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HYERS-ULAM STABILITY OF WEIGHTED COMPOSITION OPERATORS ON L^p-SPACES

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ABSTRACT. For weighted composition operator $uC_\varphi : f \longmapsto u.(f \circ \varphi)$ on $L^p(\Sigma)$, we give a necessary and sufficient condition to have the Hyers-Ulam stability.

1. Preliminaries and notations

Let (X, Σ, μ) be a complete σ-finite measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. Let φ be a measurable transformation from X into X. If $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$, then φ is said to be non-singular. Let h be the Radon-Nikodym derivative $d\mu \circ \varphi^{-1}/d\mu$. We will always assume that h is almost everywhere finite-valued or equivalently, $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite. As usual, Σ is said to be φ-invariant if $\varphi(\Sigma) \subseteq \Sigma$, where $\varphi(\Sigma) = \{\varphi(A) : A \in \Sigma\}$. The measure μ is said to be normal if $\mu(A) = 0$ implies that $\varphi(A) \in \Sigma$ and $\mu(\varphi(A)) = 0$. To examine the weighted composition operators efficiently, Lambert [6] associated with each transformation φ, the so-called conditional expectation operator $E(.|\varphi^{-1}(\Sigma)) = E(.)$. In fact $E(f)$ is defined for each non-negative

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measurable function f or for each $f \in L^p(\Sigma)$, and is uniquely determined by the following two conditions:

(i) $E(f)$ is $\varphi^{-1}(\Sigma)$-measurable.

(ii) If A is any $\varphi^{-1}(\Sigma)$-measurable set for which $\int_A f d\mu$ converges, then we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

It is easy to show that if f is any non-negative Σ-measurable function or if $f \in L^p(\Sigma)$, then there exists a Σ-measurable function g such that $E(f) = g \circ \varphi$. We can assume that the support of g, $\sigma(g) = \{x \in X : g(x) \neq 0\}$, lies in $\sigma(h)$ and there exists only one g with this property. We then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ (see [2]). For further discussions on the conditional expectation operators see the interesting papers [1], [5] and [6]. If $u : X \to \mathbb{C}$ is a measurable function, the weighted composition operator uC_φ on $L^p(\Sigma)$ induced by φ and u is given by

$$uC_\varphi(f) = u.f \circ \varphi, \qquad f \in L^p(\Sigma).$$

Here, the non-singularity of φ guarantees that uC_φ is well defined as a mapping of equivalence classes of functions on $\sigma(u)$. Boundedness of weighted composition operators on $L^p(\Sigma)$ spaces has already been studied in [5]. Namely, uC_φ is bounded if and only if $hE(|u|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$.

Let $\mathcal{B}$ be a Banach space and let T be a mapping from $\mathcal{B}$ into itself. We say that T has the Hyers-Ulam stability, if there exists a constant K such that:

(*) For any $g \in T(\mathcal{B})$, $\varepsilon > 0$ and $f \in \mathcal{B}$ satisfying $\|Tf - g\| \leq \varepsilon$, we can find $f_0 \in \mathcal{B}$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

We call K a HUS constant for T, and denote the infimum of all HUS constants for T by K_T. A subspace M of $\mathcal{B}$ is said to be proximinal, if for any $f \in \mathcal{B}$, there exists $g \in M$ such that $\|f - g\| = \|f + M\|$. We refer the reader for the Hyers-Ulam stability of substitution operators on function spaces to [3], [4], [8], [9], [10] and [11].

From now on, by an operator we will mean a non-zero linear operator. The linearity of T implies that the condition (*) is equivalent to stating that for any $\varepsilon > 0$ and $f \in \mathcal{B}$ with $\|Tf\| \leq \varepsilon$ there exists $f_0 \in \mathcal{B}$ such that $Tf_0 = 0$ and $\|f - f_0\| \leq K\varepsilon$. For a bounded operator $T : \mathcal{B} \to \mathcal{B}$, we denote the null space of T by $\mathcal{N}(T)$ and the range of T by $\mathcal{R}(T)$. When T is not one-to-one, one may consider the operator $\widetilde{T}$ from $\mathcal{B}/\mathcal{N}(T)$ into $\mathcal{B}$ defined by $\widetilde{T}(f+\mathcal{N}(T)) = Tf$, for all $f \in \mathcal{B}$. Clearly $\widetilde{T}$ is a one-to-one operator and $\mathcal{R}(\widetilde{T}) = \mathcal{R}(T)$.

Takagi, Miura and Takahasi [11] investigated the relation of the Hyers-Ulam stability of T and the inverse operator $\widetilde{T}^{-1}$ from $\mathcal{R}(T)$ into $\mathcal{B}/\mathcal{N}(T)$ in the following sense.

Theorem A ([11], Theorem 2). *For a bounded linear operator T on a Banach space, the following statements are equivalent:*

(a) *T has the Hyers-Ulam stability.*
(b) *T has closed range.*
(c) *$\widetilde{T}^{-1}$ is bounded.*

Moreover, in this case $K_T = \|\widetilde{T}^{-1}\|$.

2. Main results

In this section for a weighted composition operator $uC_\varphi : L^p(\Sigma) \to L^p(\Sigma)$, we give a necessary and sufficient condition for uC_φ to have the Hyers-Ulam stability and then we show that K_{uC_φ} is a HUS constant for uC_φ.

Theorem 2.1. *Let $1 \leq p < \infty$ and let Σ be φ-invariant. If μ is normal and uC_φ is a bounded weighted composition operator on $L^p(\Sigma)$, then the following assertions are equivalent:*

(i) *uC_φ has the Hyers-Ulam stability.*
(ii) *uC_φ has closed range.*
(iii) *There exists $r > 0$ such that $J(x) := (h(x)E(|u|^p) \circ \varphi^{-1}(x))^{1/p} \geq r$ for μ-almost all $x \in \sigma(J)$.*
(iv) *There exists $r > 0$ such that $\varphi(\sigma(u)) \subseteq \{x \in X : J(x) \geq r\}$.*
(v) *There exists $K > 0$ such that $\|f + \mathcal{N}(uC_\varphi)\| \leq K\|uC_\varphi f\|$, for all $f \in L^p(\Sigma)$.*

For the proof of this theorem, we need the following lemma.

Lemma 2.2. *Let $1 \leq p < \infty$ and Σ be φ-invariant. If uC_φ is a bounded weighted composition operator on $L^p(\Sigma)$, then we have*

$$\|f + \mathcal{N}(uC_\varphi)\|^p = \int_{\varphi(\sigma(u))} |f|^p d\mu \ .$$

Proof. Put $S = \varphi(\sigma(u))$ and $S^c = X \setminus S$. Then we can write

$$L^p(X, \Sigma, \mu) = L^p(S, \Sigma_1, \mu) \oplus L^p(S^c, \Sigma_2, \mu),$$

where $\Sigma_1 = \Sigma \cap S$ and $\Sigma_2 = \Sigma \cap S^c$. Here

$$\mathcal{N}(uC_\varphi) = \{f \in L^p(\Sigma) : f = 0 \ \text{ on } S\} = L^p(\Sigma_2).$$

If uC_φ is one-to-one, then $\mu(S^c) = 0$ and hence there is nothing to prove. Choose $h \in \mathcal{N}(uC_\varphi)$ arbitrarily. For each $f \in L^p(\Sigma)$ we have

$$\int_S |f|^p d\mu = \int_S |f + h|^p d\mu \leq \int_X |f + h|^p d\mu = \|f + h\|^p.$$

Hence $\int_S |f|^p d\mu \leq \|f + \mathcal{N}(uC_\varphi)\|^p$. On the other hand, put $h = -\chi_{S^c} f$. Clearly, $h \in \mathcal{N}(uC_\varphi)$. Then we have

$$\|f + \mathcal{N}(uC_\varphi)\|^p \leq \|f + h\|^p = \|f(1 - \chi_{S^c})\|^p = \|f\chi_S\|^p = \int_S |f|^p d\mu,$$

for all $f \in L^p(\Sigma)$. Thus the lemma is proved. □

Proof of Theorem 2.1. The implications (i)⇒(ii) and (v)⇒(i) are direct consequences of Theorem A and definition of the Hyers-Ulam stability. We show (ii)⇒(iii)⇒(iv)⇒(v).

(ii)⇒(iii) Let $f \in L^p(\Sigma)$. Applying the properties of the conditional expectation and using the change of variable formula we have

$$\|uC_\varphi f\|^p = \int_X |u.f \circ \varphi|^p d\mu = \int_X E(|u|^p)|f|^p \circ \varphi d\mu$$
$$= \int_X E(|u|^p) \circ \varphi^{-1} |f|^p d\mu \circ \varphi^{-1} = \int_X (hE(|u|^p) \circ \varphi^{-1})|f|^p d\mu$$
$$= \int_X |Jf|^p d\mu = \|M_J f\|^p \ ,$$

where $J^p = hE(|u|^p) \circ \varphi^{-1}$. Hence we conclude that the pair (u, φ) induces a weighted composition operator $uC_\varphi : L^p(\Sigma) \to L^p(\Sigma)$ if and only if J induces a multiplication operator $M_J : L^p(\Sigma) \to L^p(\Sigma)$ and $\|uC_\varphi\| = \|M_J\| = \|J\|_\infty$. It is a well-known fact that the bounded multiplication operator M_θ on $L^p(\Sigma)$ has closed range if and only if θ is bounded away from zero on $\sigma(\theta)$ (for example see[7]). Therefore, uC_φ on $L^p(\Sigma)$ has closed range if and only if there exists $r > 0$ such that $J \geq r$ a.e. on $\sigma(J)$.

(iii)⇒(iv) Suppose $J \geq r$ on $\sigma(J)$ for some $r > 0$. It is enough to prove that $S \subseteq \sigma(J)$. If uC_φ is one-to-one, then $\sigma(J) = X$ and hence there is nothing to prove. If $S \nsubseteq \sigma(J)$, then we can choose $A \subseteq \sigma(u)$ with $0 < \mu(\varphi(A)) < \infty$ such that $\varphi(A) \cap \sigma(J) = \emptyset$. Then we have

$$0 = \int_X |\chi_{\varphi(A)} J|^p d\mu = \int_X \chi_{\varphi^{-1}(\varphi(A))} |u|^p d\mu.$$

Hence $\mu(A) = \mu(A \cap \sigma(u)) \leq \mu(\varphi^{-1}(\varphi(A)) \cap \sigma(u)) = 0$. Since μ is normal, we have $\mu(\varphi(A)) = 0$. But this is a contradiction.

(iv)⇒(v) Put $A = \{x : J(x) \geq r\}$. By Lemma 2.2, we have

$$\begin{aligned} \|f + \mathcal{N}(uC_\varphi)\|^p &= \int_S |f|^p d\mu \leq \int_A |f|^p d\mu \leq \frac{1}{r^p} \int_A |Jf|^p d\mu \\ &\leq \frac{1}{r^p} \int_X |M_J f|^p d\mu = \frac{1}{r^p} \|M_J f\|^p = \frac{1}{r^p} \|uC_\varphi f\|^p, \end{aligned}$$

for all $f \in L^p(\Sigma)$. Hence there is a constant $K = 1/r$, such that $\|f + \mathcal{N}(uC_\varphi)\| \leq K \|uC_\varphi f\|$. □

Theorem 2.3. *Under the same assumptions as in Theorem 2.1, if* $R = \sup\{r > 0 : \varphi(\sigma(u)) \subseteq \{J \geq r\}\}$, *then* $K_{uC_\varphi} = 1/R$.

Proof. Put $S = \varphi(\sigma(u))$. By theorem 2.1, if r is taken over all numbers satisfying $S \subseteq \{J \geq r\}$, we obtain $K_{uC_\varphi} = \|\widetilde{uC_\varphi}^{-1}\| \leq 1/R$. For the opposite inequality, assume that $\|\widetilde{uC_\varphi}^{-1}\| < 1/r$ and $S \nsubseteq \{J \geq r\}$ for some $r > 0$. Then we can choose $A \subseteq S$, with $0 < \mu(A) < \infty$ such that $J_{|A} < r$. Put $f_0 = \chi_A / \mu(A)^{1/p}$. Then we have $\|uC_\varphi f_0\| = \|M_J f_0\| = \|Jf_0\| < r$. Hence we obtain

$$\begin{aligned} 1 = \|f_0\| = \left(\int_S |f_0|^p d\mu\right)^{\frac{1}{p}} &= \|f_0 + \mathcal{N}(uC_\varphi)\| = \|\widetilde{uC_\varphi}^{-1}(uC_\varphi f_0)\| \\ &\leq \|\widetilde{uC_\varphi}^{-1}\| \; \|uC_\varphi f_0\| < r\frac{1}{r} = 1, \end{aligned}$$

which is a contradiction. Thus we conclude that if $\|\widetilde{uC_\varphi}^{-1}\| < 1/r$ then $S \subseteq \{J \geq r\}$. This implies $1/R \leq \|\widetilde{uC_\varphi}^{-1}\|$. □

Remark 2.4. (a) If we omit the φ-invarian of Σ and normality of μ in Theorem 2.1, then the implications (i)$\Longleftrightarrow$(ii) $\Longleftrightarrow$(iii) are still valid.

(b) Combining Theorem 3 with Proposition 3 in [8], we see the following fact: Suppose that the measure space (X, Σ, μ) is nonatomic and let uC_φ be a weighted composition operator on $L^p(\Sigma)$ $(1 \leq p < \infty)$. Assume that $u(x) \neq 0$ for μ-almost all $x \in X$ and that a composition operator C_φ is invertible. Then uC_φ has the Hyers-Ulam stability if and only if uC_φ is a Fredholm operator.

(c) Since for $1 < p < \infty$, $L^p(\Sigma)$ is reflexive and $\mathcal{N}(uC_\varphi)$ is a closed subspace of $L^p(\Sigma)$, it follows that $\mathcal{N}(uC_\varphi)$ is proximinal. Hence K_{uC_φ} is also a HUS constant for uC_φ. Now, consider the case $p = 1$. Pick $f \in L^1(\Sigma)$. Since $L^1(\Sigma) = L^1(\Sigma_1) \oplus L^1(\Sigma_2)$, so $g = f\chi_S \in L^1(\Sigma_1)$. Hence we have

$$\|f + \mathcal{N}(uC_\varphi)\| = \int_S |f| d\mu = \int_X |g| d\mu = \|g\| = \|f - (f - g)\|.$$

Moreover, $f - g = 0$ on S, which implies $f - g \in \mathcal{N}(uC_\varphi)$. Thus $\mathcal{N}(uC_\varphi)$ is proximinal. Therefore by Corollary 1 of [3], K_{uC_φ} is a HUS constant for uC_φ on $L^1(\Sigma)$. For more details see [3]. Also, we note that every bounded composition operator on $L^\infty(\Sigma)$ has the Hyers-Ulam stability.

Example 2.5. Let $X = [0, 1]$ with the Lebesgue measure μ, and let $\varphi : [0, 1] \to [0, 1]$ be defined by

$$\varphi(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

If we consider $uC_\varphi : L^2(\Sigma) \to L^2(\Sigma)$ as $uC_\varphi f(x) = xf(\varphi(x))$, then a simple computation gives

$$J(x) = \begin{cases} (2x^2 - x + \frac{1}{4})^{1/2} & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Since $J(x) \geq 1/\sqrt{8}$ on $\sigma(J) = [0, 1/2)$, uC_φ has the Hyers-Ulam stability and $K_{uC_\varphi} = \sqrt{8}$.

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