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A NOTE ON WEIGHTED COMPOSITION OPERATORS ON L^p-SPACES

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Abstract. In this paper we will consider the weighted composition operators uC_φ between two different $L^p(X, \Sigma, \mu)$ spaces, generated by measurable and non-singular transformations φ from X into itself and measurable functions u on X. We characterize the functions u and transformations φ that induce weighted composition operators between L^p-spaces by using some properties of conditional expectation operator, pair (u, φ) and the measure space (X, Σ, μ). Also, some other properties of these types of operators will be investigated.

1. Preliminaries And Notation

Let (X, Σ, μ) be a sigma finite measure space. By $L(X)$, we denote the linear space of all Σ-measurable functions on X. When we consider any subsigma algebra $\mathcal{A}$ of Σ, we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|.\|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. Here functions which are equal μ-almost everywhere are

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identical. An atom of the measure μ is an element $B \in \Sigma$ with $\mu(B) > 0$ such that for each $F \in \Sigma$, if $F \subset B$ then either $\mu(F) = 0$ or $\mu(F) = \mu(B)$. A measure with no atoms is called non-atomic. We can easily check the following well known facts (see [9]):

(a) Every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is a non-atomic over B and Z is a countable union of atoms of finite measure.

(b) For each $f \in L^r(\Sigma)$, there exist two functions $f_1 \in L^p(\Sigma)$ and $f_2 \in L^q(\Sigma)$ such that $f = f_1 f_2$ and $\|f\|_r^r = \|f_1\|_p^p = \|f_2\|_q^q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Associated with each sigma algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$, which is called *conditional expectation* operator, on the set of all non-negative measurable functions f or for each $f \in L^p$ for any p, $1 \leq p \leq \infty$, and is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is $\mathcal{A}$- measurable, and

(ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$.

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^{\mathcal{A}}(f.g \circ T) = E^{\mathcal{A}}(f)(g \circ T)$.
E2. $E^{\mathcal{A}}(1) = 1$.
E3. $|E^{\mathcal{A}}(fg)|^2 \leq E^{\mathcal{A}}(|f|^2) E^{\mathcal{A}}(|g|^2)$.
E4. If $f > 0$ then $E^{\mathcal{A}}(f) > 0$.

Properties E1. and E2. imply that $E^{\mathcal{A}}(\cdot)$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma))$
$= L^p(\mathcal{A})$. Suppose that φ is a mapping from X into X which is measurable, (i.e., $\varphi^{-1}(\Sigma) \subseteq \Sigma$) such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual). Let h be the Radon-Nikodym derivative $h = \frac{d\mu \circ \varphi^{-1}}{d\mu}$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ $(p \geq 1)$, there exists a Σ-measurable function g such that $E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$, though we

make no assumptions regarding the invertibility of φ (see [2]). For a deeper study of the properties of E see the paper [6].

2. Some Results On Weighted Composition Operators Between Two L^p-Spaces

Let $1 \leq q \leq p < \infty$ and we define $\mathcal{K}_{p,q}$ or $\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ as follows:

$$\mathcal{K}_{p,q} = \{u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma)\}.$$

$\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ is a vector subspace of $L(X)$. Also note that if $1 \leq q = p < \infty$, then $L^\infty(\Sigma) \subseteq \mathcal{K}_{p,p}(\mathcal{A}, \Sigma)$ and $\mathcal{K}_{p,p}(\Sigma, \Sigma) = L^\infty(\Sigma)$ (see [3]; problem 64, 65).

For $u \in L(X)$, let M_u from $L^p(\mathcal{A})$ into $L(X)$ defined by $M_u f = u.f$ be the corresponding linear transformation. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each L^p convergent sequence assures us that for each $u \in \mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$, the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is a multiplication operator (bounded linear transformation).

We shall find the relationship between a sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and the set of multiplication operators which map $L^p(\mathcal{A})$ into $L^q(\Sigma)$. Our first task is the description of the members of $\mathcal{K}_{p,q}$ in terms of the conditional expectation induced by $\mathcal{A}$.

Theorem 1.1. *Suppose* $1 \leq q < p < \infty$ *and* $u \in L(X)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, *where* $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

Proof. To prove the theorem, we adopt the method used by Axler [1]. Suppose $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, so $E^{\mathcal{A}}(|u|^q) \in L^{\frac{r}{q}}(\mathcal{A})$. For each $f \in L^p(\mathcal{A})$, we have $|f|^q \in L^{\frac{p}{q}}(\mathcal{A})$. Since $\frac{q}{p} + \frac{q}{r} = 1$, Hölder's inequality yields

$$\|u.f\|_q = \left\{\int |u|^q |f|^q d\mu\right\}^{\frac{1}{q}} = \left\{\int E^{\mathcal{A}}(|u|^q)|f|^q d\mu\right\}^{\frac{1}{q}}$$

$$\leq \left\{ \left(\int \left(E^{\mathcal{A}}(|u|^q) \right)^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} \left(\int (|f|^q)^{\frac{p}{q}} d\mu \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} = \|(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}}\|_r \|f\|_p.$$

Hence $u \in \mathcal{K}_{p,q}$. Now suppose only that $u \in \mathcal{K}_{p,q}$. So the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ given by $M_u f = u.f$ is a bounded linear operator. Let φ be a nonnegative integrable simple function then

$$\int E^{\mathcal{A}}(|u|^q)\, \varphi d\mu \leq \|M_u\|^q \left(\int \varphi^{\frac{p}{q}} d\mu \right)^{\frac{q}{p}} = \|M_u\|^q \, \|\varphi\|_{\frac{p}{q}}$$

It follows that $E^{\mathcal{A}}(|u|^q) \in L^{\left(\frac{p}{q}\right)'}(X, \mathcal{A}, \mu_{|\mathcal{A}}) \simeq L^{\frac{r}{q}}(X, \mathcal{A}, \mu_{|\mathcal{A}})$. □

Corollary 2.2. *Suppose* $1 \leq q < p < \infty$ *and* $u \in L(X)$. *Then* M_u *from* $L^p(\Sigma)$ *into* $L^q(\Sigma)$ *is bounded linear operator if and only if* $u \in L^{\frac{pq}{p-q}}(\Sigma)$. *In this case* $\|M_u\| = \|u\|_{\frac{pq}{p-q}}$.

Proof. Put $\mathcal{A} = \Sigma$ in the previous theorem. Then we will have $E^{\mathcal{A}} = I$ (identity operator). Then the proof holds.

Let $u \in L(X)$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. If $p = q$ then r must be ∞. So $M_u(L^p(\Sigma)) \subseteq L^p(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$. In this case $\|M_u\| = \|u\|_\infty$. This fact is well-known. For the direct proof, see [3].

Take a function u in $L(X)$ and let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair (u, φ) induces a linear operator uC_φ from $L^p(\Sigma)$ into $L(X)$ defined by

$$uC_\varphi(f) = u.f \circ \varphi \qquad (f \in L^p(\Sigma)).$$

Here, the non-singularity of φ guarantees that uC_φ is well defined as a mapping of equivalence classes of functions on support u. If uC_φ takes $L^p(\Sigma)$ into $L^q(\Sigma)$, then we call uC_φ a weighted composition operator $L^p(\Sigma)$ into $L^q(\Sigma)$ $(1 \leq q \leq \infty)$.

Boundedness of composition operators in $L^p(\Sigma)$ spaces $(1 \leq p \leq \infty)$ where measure spaces are sigma finite appeared already in Singh paper [7] and for two different $L^p(\Sigma)$ spaces in the paper [8]. Also boundedness of weighted composition operators on

$L^p(\Sigma)$ spaces has already been studied in [4]. Namely, for a non-singular measurable transformation φ and complex valued measurable weight function u on X, uC_φ is bounded if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^p)\circ\varphi^{-1} \in L^\infty(\Sigma)$. In the following theorem we give a necessary and sufficient condition for boundedness of weighted composition operators from $L^p(\Sigma)$ into $L^q(\Sigma)$, where $p > q$ as follows:

Theorem 2.3. *Suppose $1 \leq q < p < \infty$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Let $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair (u, φ) induces a weighted composition operator uC_φ from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $J = hE^{\varphi^{-1}(\Sigma)}(|u|^q)\circ\varphi^{-1} \in L^{\frac{r}{q}}(\Sigma)$.*

Proof. Let $f \in L^p(\Sigma)$. We will have

$$\|uC_\varphi f\|_q^q = \int |u.f\circ\varphi|^q d\mu = \int hE^{\varphi^{-1}(\Sigma)}(|u|^q)\circ\varphi^{-1}|f|^q d\mu$$

$$= \int |\sqrt[q]{J}f|^q d\mu = \|M_{\sqrt[q]{J}}f\|_q^q.$$

So by Corollary 2.2, uC_φ is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $\sqrt[q]{J} \in L^r(\Sigma)$ or equivalently $J \in L^{\frac{r}{q}}(\Sigma)$. □

Corollary 2.4. *Suppose $1 \leq p \leq \infty$, $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair (u, φ) induces a weighted composition operator uC_φ from $L^p(\Sigma)$ into $L^p(\Sigma)$ if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^p)\circ\varphi^{-1} \in L^\infty(\Sigma)$.*

Corollary 2.5. *Under the same assumptions as in theorem 2.3, φ induces a composition operator $C_\varphi : L^p(\Sigma) \to L^q(\Sigma)$ if and only if $h \in L^{\frac{r}{q}}(\Sigma)$.*

Remark 2.6. One of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let X be $[0, 1]$, Σ the Borel sets, and μ Lebesgue measure.

Let φ be the map $\varphi(x) = x^3$ on $[0,1]$. A simple computation shows that $h = 1/3x^{-2/3} \notin L^3(\Sigma)$. Then C_φ dos not define a bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$. However with $u(x) = x$, we have $\varphi^{-1}(\Sigma) = \Sigma$ (so $E = I$) and $J = 1/3 \in L^3(\Sigma)$. Hence $uC_\varphi = M_u \circ C_\varphi$ is bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$.

The procedure which Axler has used for the case $p < q$ in [1], when X is the interval $[-\pi, \pi]$, can also be used here.

At this stage we investigate a necessary and sufficient condition for a multiplication operator to be fredholm. For a bounded linear operator A on a Banach space, we use the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote the kernel and the range of A, respectively. We recall that A is said to be a Fredholm operator if $\mathcal{R}(A)$ is closed and if $\dim\mathcal{N}(A) < \infty$ and $\text{codim}\mathcal{R}(A) < \infty$. Now we attempt to prove a theorem which is likely to be found elsewhere.

Theorem 2.7. *Suppose that μ is a non-atomic measure on $L^2(\Sigma)$. Then the following conditions are equivalent:*
(a) M_u is an invertible operator.
(b) M_u is a Fredholm operator.
(c) $\mathcal{R}(M_u)$ is closed and codim$\mathcal{R}(M_u) < \infty$.
(d) $|u| \geq \delta$ almost everywhere on X for some $\delta > 0$.

Proof. The implications (d) $\Longrightarrow$ (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are obvious. We show (c) $\Longrightarrow$ (d).

Suppose that $\mathcal{R}(M_u)$ is closed and codim $\mathcal{R}(M_u) < \infty$. Then there exists a $\delta > 0$ such that $|u| \geq \delta$ on $\sigma(u)$. So it is enough to show that $\mu(\sigma(u)^c) = 0$. First of all we prove that M_u is onto. Let $0 \neq f_0 \in \mathcal{R}(M_u)^\perp$, therefore, for any $f \in L^2(\Sigma)$ we have $(M_u f, f_0) = 0$. Now we choose $t > 0$ such that the set

$$Z_t = \{s \in X : |f_0|^2(x) \geq t\}$$

is of positive measure. Since μ is a non-atomic measure we may choose a sequence of disjoint subsets Z_n of Z_t such that $0 < \mu(Z_n) < \infty$. Now let $g_n = \chi_{Z_n}.f_0$. It is clear that each g_n is non-zero element of $L^2(\Sigma)$, and for $n \neq m$, $(g_n, g_m) = 0$. Therefore, for $f \in L^2(\Sigma)$

we have

$$(f, M_u^* g_n) = (M_u f, \chi_{Z_n} f_0) = (M_u \chi_{Z_n f}, f_0) = 0.$$

So $g_n \in \mathcal{N}(M_u^*)$ for any n. Therefore, $\{g_n\}$ is a linearly independent subset of $\mathcal{N}(M_u^*)$, which is a contradiction to $\dim \mathcal{N}(M_u^*) = \text{codim } \mathcal{R}(M_u) < \infty$. If $\mu(\sigma(u)^c) > 0$, then there exists a set $Z \subset \sigma(u)^c$ such that $0 < \mu(Z) < \infty$, so we conclude that $\chi_Z \in L^2(\Sigma) \backslash \mathcal{R}(M_u)$, which contradicts the fact that M_u is onto. Therefore $\mu(\sigma(u)^c) = 0$. □

Corollary 2.8. *M_u is a Fredholm operator if and only if M_u^n $(= M_{u^n})$ is also Fredholm.*

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