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WEIGHTED COMPOSITION OPERATORS BETWEEN L^p-SPACES

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Abstract. In this paper we will consider the weighted composition operator $W = uC_\varphi$ between two different $L^p(X, \Sigma, \mu)$ spaces, generated by measurable and non-singular transformations φ from X into itself and measurable functions u on X. We characterize the functions u and transformations φ that induce weighted composition operators between L^p-spaces by using some properties of conditional expectation operator, pair (u,φ) and the measure space (X, Σ, μ). Also, Fredholmness of these type operators will be investigated.

1. Preliminaries and notations

Takagi in [9] has characterized the boundedness of multiplication and composition operators on $L^p(\Sigma)$ spaces in $1 \le p < q$ and $1 \le q < p$ cases. In [4], boundedness of weighted composition operators has been investigated in $1 \le q \le p < \infty$ case. In the next section we will give the necessary and sufficient condition for boundedness of weighted composition operators in $1 \le p < q < \infty$ case. In section 3 we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\varphi$ to be Fredholm. Fredholm weighted composition operators have been studied by H. Takagi[8] in the $L^p(\Sigma)$ setting. By using some properties of conditional expectation operator we omit the continuity hypothesis of M_u. In other words, we do not require that $u \in L^\infty(\Sigma)$. This is stated as a hypothesis in [8].

Let (X, Σ, μ) be a σ-finite measure space. By $L(X)$, we denote the linear space of all Σ-measurable functions on X. When we consider any sub-σ-algebra $\mathcal{A}$ of Σ, we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any σ-finite algebra $\mathcal{A} \subseteq \Sigma$ and

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$1 \le p \le \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|\cdot\|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. It is easy to see that every $\mathcal{A}$-measurable function $f \in L(X)$ is constant μ-almost everywhere on an atom A. So for each $f \in L(X)$ and each atom A we have $\int_A f d\mu = f(A)\mu(A)$. A measure with no atoms is called non-atomic. We can easily check the following well known facts (see [11]):

(a) Every σ-finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is a non-atomic over B and Z is a countable union of atoms of finite measure. So we can write $X = B \cup \left(\bigcup_{n \in N} A_n\right)$, where $\{A_n\}_{n \in N}$ is a countable collection of disjoint atoms and B is a non-atomic set.

(b) Suppose $1 \le p < q < \infty$. If an $\mathcal{A}$-measurable set K is non-atomic and that $\mu(K) > 0$, there exists a function $f_0 \in L^p(\mathcal{A})$ such that $\int_K |f_0|^q d\mu < \infty$.

Associated with each σ-algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$ on the set of all non-negative measurable functions f or on the set of all functions $f \in L^p(\Sigma)$, $1 \le p \le \infty$, that is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is $\mathcal{A}$-measurable, and

(ii) if A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$.

The operator $E^{\mathcal{A}}$ is called the conditional expectation operator. This operator is at the central idea of our work, and we list here some of its useful properties:

E1. If g is $\mathcal{A}$-measurable then $E^{\mathcal{A}}(fg) = E^{\mathcal{A}}(f)g$.

E2. $E^{\mathcal{A}}(1) = 1$.

E3. $|E^{\mathcal{A}}(fg)|^2 \le E^{\mathcal{A}}(|f|^2)E^{\mathcal{A}}(|g|^2)$.

E4. If $f > 0$, then $E^{\mathcal{A}}(f) > 0$.

The properties E1 and E2 imply that $E^{\mathcal{A}}(\cdot)$ is an idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. So when $\mathcal{A} = \Sigma$, we have $E^{\Sigma} = I$ where I is the identity operator. Suppose that φ is a mapping from X into X which is measurable, (i.e., $\varphi^{-1}(\Sigma) \subseteq \Sigma$) and $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ ($\mu \circ \varphi^{-1} \ll \mu$). Let h be the Radon-Nikodym derivative,

$h = d\mu \circ \varphi^{-1}/d\mu$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ $(p \geq 1)$, there exists a Σ-measurable function g such that $E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$, though we make no assumption regarding the invertibility of φ (see [2]). For a deeper study of the properties of E see the paper [6].

Take a function u in $L(X)$ and let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair (u, φ) induces a linear operator uC_φ from $L^p(\Sigma)$ into $L(X)$ defined by

$$uC_\varphi(f) = u.f \circ \varphi, \quad (f \in L^p(\Sigma)).$$

Here, the non-singularity of φ guarantees that uC_φ as a mapping of equivalence classes of functions on support u is well defined. If uC_φ takes $L^p(\Sigma)$ into $L^q(\Sigma)$ or uC_φ is equivalently bounded, then we say that uC_φ is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ $(1 \leq p, q \leq \infty)$. When $u \equiv 1$, we just have the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$. For more details see [7].

2. Boundedness of weighted composition operators in $1 \leq p < q < \infty$ case

Let $1 \leq p < q < \infty$. In this section we characterize the functions u and transformations φ that induce weighted composition operators $uC_\varphi : L^p(\Sigma) \to L^q(\Sigma)$ by using some properties of conditional expectation operator, pair (u, φ) and the measure space (X, Σ, μ).

Case: $1 \leq q \leq p < \infty$

Let $1 \leq q \leq p < \infty$. In [4] we examined the set

$$\mathcal{K}_{p,q} = \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) = \{u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma)\}.$$

$\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ is a vector subspace of $L(X)$. Also note that if $1 \leq q = p < \infty$, then $L^\infty(\Sigma) \subseteq \mathcal{K}_{p,p}(\mathcal{A}, \Sigma)$ and $\mathcal{K}_{p,p}(\Sigma, \Sigma) = L^\infty(\Sigma)$.

For $u \in L(X)$, let M_u from $L^p(\mathcal{A})$ into $L(X)$ defined by $M_u f = u.f$ be the corresponding linear transformation. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each L^p convergent sequence assures us that for each $u \in \mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$, the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is a bounded multiplication operator. Boundedness of weighted composition operators on $L^p(\Sigma)$

spaces has already been studied in [3]. Namely, for a non-singular measurable transformation φ and complex valued measurable weight function u on X, uC_φ is bounded if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^p)\circ\varphi^{-1} \in L^\infty(\Sigma)$. The following two results are established in [4].

THEOREM 2.1. *Suppose* $1 \leq q < p < \infty$ *and* $u \in L(X)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, *where* $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

THEOREM 2.2. *Suppose* $1 \leq q < p < \infty$ *and* $\frac{1}{p}+\frac{1}{r} = \frac{1}{q}$. *Let* $u \in L(X)$ *and* $\varphi : X \to X$ *be a non-singular measurable transformation. Then the pair* (u, φ) *induces a weighted composition operator* uC_φ *from* $L^p(\Sigma)$ *into* $L^q(\Sigma)$ *if and only if* $J = hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1} \in L^{\frac{r}{q}}(\Sigma)$.

Case: $1 \leq p < q < \infty$

In this case we shall find the relationship between a σ-finite algebra $\mathcal{A} \subseteq \Sigma$ and the set of multiplication operators which map $L^p(\mathcal{A})$ into $L^q(\Sigma)$. Our first task is the description of the members of $\mathcal{K}_{p,q}$ in terms of the conditional expectation induced by $\mathcal{A}$.

THEOREM 2.3. *Suppose* $1 \leq p < q < \infty$ *and* $u \in L(X)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* u *satisfies the following two conditions:*

(i) $E^{\mathcal{A}}(|u|^q) = 0$ *on* B.

(ii) $\sup_{n\in N} \left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}/\mu(A_n) < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Proof. To prove the theorem, we adopt the methods used by Axler [1] and Takagi [9]. Suppose that both (i) and (ii) hold. Put $b = \sup_{n\in N} \left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}/\mu(A_n)$. Then, for each $f \in L^p(\mathcal{A})$ with $\|f\|_p \leq 1$ we have

$$\begin{aligned}
\|u.f\|_q^q &= \int_X E^{\mathcal{A}}(|u|^q)|f|^q d\mu \\
&= \sum_{n\in N} \int_{A_n} E^{\mathcal{A}}(|u|^q)|f|^q d\mu \\
&= \sum_{n\in N} \left(\frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}}{\mu(A_n)} \right)^{\frac{q}{s}} \left(|f(A_n)|^p \mu(A_n)\right)^{\frac{q}{p}} \\
&\leq b^{\frac{q}{s}} \sum_{n\in N} \left(|f(A_n)|^p \mu(A_n)\right) \\
&= b^{\frac{q}{s}} \sum_{n\in N} \int_{A_n} |f|^p d\mu
\end{aligned}$$

$$\begin{aligned} &\le b^{\frac{q}{s}} \int_X |f|^p d\mu \\ &\le b^{\frac{q}{s}} \|f\|_p^p \\ &\le b^{\frac{q}{s}}. \end{aligned}$$

Hence $u \in \mathcal{K}_{p,q}$. Conversely, suppose that $u \in \mathcal{K}_{p,q}$. So the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ given by $M_u f = u.f$ is a bounded linear operator on $X = B \cup \left(\bigcup_{n\in N} A_n\right)$. Assume that $\mu(\{x \in B : \ E^{\mathcal{A}}(|u(x)|^q) \neq 0\}) > 0$. Then there exists a positive number δ such that $\mu(\{x \in B : E^{\mathcal{A}}(|u(x)|^q) \geq \delta\}) > 0$. Put $K = \{x \in B : \ E^{\mathcal{A}}(|u(x)|^q) \geq \delta\}$. Since K is non-atomic, by (b) we can find $f_0 \in L^p(\mathcal{A})$ such that $\int_K |f_0|^q d\mu = \infty$. Then we have

$$\infty > \|M_u f_0\|_q^q \geq \int_K E^{\mathcal{A}}(|u|^q)|f_0|^q d\mu \geq \delta \int_K |f_0|^q d\mu = \infty,$$

which is a contradiction. In other words, $E^{\mathcal{A}}(|u|^q) = 0$ on B. Now we prove that (ii) also holds. For any $n \in N$ put $f_n = (1/\mu(A_n)^{\frac{1}{p}})\chi_{A_n}$. It is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Hence we have

$$\begin{aligned} \frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{1}{q}}}{\mu(A_n)^{\frac{1}{s}}} &= \left\{\frac{1}{\mu(A_n)^{\frac{q}{p}}} E^{\mathcal{A}}(|u(A_n)|^q)\mu(A_n)\right\}^{\frac{1}{q}} \\ &= \left\{\frac{1}{\mu(A_n)^{\frac{q}{p}}} \int_{A_n} E^{\mathcal{A}}(|u|^q) d\mu\right\}^{\frac{1}{q}} \\ &= \left\{\int_X E^{\mathcal{A}}(|uf_n|^q) d\mu\right\}^{\frac{1}{q}} \\ &= \|M_u f_n\|_q \leq \|M_u\|. \end{aligned}$$

Since this holds for any $n \in N$, it follows that $b \leq \|M_u\|^s < \infty$. □

The next corollary follows immediately from Theorem 2.3 and the known fact that when $\mathcal{A} = \Sigma$ then $E^{\mathcal{A}} = I$ (identity operator).

Corollary 2.4. *Suppose $1 \leq p < q < \infty$ and $u \in L(X)$. Then the operator M_u from $L^p(\Sigma)$ into $L^q(\Sigma)$ is a bounded linear operator if and only if u satisfies the following two conditions:*

(i) *$u = 0$ on B,*
(ii) *$\sup_{n\in N} |u(A_n)|^s/\mu(A_n) < \infty$, where $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.*

In the following theorem we give a necessary and sufficient condition for boundedness of weighted composition operators from $L^p(\Sigma)$ into $L^q(\Sigma)$, where $1 \leq p < q < \infty$.

THEOREM 2.5. *Suppose* $1 \leq p < q < \infty$, $u \in L(X)$ *and* $\varphi : X \to X$ *is a non-singular measurable transformation. Then the pair* (u, φ) *induces a weighted composition operator* uC_φ *from* $L^p(\Sigma)$ *into* $L^q(\Sigma)$ *if and only if the following conditions hold:*

(i) $J = 0$ *on* B,
(ii) $\sup_{n\in N} |J(A_n)|^{\frac{s}{q}}/\mu(A_n) < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Proof. Let $f \in L^p(\Sigma)$. We will have

$$\|uC_\varphi f\|_q^q = \int hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1}|f|^q d\mu = \int |\sqrt[q]{J}f|^q d\mu = \|M_{\sqrt[q]{J}}f\|_q^q.$$

So by Corollary 2.4 the theorem holds. □

COROLLARY 2.6. *Under the same assumptions as in Theorem 2.5,* φ *induces a composition operator* $C_\varphi : L^p(\Sigma) \to L^q(\Sigma)$ *if and only if the following conditions hold:*

(i) $h = 0$ *on* B,
(ii) $\sup_{n\in N} |h(A_n)|^{\frac{s}{q}}/\mu(A_n) < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

3. Fredholm weighted composition operators on L^p-spaces

Let $1 \leq p < \infty$, $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then it is well-known fact that each $g^* \in L^q(\Sigma)$ defines a bounded linear functional F_{g^*} on $L^p(\Sigma)$ by

$$F_{g^*}(f) = \int fg^* d\mu \quad (f \in L^p(\Sigma)).$$

Moreover, the mapping $g^* \to F_{g^*}$ is an isometry from $L^q(\Sigma)$ onto $(L^p)^*(\Sigma)$, so the norm dual of $L^p(\Sigma)$ can be identified with $L^q(\Sigma)$. In the following theorem we compute the adjoint of uC_φ.

PROPOSITION 3.1. *Let* $W = uC_\varphi$ *be a weighted composition operator on* $L^p(\Sigma)$ *and* $\frac{1}{p}+\frac{1}{q} = 1$. *Then* $W^*g^* = hE(u.g^*)\circ\varphi^{-1}$ *for all* $g^* \in L^q(\Sigma)$.

Proof. Take $A \in \Sigma$ such that $0 < \mu(A) < \infty$. For $g^* \in L^q(\Sigma)$ consider a bounded linear functional F_{g^*} on $L^p(\Sigma)$ as above. Then we have

$$(W^* F_{g^*})(\chi_A) = F_{g^*}(W\chi_A) = \int (W\chi_A) g^* \, d\mu$$

$$= \int u.\chi_A \circ \varphi \; g^* \, d\mu = \int hE(u.g^*) \circ \varphi^{-1} \chi_A \, d\mu = F_{hE(u.g^*)\circ\varphi^{-1}} \chi_A.$$

Hence, $W^* F_{g^*} = F_{hE(u.g^*)\circ\varphi^{-1}}$. After identifying $(L^p)^*(\Sigma)$ with $L^q(\Sigma)$ and g^* with F_{g^*}, we can write $W^* g^* = hE(u.g^*) \circ \varphi^{-1}$ for all $g^* \in L^q(\Sigma)$. □

In the following theorem we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\varphi$ to be Fredholm. The proof of the theorem follows a similar method of proof as was used to prove Theorem 4.2 in [5] which is similar to a theorem of Takagi[8]. We use the symbols $\mathcal{N}(W)$ and $\mathcal{R}(W)$ to denote the kernel and the range of W, respectively. We recall that W is said to be a Fredholm operator if $\mathcal{R}(W)$ is closed and if $\dim \mathcal{N}(W) < \infty$ and $\operatorname{codim} \mathcal{R}(W) < \infty$.

THEOREM 3.2. *Suppose that μ is a non-atomic measure. Let $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$. Then W is a Fredholm operator if and only if $J = hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} \geq \delta$ almost every where on X for some $\delta > 0$.*

Proof. Suppose that W is a Fredholm operator. We first claim that W is onto and takes an $f_o \in L^p(\Sigma) \backslash \mathcal{R}(W)$. Since $\mathcal{R}(W)$ is closed, we can find a functional L_{g^*} on $L^p(\Sigma)$ corresponding to $g^* \in L^q(\Sigma)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ which is defined as

$$(1) \quad L_{g^*}(f) = \int_X f g^* d\mu \text{ such that } L_{g^*}(f_0) = 1 \text{ and } L_{g^*}(\mathcal{R}(W)) = 0.$$

Hence the set $E_\delta = \{x \in X : \operatorname{Re}(f_0 g^*)(x) \geq \delta\}$ must have positive measure for some $\delta > 0$. Since μ is non-atomic we can choose a sequence $\{E_n\}$ of subsets of E_δ with $0 < \mu(E_n) < \mu(E_\delta)$ and $E_n \cap E_m = \emptyset$ for $n \neq m$. Let $g_n^* = \chi_{E_n} g^*$. Then $g_n^* \in L^q(\Sigma)$ and is nonzero because

$$\operatorname{Re} \int_X f_0 g_n^* d\mu \geq \delta \mu(E_n) > 0.$$

Evidently for any $f \in L^p(\Sigma)$, $\chi_{E_n} f$ is in $L^p(\Sigma)$, and so the right equality of (1) yields

$$\int_X f(W^* g_n^*) d\mu = \int_X f hE(u g_n^*) \circ \varphi^{-1} d\mu = \int_{E_n} f E(u g^*) \circ \varphi^{-1} d\mu \circ \varphi^{-1}$$

$$\int_{\varphi^{-1}(E_n)} f\circ\varphi E(ug^*)d\mu = \int_{\varphi^{-1}(E_n)} ug^* f\circ\varphi d\mu = \int_X g^* uf\circ\varphi(\chi_{E_n}\circ\varphi)d\mu$$

$$\int_X g^* u(f\chi_{E_n})\circ\varphi d\mu = \int_X g^* W(f\chi_{E_n})d\mu = 0.$$

This implies that $g_n^* \in \mathcal{N}(W^*)$. Thus the sequence $\{g_n^*\}$ forms a linearly independent subset of $\mathcal{N}(W^*)$. This contradicts the fact that $\dim\mathcal{N}(W^*) = \operatorname{codim}\mathcal{R}(W) < \infty$. Hence W is onto. Next we put $Z(J) = \{x : J(x) = 0\}$. Now we claim that $\mu(Z(J)) = 0$. For, if $\mu(Z(J)) > 0$, there exists a subset F of $Z(J)$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(W)$, then there exists $f \in L^p(\Sigma)$ such that $\chi_F = Wf$. Then

$$\mu(F) = \int_F |Wf|^p d\mu \int_F J|f|^p d\mu = 0$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma)\setminus\mathcal{R}(W)$, which contradicts the fact that W is onto. Also since $\mu(Z(J)) = 0$ and $\mu\circ\varphi^{-1} \ll \mu$ we have $\mu(Z(J\circ\varphi)) = 0$. For each $n = 1, 2, \ldots$, let

$$H_n = \left\{x \in X : \frac{\|J\circ\varphi\|_\infty}{(n+1)^2} < J\circ\varphi(x) \le \frac{\|J\circ\varphi\|_\infty}{n^2}\right\},$$

and $H = \{n : \mu(H_n) > 0\}$. Then the H_n's are pairwise disjoint and $X = \bigcup_{n=1}^\infty H_n$. Define

$$f(x) = \begin{cases} (J\circ\varphi(x)/\mu(H_n))^{\frac{1}{p}} & \text{if } x \in H_n,\ n \in H, \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\begin{aligned} \int_X |f|^p d\mu &= \sum_{n\in H}\int_{H_n} \frac{J\circ\varphi(x)}{\mu(H_n)}d\mu \\ &\le \sum_{n\in H} \frac{\|J\circ\varphi\|_\infty}{n^2} \le \|J\circ\varphi\|_\infty \sum_{n=1}^\infty \frac{1}{n^2} < \infty, \end{aligned}$$

so $f \in L^p(\Sigma)$. If $g \in L^p(\Sigma)$ is such that $Wg = f$, then

$$\int_X E^{\varphi^{-1}(\Sigma)}(|u|^p)|g|^p\circ\varphi d\mu = \int_X E^{\varphi^{-1}(\Sigma)}(|f|^p)d\mu.$$

It follows that

$$\int_X hE^{\varphi^{-1}(\Sigma)}(|u|^p)\circ\varphi^{-1}|g|^p d\mu = \int_X hE^{\varphi^{-1}(\Sigma)}(|f|^p)\circ\varphi^{-1}d\mu.$$

Thus $|g|^p = hE^{\varphi^{-1}(\Sigma)}(|f|^p) \circ \varphi^{-1}/J$ on off $Z(J)$. Since $\mu(Z(J)) = 0$, it follows that

$$\int_X |g|^p d\mu = \int_X \frac{E^{\varphi^{-1}(\Sigma)}(|f|^p) \circ \varphi^{-1}}{J} d\mu \circ \varphi^{-1} = \int_X \frac{E^{\varphi^{-1}(\Sigma)}(|f|^p)}{J \circ \varphi} d\mu$$
$$= \int_X \frac{|f|^p}{J \circ \varphi} d\mu = \sum_{n \in H} \int_{H_n} \frac{d\mu}{\mu(H_n)} = \sum_{n \in H} 1.$$

This implies that H must be finite set. Thus there is an n_o such that $n \geq n_o$ implies $\mu(H_n) = 0$ and so

$$\mu \left(\left\{ x \in X : J \circ \varphi(x) \leq \frac{\|J \circ \varphi\|_\infty}{n_o^2} \right\} \right) = \mu \left(\bigcup_{n=n_o}^{\infty} H_n \cup Z(J \circ \varphi) \right) = 0.$$

Therefore we obtain $J \circ \varphi \geq \|J \circ \varphi\|_\infty / n_o^2$ almost everywhere on X. Since $\mathcal{N}(W) = L^p(Z(J))$, $\mu(Z(J)) = 0$ so $\dim \mathcal{N}(W) = \{0\}$ and then φ is essentially surjective. Hence $J \geq \|J\|_\infty / n_o^2$ $(= \delta)$ almost everywhere on X.

Conversely, suppose that $J \geq \delta$ almost everywhere for some $\delta > 0$. Since $h > 0$ and for each $f \in L^p(\Sigma)$, $\|Wf\|_p = (\int_X J|f|^p d\mu)^{1/P} \geq \delta^{1/p}\|f\|_p$, it follows that W and C_φ are injective and $\mathcal{R}(W)$ is closed. Also since $W = M_u C_\varphi$ we deduce that M_u is injective and so $\mu(Z(u)) = 0$. Now let $g^* \in \mathcal{N}(W^*)$. Then $W^* g^* = hE^{\varphi^{-1}(\Sigma)}(ug^*) \circ \varphi^{-1} = 0$ and so $E^{\varphi^{-1}(\Sigma)}(ug^*) \circ \varphi^{-1} = 0$. It follows that $g^* = 0$. Thus $\operatorname{codim} \mathcal{R}(W) = \dim \mathcal{N}(W^*) = 0$. Therefore the theorem is proved. □

Remark 3.3. One of the interesting features of a weighted composition operator is that the multiplication operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let X be $(0, 1)$, Σ be the Borel sets, and μ be the Lebesgue measure. Let φ be the map $\varphi(x) = \sqrt[3]{x}$ and $u(x) = 1/\sqrt{x}$ on $(0, 1)$. Then M_u dos not define a bounded operator from $L^1(\Sigma)$ into $L^1(\Sigma)$. However a simple computation shows that $J(x) = 3\sqrt{x} \in L^\infty(\Sigma)$ and so $Wf(x) = 1/\sqrt{x} f(\sqrt[3]{x})$ is bounded operator on $L^1(\Sigma)$.

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