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M-Hyponormal Powers of Weighted Composition Operators

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Abstract. In this note, M-hyponormality powers of weighted composition operators on L^2-spaces are characterized and their various properties are studied.

Keywords: Weighted composition operator; Conditional expectation; M-hyponormal.

1. Preliminaries And Notations

Let (X, Σ, μ) be a complete σ-finite measure space and suppose that T is a measurable transformation from X into X such that $\mu \circ T^{-1}$ is absolutely continuous with respect to μ and write $\mu \circ T^{-1} \ll \mu$. Let h be the Radon-Nikodym derivative $d\mu \circ T^{-1}/d\mu$ and we always assume that h is almost everywhere finite-valued or, equivalently $(X, T^{-1}(\Sigma), \mu)$ is σ-finite. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. To examine the weighted composition operators efficiently, Lambert in [8] associated with each transformation T, the so-called conditional expectation operator $E(\bullet|T^{-1}(\Sigma)) = E_i(\bullet)$. $E(f)$ is defined for each non-negative measurable function f or for each $f \in L^2(\Sigma)$, and is uniquely determined by the conditions:

(i) $E(f)$ is $T^{-1}(\Sigma)$-measurable and

(ii) If A is any $T^{-1}(\Sigma)$-measurable set for which $\int_A f d\mu$ converges we have

$$\int_{T^{-1}(A)} f d\mu = \int_{T^{-1}(A)} E(f) d\mu.$$

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As an operator on $L^2(\Sigma)$, $E(\bullet)$ is the contractive orthogonal projection onto $L^2(T^{-1}(\Sigma)) = \overline{R(C_T)}$, the closure of the range of C_T, used by Harrington and Whitley in [5]. It is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^2(\Sigma)$, there exists a Σ-measurable function g such that $E(f) = g \circ T$. We can assume that the support of g, $\sigma(g) = \{x \in X : g(x) \neq 0\}$, lies in the $\sigma(h)$ and there exists only one g with this property. We then write $g = E(f) \circ T^{-1}$ though we make no assumptions regarding the invertibility of T (see [2]). For further discussion of the conditional expectation operator see the interesting papers [6], [8] and [3]. If $u : X \to C$ is a measurable function, the weighted composition operator $W_T = uC_T$ on $L^2(\Sigma)$ induced by T and u is given by

$$W_T(f) = u.f \circ T, \qquad f \in L^2(\Sigma).$$

Here, the non-singularity of T guarantees that W_T is well defined as a mapping of equivalence classes of functions on $\sigma(u)$. In this case, the adjoint W_T^* is given by

$$W_T^* f = hE(uf) \circ T^{-1}, \qquad f \in L^2(\Sigma).$$

If $W_T(L^2(\Sigma)) \subseteq L^2(\Sigma)$, by the closed graph theorem W_T is bounded. Boundedness of weighted composition operators on $L^2(\Sigma)$ spaces already being studied in [6]. Namely, W_T is bounded if $J = hE(|u|^2) \circ T^{-1} \in L^\infty(\Sigma)$.

Let $B(H)$ denote the Banach space of all bounded linear operators on the Hilbert space H. An operator $T \in B(H)$ is called M-hyponormal if there exists some $M > 0$ such that $\|T^*x\| \leq M\|Tx\|$ for all $x \in H$.

Power-hyponormality of composition operators in $L^2(\Sigma)$ appeared already in [4] and for adjoint of weighted composition operators in the paper [3]. Also, the analogous results for M-cohyponormality of composition operators has been studied in [7]. Namely, if $C_{T_1}^*$ and $C_{T_2}^*$ are both M-hyponormal with $h_1 \leq M^2(h_2 \circ T_2)$ a.e. and $h_2 \leq M^2(h_1 \circ T_1)$ a.e., then for all positive integers m, n and p, $[(C_{T_1}^m C_{T_2}^n)^p]^*$ is $M^{p^2(m+n)^2}$-hyponormal. The aim of this paper is to generalize the results obtained for composition operators in [7] to the weighted composition operators.

2. Lemmas and Main Result

In the following we list some facts that will be applied often in this article (see [1, 8]):

- $\int_X f \circ T d\mu = \int_X h.f d\mu$ for all $f \in L^1(\Sigma)$ (change of variables formula).
- $\|W_T f\| = \|M_{\sqrt{J}} f\|$ where $J = hE(|u|^2) \circ T^{-1} \in L^\infty(\Sigma)$ and $f \in L^2(\Sigma)$.
- If u is in $L^\infty(T^{-1}(\Sigma))$, then $E(uf) = uE(f)$.
- For any f and g in $L^2(\Sigma)$ and any $T^{-1}(\Sigma)$-measurable set A, $\int_A E(f)E(g)d\mu = \int_A fE(g)d\mu$.
- For any f and g in $L^2(\Sigma)$, $|E(fg)|^2 \leq E(|f|^2)E(|g|^2)$.

Proposition 2.1. *[4] Let T_i be a measurable transformation of X such that $\mu\circ T_i^{-1}$ is absolutely continuous with respect to μ and $h_i = d\mu\circ T_i^{-1}/d\mu \in L^\infty(\Sigma)$ for $i = 1, 2$, and $T_3 : X \to X$ given by $T_3 = T_1\circ T_2$. Then $h_3 = d\mu\circ T_3^{-1}/d\mu$ is absolutely continuous with respect to μ and $h_3 = h_1E_1(h_2)\circ T_1^{-1}$ where $E(\bullet|T_i^{-1}(\Sigma)) = E_i(\bullet)$.*

Proof. Let $A \in \Sigma$. By using of conditional expectation operator and change of variables formula we have

$$\int_A h_3 d\mu = \int_A \frac{d\mu\circ(T_1\circ T_2)^{-1}}{d\mu} d\mu = \int_A d\mu\circ T_2^{-1}\circ T_1^{-1}$$

$$= \int_{T_1^{-1}(A)} d\mu\circ T_2^{-1} = \int_{T_1^{-1}(A)} h_2 d\mu = \int_{T_1^{-1}(A)} E_1(h_2)d\mu$$

$$= \int_A E_1(h_2)\circ T_1^{-1} d\mu\circ T_1^{-1} = \int_A h_1E_1(h_2)\circ T_1^{-1}d\mu.$$

Since (X, Σ, μ) is a σ-finite measure space, so the proof is therefore complete. ■

For a generalization of the above fact, we define the measure $\mu_{T_i,u}(E) = \int_{T_i^{-1}(E)} |u|^2 d\mu$, for $E \in \Sigma$ and $i = 1, 2$. Then it is easy to see that $\mu_{T_i,u} \ll \mu$. Put $H_i = d\mu_{T_i,u}/d\mu$ which, of course, is a non-negative Σ-measurable function. By simple calculation we have $H_i = h_iE_i(|u|^2)\circ T_i^{-1}$ $(i = 1, 2)$ and $H_3 = h_1E_1(H_2)\circ T_1^{-1}$.

Example 2.2. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and Σ the Lebesgue sets. Define $T_i : X \to X$ by $T_1(x) = (\sqrt{1+x} - 1)\chi_{[-1,0]} + (1 - \sqrt{1-x})\chi_{(0,1]}$ and $T_2(x) = \sqrt[3]{3x}$. One easily verifies that $E_1(f) = (f(x) + f(-x))/2$ for all positive measurable function f on X. Direct computation shows that $h_1(x) = (2 + 2x)\chi_{[-1,0]} + (2 - 2x)\chi_{(0,1]}$, $E_1(h_2(x)) = x^2$ and $(h_1E_1(h_2)\circ T_1^{-1})(x) = (2 + 2x)(2x + x^2)\chi_{[-1,0]} + (2 - 2x)(2x - x^2)\chi_{(0,1]} = h_3(x)$. Now, if we take $u(x) = x$, then $H_2(x) = x^8/9$ and $(h_1E_1(H_2)\circ T_1^{-1})(x) = 1/9(2 + 2x)(2x + x^2)^8\chi_{[-1,0]} + 1/9(2 - 2x)(2x - x^2)^8\chi_{(0,1]} = H_3(x)$.

In order to prove our main result, it is necessary to state and prove several lemmas.

Lemma 2.3. *If W_T^* is M-hyponormal, then $J \le M^2(J\circ T)$, where $J = hE(|u|^2)\circ T^{-1}$.*

Proof. Since W_T^* is M-hyponormal, then

$$\| W_T f \|^2 \le M^2 \| W_T^* f \|^2, \quad \text{for all } f \in L^2(\Sigma).$$

To assert the inequality it suffices to show that $\| W_T f \|^2 = (Jf, f)$ and $\|W_T^* f\|^2 \leq ((J \circ T)f, f)$. For all $f \in L^2(\Sigma)$, we have

$$\| W_T f \|^2 = \int |uf \circ T|^2 d\mu = \int |u|^2 |f|^2 \circ T d\mu = \int E(|u|^2)|f|^2 \circ T d\mu$$

$$= \int hE(|u|^2) \circ T^{-1}|f|^2 d\mu = (hE(|u|^2) \circ T^{-1} f, f) = (Jf, f);$$

therefore, the first identity holds. On the other hand, we have

$$\begin{aligned} \| W_T^* f \|^2 &= (W_T^* f, W_T^* f) = (W_T W_T^* f, f) = (uh \circ TE(\overline{u}f), f) \\ &= \int uh \circ TE(\overline{u}f)\overline{f} d\mu = \int (h \circ T)E(\overline{u}f)E(u\overline{f})d\mu \\ &= \int (h \circ T)|E(\overline{u}f)|^2 d\mu \leq \int (h \circ T)E(|u|^2)E(|f|^2)d\mu \end{aligned}$$

$$\int (h \circ T)E(|u|^2)f\overline{f} d\mu = \int (J \circ T)f\overline{f} d\mu = ((J \circ T)f, f).$$

Hence $(Jf, f) \leq M^2((J \circ T)f, f)$. Since f is an arbitrary element of $L^2(\Sigma)$, we have $J \leq M^2(J \circ T)$; and the proof is therefore complete. ∎

Example 2.4. Let $X = [0, 1]$, $d\mu = dx$ and Σ the Lebesgue sets. Let $T : X \to X$ be defined by $T(x) = 4x - 4x^2$ and let u be the map $u(x) = (4 - 8x)(\chi_{[0,1/2]} - \chi_{(1/2,1]})$ on X. Then a simple computation gives $\|W_T^* f\| = \|f((1+\sqrt{1-x})/2) + f((1-\sqrt{1-x})/2)\|$, $\|W_T f\| = \|\sqrt[4]{64(1-x)}f\|$ and $J \circ T(x) = 8|2x - 1|$. So for each $M > 2^{-3/4}$, $J \geq M^2(J \circ T)$ and hence by lemma 2.3, W_T is not M-hyponormal operator on $L^2(\Sigma)$.

Lemma 2.5. *If W_T^* is M-hyponormal, then for $f \in L^2(\Sigma)$, $((J \circ T)Ef, f) = ((J \circ T)f, f)$ where $J = hE(|u|^2) \circ T^{-1}$.*

Proof. Let $f \in L^2(\Sigma)$, then we have

$$\begin{aligned} ((J \circ T)Ef, f) &= \int (J \circ T)(Ef)\overline{f} d\mu = \int (J \circ T)EfE\overline{f} d\mu \\ &= \int (J \circ T)|Ef|^2 d\mu \leq \int (J \circ T)E(|f|^2)d\mu \\ &= \int (J \circ T)|f|^2 d\mu = ((J \circ T)f, f) \end{aligned}$$

Hence $((J \circ T)Ef, f) \leq ((J \circ T)f, f)$. On the other hand, since W_T^* is M-hyponormal, then

$$KerE = \overline{(R(W_T))}^{\perp} = KerW_T^* \subseteq Ker(W_T).$$

Now, since for all $f \in L^2(\Sigma)$, $Ef - f \in KerE$, then $Ef - f \in Ker(W_T)$. This shows that

$$\int J(Ef - f)d\mu = \int hE(|u|^2) \circ T^{-1}(Ef - f)d\mu$$

$$= \int E(|u|^2)(Ef - f) \circ Td\mu = \int |u|^2(Ef - f) \circ Td\mu = 0.$$

It follows that $J(Ef - f) = 0$. Thus, it follows from Lemma 2.3 that for all $g \in L^2(\Sigma)$, we have

$$((J \circ T)(Ef - f), g) = \int (J \circ T)(Ef - f)\overline{g}d\mu \geq \frac{1}{M^2} \int J(Ef - f)\overline{g}d\mu = 0.$$

We conclude that $((J \circ T)(Ef - f), g) \geq 0$, hence $(J \circ T)Ef \geq (J \circ T)f$. ■

Lemma 2.5 can be extended to give the following result.

Lemma 2.6. *If W_T^* is M-hyponormal, then for all $f \in L^2(\Sigma)$ and $n \in N$*

$$((J^n \circ T)Ef, f) = ((J^n \circ T)f, f).$$

Lemma 2.7. *If $J \leq M^2(h \circ T)$, for all $f \in L^2(\Sigma)$ and $r, m \in N$, then*

$$((J \circ T)^r W_T^m f, W_T^m f) \leq M^{(m-1)(2r+m)}(J^{r+m}f, f).$$

Proof. We shall prove the result by induction on m and fixed r. For $m = 1$ and $f \in L^2(\Sigma)$,

$$((J \circ T)^r W_T f, W_T f) = \int (J \circ T)^r uf \circ T.\overline{uf \circ T}d\mu$$

$$= \int (J \circ T)^r |u|^2 |f|^2 \circ Td\mu = \int J^r \circ TE(|u|^2)|f|^2 \circ Td\mu$$

$$= \int J^r hE(|u|^2) \circ T^{-1}|f|^2 d\mu = \int J^r Jf\overline{f}d\mu = (J^{r+1}f, f),$$

which shows that the Lemma holds for $m = 1$. Now assuming that the Lemma holds for $m = 1, 2, ..., k$ and $f \in L^2(\Sigma)$. Then we have

$$
\begin{aligned}
&((J \circ T)^r W_T^{k+1} f, W_T^{k+1} f) = ((J \circ T)^r W_T^k W_T f, W_T^k W_T f) \\
&\leq M^{(k-1)(2r+k)} (J^{r+k} W_T f, W_T f) = M^{(k-1)(2r+k)} \int J^{r+k} |u|^2 |f|^2 \circ T d\mu \\
&= M^{(k-1)(2r+k)} \int J^{r+k} E(|u|^2) |f|^2 \circ T d\mu \\
&\leq M^{(k-1)(2r+k)} M^{2(r+k)} \int (J \circ T)^{r+k} E(|u|^2) |f|^2 \circ T d\mu \\
&= M^{k(2r+k+1)} \int J^{r+k} h E(|u|^2) \circ T^{-1} |f|^2 d\mu \\
&= M^{k(2r+k+1)} \int J^{r+k} J f \overline{f} d\mu = M^{k(2r+k+1)} (J^{r+k+1} f, f),
\end{aligned}
$$

which shows that the result holds for $m = k + 1$. Thus the result holds for all $r, m \in N$ and $f \in L^2(\Sigma)$. ∎

Lemma 2.8. *If W_T^* is M-hyponormal and $u \in L^\infty(T^{-1}\Sigma)$ then for all $f \in L^2(\Sigma)$ and $r, m \in N$, we have*

$$M^{(m-1)(2r+m)} (J^r (W_T^m)^* f, (W_T^m)^* f) \geq ((J \circ T)^{r+m} f, f).$$

Proof. We shall prove the result by induction on m and fixed r. Suppose that $m = 1$ and $f \in L^2(\Sigma)$. Then

$$
\begin{aligned}
(J^r W_T^* f, W_T^* f) &= ((J^r \circ T) W_T W_T^* f, f) = ((J^r \circ T) u h \circ T E(\overline{u} f), f) \\
&= ((J^r \circ T) |u|^2 h \circ T E f, f) = ((J^r \circ T) E(|u|^2) h \circ T E f, f) \\
&= ((J^r \circ T)(J \circ T) E f, f) = ((J^{r+1} \circ T) f, f),
\end{aligned}
$$

which shows that the Lemma holds for $m = 1$. The rest of the proof can be repeated as in Lemma 2.8 in [7]. ∎

Lemma 2.9. *If $J \leq M^2(J \circ T)$, then for all $n \in N$ and $f \in L^2(\Sigma)$,*

$$((W_T^n)^* (W_T^n) f, f) \leq M^{n(n-1)} (J^n f, f).$$

Proof. Since $W_T^* W_T f = J f$, hence the result holds for $n = 1$. Let us suppose that the result is true for $n = r$ and $f \in L^2(\Sigma)$. First we show that

$$(J^k W_T f, W_T f) \leq M^{2k} (J^{k+1} f, f). \tag{1}$$

Suppose $f \in L^2(\Sigma)$. Then

$$(J^k W_T f, W_T f) = \int J^k W_T f \overline{W_T f} d\mu = \int J^k |u|^2 |f|^2 \circ T d\mu$$

$$= \int J^k E(|u|^2)|f|^2 \circ T d\mu \leq M^2 \int (J^k \circ T) E(|u|^2)|f|^2 \circ T d\mu$$

$$= M^{2k} \int J^k h E(|u|^2) \circ T^{-1} f\overline{f} d\mu = M^{2k} \int J^{k+1} f\overline{f} d\mu = M^{2k}(J^{k+1}f, f).$$

Hence, by induction hypothesis and (1), we have

$$\begin{aligned} &((W_T^{k+1})^*(W_T^{k+1})f, f) = ((W_T^k)^*(W_T^k)W_T f, W_T f) \\ \leq\ & M^{k(k-1)}(J^k W_T f, W_T f) \leq M^{k(k-1)} M^{2k}(J^{k+1} f, f) \\ =\ & M^{k(k+1)}(J^{k+1} f, f). \end{aligned}$$

■

Lemma 2.10. *If $(uC_T)^*$ is M-hyponormal and $u \in L^\infty(T^{-1}(\Sigma))$ then for all $f \in L^2(\Sigma)$ and $n \in N$*

$$M^{n(n-1)}((W_T^n)^(W_T^n)^* f, f) \geq ((J \circ T)^n f, f) \ .$$

Proof. We shall prove the result by induction on n. Suppose that $n = 1$ and $f \in L^2(\Sigma)$. Then

$$(W^T W_T^* f, f) = \int uh \circ T E(\overline{u} f)\overline{f} d\mu = \int h \circ T |u|^2 E(|f|^2) d\mu$$

$$= \int h \circ T E(|f|^2) E(|u|^2) d\mu = \int (J \circ T) f\overline{f} d\mu = ((J \circ T)f, f).$$

Now, assume that the Lemma holds for n. Then by using Lemma 2.8, we have

$$\begin{aligned} &((W_T^{n+1})(W_T^{n+1})^* f, f) = (W_T W_T^n (W_T^n)^* W_T^* f, f) \\ =\ & (W_T^n (W_T^n)^* W_T^* f, W_T^* f) \geq \frac{1}{M^{n(n-1)}}((J \circ T)^n W_T^* f, W^* f) \\ \geq\ & \frac{1}{M^{n(n-1)}} \frac{1}{M^{2n}}((J)^n W_T^* f, W^* f) = \frac{1}{M^{n(n+1)}}((J^{n+1} \circ T) f, f), \end{aligned}$$

which completes the induction step and the Lemma is proved. ■

If we change the role of h and C_T with J and W_T respectively and using previous lemmas in this paper, we can prove the following theorem similar to the proof used in [7].

Theorem 2.11. (a) *If* W_T^* *is* M*-hyponormal, then for all* $n \in N, (W_T^*)^n$ *is* M^{n^2}*-hyponormal.*

(b) *Put* $A = W_{T_1}$ *and* $B = W_{T_2}$*. If* A^* *and* B^* *are* M*-hyponormal such that*

$$J_1 \leq M^2(J_2 \circ T_2), \quad (J_1 = h_1 E_1(|u|^2) \circ T_1^{-1})$$

and

$$J_2 \leq M^2(J_1 \circ T_1), \quad (J_2 = h_2 E_2(|u|^2) \circ T_2^{-1}).$$

Then $(A^m B^n)^*$ *is* $M^{(m+n)^2}$*-hyponormal for all* $m, n \in N$*.*

(c) *Under the hypothesis of (b),* $[(A^m B^n)^p]^*$ *is* $M^{p^2(m+n)^2}$*-hyponormal.*

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