alpha=0.4$		$\alpha=0.5$		$\alpha=0.6$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	7.5666e−004	–	0.0010	–	0.0023	–
1/20	3.7222e−004	1.02	5.5011e−004	0.86	9.3774e−004	1.29
1/40	1.6362e−004	1.18	2.5898e−004	1.08	4.1214e−004	1.19
1/80	6.6876e−005	1.29	1.0805e−004	1.26	1.8446e−004	1.16
$h=\kappa$	$\alpha=0.7$		$\alpha=0.8$		$\alpha=0.9$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0045	–	0.0087	–	0.0182	–
1/20	0.0021	1.10	0.0046	0.92	0.0114	0.67
1/40	8.9851e−004	1.22	0.0022	1.06	0.0063	0.86
1/80	3.6145e−004	1.31	9.9691e−004	1.14	0.0032	0.98

Table 2 The absolute errors and the convergence orders of the second method (4.17) for Example 1 (5.1–5.2)

$h=\kappa$	$\alpha=0.1$		$\alpha=0.2$		$\alpha=0.3$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	1.3969e−004	–	3.3485e−004	–	6.4809e−004	–
1/20	4.3384e−005	1.69	1.1004e−004	1.61	2.2397e−004	1.53
1/40	1.4348e−005	1.60	3.7777e−005	1.54	7.5563e−005	1.57
1/80	4.5825e−006	1.65	1.2889e−005	1.55	2.7245e−005	1.47
$h=\kappa$	$\alpha=0.4$		$\alpha=0.5$		$\alpha=0.6$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0011	–	0.0020	–	0.0033	–
1/20	4.1963e−004	1.39	7.6230e−004	1.39	0.0014	1.24
1/40	1.4705e−004	1.51	2.8322e−004	1.43	5.4917e−004	1.35
1/80	5.2423e−005	1.49	1.0303e−004	1.46	2.1277e−004	1.37
$h=\kappa$	$\alpha=0.7$		$\alpha=0.8$		$\alpha=0.9$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0057	–	0.0102	–	0.0206	–
1/20	0.0026	1.13	0.0052	0.98	0.0122	0.76
1/40	0.0011	1.24	0.0024	1.12	0.0066	0.89
1/80	4.5943e−004	1.26	0.0011	1.26	0.0033	1.00

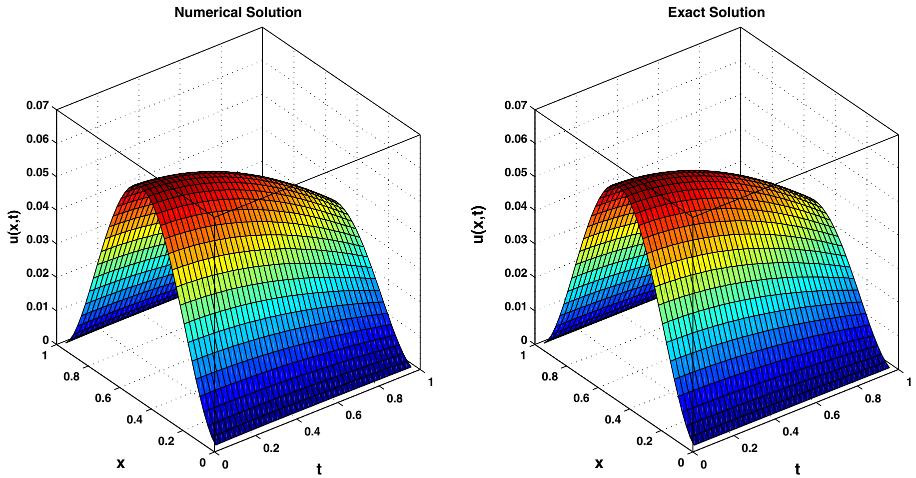

Fig. 2 The numerical and exact solutions by (4.17) for Example 2 (5.3–5.4), at $T = 1$ and $\alpha = 0.5$

where $0 < \alpha < 1$ and

$$g(x,t) = -x^2(1-x)^2\sin(t) + \frac{\cos(t)}{2\cos(\frac{\alpha\pi}{2})}\left[\frac{\Gamma(5)\left[(1-x)^{4-\alpha}+x^{4-\alpha}\right]}{\Gamma(5-\alpha)} - \frac{2\Gamma(4)\left[(1-x)^{3-\alpha}+x^{3-\alpha}\right]}{\Gamma(4-\alpha)} + \frac{\Gamma(3)\left[(1-x)^{2-\alpha}+x^{2-\alpha}\right]}{\Gamma(3-\alpha)}\right].$$

The exact solution of this problem is $u(x,t) = \cos(t)x^2(1-x)^2$.

The outcomes of Example 2 are presented as:

Figure 2, shows the numerical and exact solutions by (4.17) for Example 2 (5.3–5.4), at $T = 1$ and $\alpha = 0.5$. Tables 3 and 4 show the absolute errors and the convergence orders of the first method (4.9) and the second method (4.17) respectively for Example 2 (5.3–5.4). This result verifies that approximate solution by proposed schemes are in good agreement with the exact solution. Figure 4 shows the numerical solution by (4.9) for Example 2 (5.3–5.4), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$. Also, Fig. 5 shows the numerical solution by (4.17) for Example 2 (5.3–5.4), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$.

Example 3 The following fractional kinetics equation with a source term $f(x,t)$ is considered, as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t); \tag{5.5}$$

subject to the initial condition:

$$\begin{aligned} &u(x,0) = 0, \quad 0 \le x \le 1, \\ &u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1, \end{aligned} \tag{5.6}$$

where $0 < \alpha < 1$ and the exact solution of this problem is $u(x,t) = t\,[x^\alpha - x]$.

For numerical results of Example 3, we use the approximation of f in proposed methods. Since the Riemann-Liouville derivative of our exact solution (singular function) is not

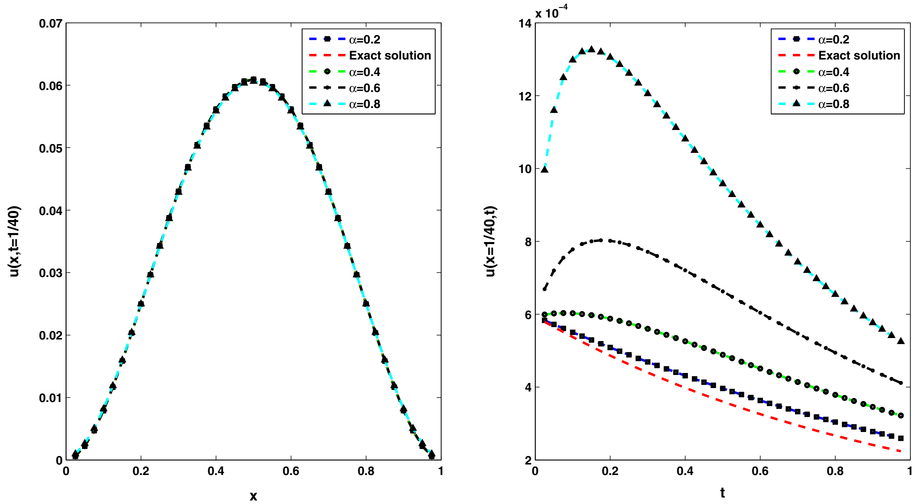

Fig. 3 The numerical solution by (4.17) for Example 1 (5.1–5.2), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$

possible. We use the exact solution($u(x,t) = t\,[x^\alpha - x]$) to approximate the value of source term $f(x,t)$ at (x_{n+1}, t_j). Therefore, we take

$$
\begin{aligned}
f(x_{n+1}, t_j) &\approx \frac{\partial u(x_{n+1}, t_j)}{\partial t} - \frac{\partial^\alpha u(x_{n+1}, t_j)}{\partial |x|^\alpha} \\
&= \left[x_{n+1}^\alpha - x_{n+1}\right] + c_\alpha \left[\Gamma(\alpha+1) - \frac{x_{n+1}^{1-\alpha}}{\Gamma(2-\alpha)}\right] t_j \\
&+ \frac{c_\alpha h^{-\alpha}}{\Gamma(2-\alpha)} \left[\sum_{i=n+1}^{M-1} \left[(i-n-1)^{1-\alpha} - (i-n)^{1-\alpha}\right] . \left[u_{i+1}^j - u_i^j\right]\right] \\
&+ c_\alpha g(x_{n+1}, t_j).
\end{aligned} \tag{5.7}
$$

It should be noted that here starting from the expression of the exact solution, then using (5.5), we construct the source term as given above. We use the proposed numerical method to solve the fractional kinetics equation (5.5) using the above source term function (5.7).

The outcomes of Example 3 are presented as:

Figure 6 shows the exact and numerical solutions by (4.9) for Example 3 (5.5–5.6), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $T = 1$ and $h = \kappa = 1/100$. Table 5 shows the absolute errors and the convergence orders of the first method (4.9) for Example 3 (5.5–5.6). Results in Fig. 6 and Table 5 show numerical solution by proposed scheme (4.9) are in good agreement with the exact solution.

Conclusion

In this paper, we design new numerical methods by using the piecewise linear interpolation polynomial for solving a Riesz space partial fractional differential equation. In the new numerical methods, it is shown that the methods are unconditionally stable and convergent with the accuracy of $O(\kappa + h)$ and $O(\kappa^2 + h)$ respectively. Several numerical examples

Table 3 The absolute errors and the convergence orders of the first method (4.9) for Example 2 (5.3–5.4)

$h = \kappa$	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0016	–	0.0018	–	0.0021	–
1/20	7.9280e−004	1.01	8.4665e−004	1.09	9.5936e−004	1.13
1/40	3.9121e−004	1.02	4.0274e−004	1.07	4.3690e−004	1.13
1/80	1.9345e−004	1.02	1.9381e−004	1.05	2.0249e−004	1.10
$h = \kappa$	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.6$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0027	–	0.0036	–	0.0052	–
1/20	0.0012	1.17	0.0016	1.17	0.0024	1.11
1/40	5.1445e−004	1.22	6.7572e−004	1.24	0.0010	1.26
1/80	2.2813e−004	1.17	2.8872e−004	1.23	4.2534e−004	1.23
$h = \kappa$	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0081	–	0.0135	–	0.0255	–
1/20	0.0039	1.05	0.0071	0.93	0.0160	0.67
1/40	0.0017	1.20	0.0034	1.06	0.0088	0.86
1/80	7.4345e−04	1.19	0.0016	1.09	0.0045	0.97

Table 4 The absolute errors and the convergence orders of the second method (4.17) for Example 2 (5.3–5.4)

$h = \kappa$	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	1.5178e−004	–	4.2321e−004	–	8.5478e−004	–
1/20	5.7577e−005	1.40	1.4488e−004	1.55	2.9928e−004	1.51
1/40	1.9883e−005	1.53	5.1751e−005	1.49	1.0248e−004	1.55
1/80	6.3476e−006	1.65	1.7689e−005	1.55	3.6802e−005	1.48
$h = \kappa$	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.6$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0015	–	0.0026	–	0.0044	–
1/20	5.6728e−004	1.40	0.0010	1.38	0.0019	1.21
1/40	1.9962e−004	1.51	3.8459e−004	1.38	7.4313e−004	1.35
1/80	6.9640e−005	1.52	1.4011e−004	1.46	2.8820e−004	1.37
$h = \kappa$	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0076	–	0.0134	–	0.0264	–
1/20	0.0035	1.12	0.0069	0.96	0.0161	0.71
1/40	0.0015	1.22	0.0033	1.06	0.0088	0.87
1/80	6.1904e−004	1.28	0.0015	1.14	0.0044	1.00

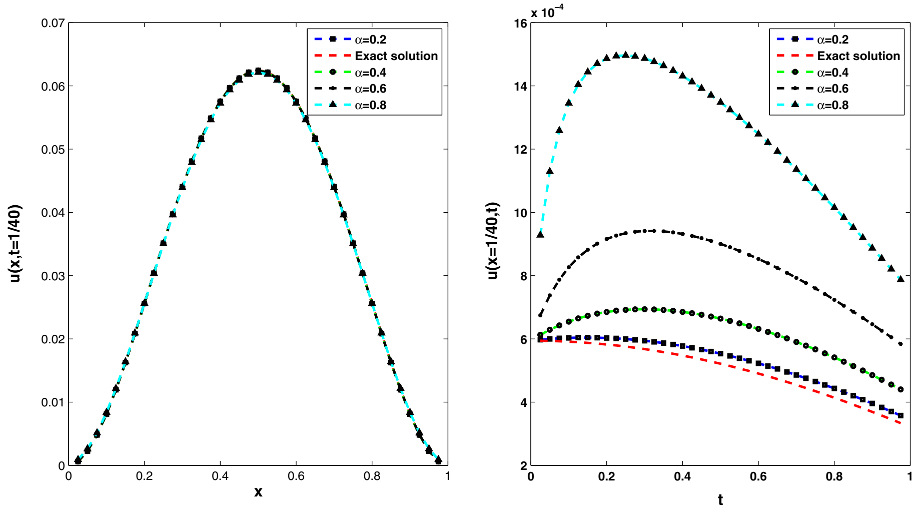

Fig. 4 The numerical solution by (4.9) for Example 2 (5.3–5.4), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$

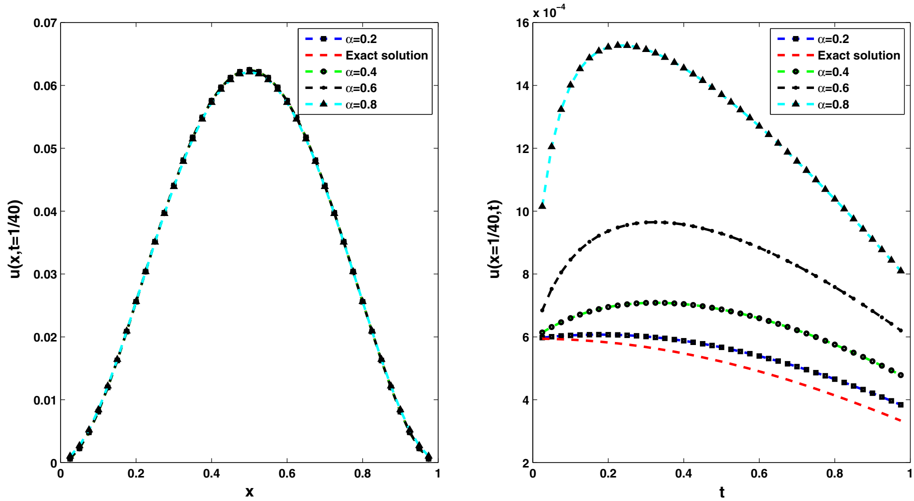

Fig. 5 The numerical solution by (4.17) for Example 2 (5.3–5.4), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when $h = \kappa = 1/40$

are implemented for the Riesz space partial fractional differential equation, during these examples, it is observed that the proposed methods are in good agreement with exact solutions. In our future work, we shall try to apply the presented method with higher-order numerical accuracy to other partial fractional differential equations.

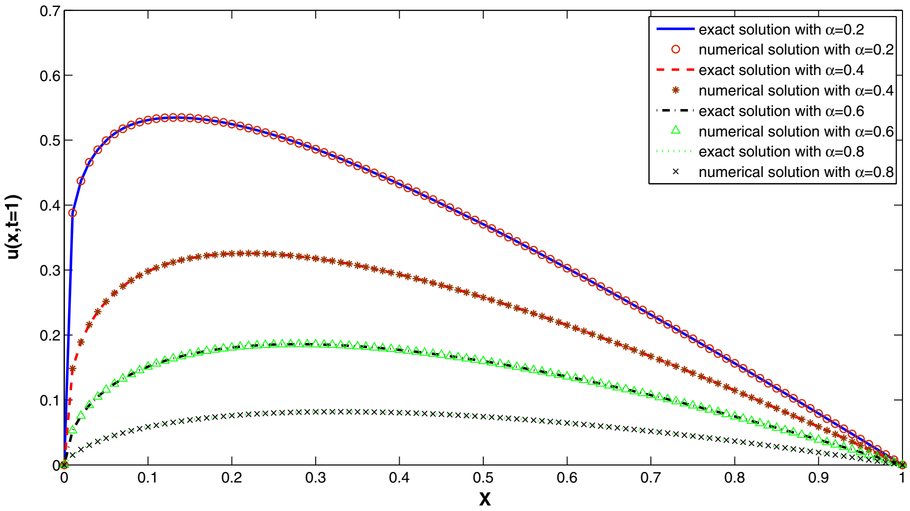

Fig. 6 The exact and numerical solutions by (4.9) for Example 3 (5.5–5.6), for various $\alpha = 0.2, 0.4, 0.6, 0.8$, when T = 1 and $h = \kappa = 1/100$

Table 5 The absolute errors and the convergence orders of the first method (4.9) for Example 3 (5.5–5.6)

$h=\kappa$	$\alpha=0.1$		$\alpha=0.2$		$\alpha=0.3$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0102	–	0.0128	–	0.0143	–
1/20	0.0071	0.52	0.0099	0.37	0.0112	0.35
1/40	0.0038	0.94	0.0058	0.76	0.0069	0.70
1/80	0.0020	0.97	0.0032	0.85	0.0039	0.80
$h=\kappa$	$\alpha=0.4$		$\alpha=0.5$		$\alpha=0.6$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0147	–	0.0141	–	0.0127	–
1/20	0.0114	0.37	0.0106	0.41	0.0091	0.48
1/40	0.0066	0.78	0.0061	0.79	0.0050	0.86
1/80	0.0036	0.86	0.0033	0.88	0.0026	0.94
$h=\kappa$	$\alpha=0.7$		$\alpha=0.8$		$\alpha=0.9$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
1/10	0.0107	–	0.0082	–	0.0051	–
1/20	0.0072	0.57	0.0051	0.68	0.0032	0.67
1/40	0.0043	0.74	0.0029	0.81	0.0019	0.75
1/80	0.0025	0.78	0.0015	0.88	0.0011	0.79

Author Contributions Lateef Saeed and Javidi wrote the main manuscript text and Saedshoar Heris plots all the figures.

Funding The authors have not disclosed any funding.

Data Availability Enquiries about data availability should be directed to the authors.

Declarations

Competing interests The authors have not disclosed any competing interests.

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