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LAMBERT MULTIPLIERS BETWEEN L^p SPACES

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Abstract. In this paper Lambert multipliers acting between L^p spaces are characterized by using some properties of conditional expectation operator. Also, Fredholmness of corresponding bounded operators is investigated.

Keywords: conditional expectation, multipliers, multiplication operators, Fredholm operator

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1. Introduction and preliminaries

Let $L(X, \Sigma, \mu)$ be a σ-finite measure space. For any complete σ-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leqslant p \leqslant \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|\cdot\|_p$. We view $L^p(\mathcal{A})$ as a Banach sub-space of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X\colon f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

To examine the weighted composition operators efficiently, Alan Lambert in [9] associated with each transformation T the so-called conditional expectation operator $E(\cdot|\mathcal{A}) = E(\cdot)$ which is defined for each non-negative measurable function f or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions

(i) $E(f)$ is $\mathcal{A}$-measurable and

(ii) if A is any $\mathcal{A}$-measurable set for which $\int_A f\,\mathrm{d}\mu$ converges then

$$\int_A f\,\mathrm{d}\mu = \int_A E(f)\,\mathrm{d}\mu.$$

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This operator will play a major role in our work, and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$.
- $|E(f)|^p \leqslant E(|f|^p)$.
- $\|E(f)\|_p \leqslant \|f\|_p$.
- If $f \geqslant 0$ then $E(f) \geqslant 0$; if $f > 0$ then $E(f) > 0$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^p(\mathcal{A})$.

As an operator on $L^p(\Sigma)$, $E(\cdot)$ is contractive idempotent and $E(L^p(\Sigma)) = L^p(\mathcal{A})$. A real-valued Σ-measurable function f is said to be conditionable with respect to $\mathcal{A}$ if $\mu(\{x \in X\colon\ E(f^+)(x) = E(f^-)(x) = \infty\}) = 0$. In this case $E(f) := E(f^+) - E(f^-)$. If f is complex-valued, then f is conditionable if both the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. In this case, $E(f) := E(\operatorname{Re} f) + \mathrm{i}E(\operatorname{Im} f)$ (see [4]). We denote the linear space of all conditionable Σ-measurable functions on X by $L^0(\Sigma)$. For f and g in $L^0(\Sigma)$, we define $f \star g = fE(g) + gE(f) - E(f)E(g)$. Let $1 \leqslant p$, $q \leqslant \infty$. A measurable function $u \in L^0(\Sigma)$ for which $u \star f \in L^q(\Sigma)$ for each $f \in L^p(\Sigma)$ is called a Lambert multiplier. In other words, $u \in L^0(\Sigma)$ is a Lambert multiplier if and only if the corresponding $\star$-multiplication operator $T_u\colon\ L^p(\Sigma) \to L^q(\Sigma)$ defined as $T_u f = u \star f$ is bounded. Note that if u is a $\mathcal{A}$-measurable function or $\mathcal{A} = \Sigma$, then $u \in K_p^\star$ if and only if the multiplication operator $M_u\colon\ L^p(\Sigma) \to L^q(\Sigma)$ is bounded.

In the next section, Lambert multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of the conditional expectation operator. In Section 3, Fredholmness of the corresponding $\star$-multiplication operators will be investigated.

2. Characterization of Lambert multipliers

Let $1 \leqslant p$, $q \leqslant \infty$. Define $K_{p,q}^\star$, the set of all Lambert multipliers from $L^p(\Sigma)$ into $L^q(\Sigma)$, as follows:

$$K_{p,q}^\star = \{u \in L^0(\Sigma)\colon\ u \star L^p(\Sigma) \subseteq L^q(\Sigma)\}.$$

$K_{p,q}^\star$ is a vector subspace of $L^0(\Sigma)$. Put $K_{p,p}^\star = K_p^\star$. In the following theorem we characterize the members of $K_{p,q}^\star$ in the case $1 \leqslant p = q < \infty$.

Theorem 2.1. *Suppose $1 \leqslant p < \infty$ and $u \in L^0(\Sigma)$. Then $u \in K_p^\star$ if and only if $E(|u|^p) \in L^\infty(\mathcal{A})$.*

Proof. Let $E(|u|^p) \in L^\infty(\mathcal{A})$ and take $f \in L^p(\Sigma)$. Since $|E(u)|^p \leqslant E(|u|^p)$ a.e., we have

$$\|E(u)f\|_p^p = \int_X |E(u)f|^p \, \mathrm{d}\mu \leqslant \int_X E(|u|^p)|f|^p \, \mathrm{d}\mu \leqslant \|E(|u|^p)\|_\infty \|f\|_p^p.$$

Hence $\|E(u)f\|_p \leqslant \|E(|u|^p)\|_\infty^{1/p} \|f\|_p$. A similar argument, using the fact that $E(fE(g)) = E(f)E(g)$, reveals that we also have

$$\begin{aligned}\|E(u)E(f)\|_p^p = \|uE(f)\|_p^p &= \int_X |uE(f)|^p \, \mathrm{d}\mu \leqslant \int_X |u|^p E(|f|^p) \, \mathrm{d}\mu \\ = \int_X E(|u|^p)E(|f|^p) \, \mathrm{d}\mu &\leqslant \|E(|u|^p)\|_\infty \int_X |f|^p = \|E(|u|^p)\|_\infty \|f\|_p^p.\end{aligned}$$

Thus $\|E(u)E(f)\|_p = \|uE(f)\|_p \leqslant \|E(|u|^p)\|_\infty^{1/p} \|f\|_p$. Accordingly, we get that

$$\|u \star f\|_p \leqslant \|E(u)f\|_p + \|uE(f)\|_p + \|E(u)E(f)\|_p \leqslant 3\|E(|u|^p)\|_\infty^{1/p} \|f\|_p.$$

It follows that $u \star f \in L^p(\Sigma)$ and hence $u \in K_p^\star$.

Now, suppose only that $u \in K_p^\star$. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each $L^p(\Sigma)$ convergent sequence ensures that the operator $T_u \colon L^p(\Sigma) \to L^p(\Sigma)$ given by $T_u f = u \star f$ is bounded. Define a linear functional φ on $L^1(\mathcal{A})$ by

$$\varphi(f) = \int_X E(|u|^p) f \, \mathrm{d}\mu, \quad f \in L^1(\mathcal{A}).$$

We shall show that φ is bounded. To this end, since for each $f \in L^1(\mathcal{A})$, $E(|f|^{1/p}) = |f|^{1/p} \in L^p(\mathcal{A})$, we have

$$\begin{aligned}|\varphi(f)| \leqslant \int_X E(|u|^p)|f| \, \mathrm{d}\mu &= \int_X (E(|u||f|^{1/p})^p \, \mathrm{d}\mu \\ &= \int_X (|u||f|^{1/p})^p \, \mathrm{d}\mu = \|T_u|f|^{1/p}\|_p^p \\ &\leqslant \|T_u\|^p \||f|^{1/p}\|_p^p = \|T_u\|^p \|f\|_1.\end{aligned}$$

Thus, φ is a bounded linear functional on $L^1(\mathcal{A})$ and $\|\varphi\| \leqslant \|T_u\|^p$. By the Riesz representation theorem, there exists a unique function $g \in L^\infty(\mathcal{A})$ such that

$$\varphi(f) = \int_X gf \, \mathrm{d}\mu, \quad f \in L^1(\mathcal{A}).$$

Therefore, we have $g = E(|u|^p)$ a.e. on X and hence $E(|u|^p) \in L^\infty(\mathcal{A})$. □

Let $\Im := \{T_u\colon u \in K_p^\star\}$ and let $\Im'$ be the commutant of $\Im$ in the algebra of all bounded linear operators. Still proceeding as in the proof of Theorem 6.6 given in [2] and Theorem 4.1 given in [6], one establishes that $\Im = \Im' = \Im''$ (see also [3]). Thus $\Im$ is maximal abelian and hence it is norm closed.

For $u \in K_p^\star$ define $\|u\|_{K_p^\star} = \|E(|u|^p)\|_\infty^{1/p}$. Then precisely the same calculation as that shown in the proof of Theorem 2.1 yields that

$$\|u \star f\|_p \leqslant 3(\|E(|u|^p)\|_\infty^{1/p}\|f\|_p) < \infty, \quad f \in L^p(\Sigma),$$

and

$$\int_X E(|u|^p)|f| \,\mathrm{d}\mu \leqslant \|T_u\|^p\|f\|_1, \quad f \in L^1(\mathcal{A}).$$

It follows that

$$\|T_u\| \leqslant 3\|E(|u|^p)\|_\infty^{1/p} \tag{2.1}$$

and

$$\|E(|u|^p)\|_\infty = \sup_{\|f\|_1\leqslant 1} \int_X E(|u|^p)|f| \,\mathrm{d}\mu \leqslant \|T_u\|^p. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\|u\|_{K_p^\star} \leqslant \|T_u\| \leqslant 3\|u\|_{K_p^\star}. \tag{2.3}$$

Consequently, $\|\cdot\|_{K_p^\star}$ and the operator norm $\|\cdot\|$ are equivalent norms on $\Im$. Also, since $\Im$ is norm closed, it follows from (2.3) that $K_p^\star$ is a Banach space with the norm $\|\cdot\|_{K_p^\star}$.

Let $1 \leqslant q < p < \infty$. Our second task is the description of the members of $K_{p,q}^\star$ in terms of the conditional expectation induced by $\mathcal{A}$.

Theorem 2.2. *Suppose $1 \leqslant q < p < \infty$ and $u \in L^0(\Sigma)$. Then $u \in K_{p,q}^\star$ if and only if $(E(|u|^q))^{1/q} \in L^r(\mathcal{A})$, where $1/p + 1/r = 1/q$.*

P r o o f. Suppose $(E(|u|^q))^{1/q} \in L^r(\mathcal{A})$. Let $f \in L^p(\Sigma)$. Using the same method as in the proof of Theorem 2.1, we have

$$\|E(u)f\|_q^q \leqslant \int_X E(|u|^q)|f|^q \,\mathrm{d}\mu = \|E(|u|^q))^{1/q} f\|_q^q \leqslant \|(E(|u|^q))^{1/q}\|_r^q\|f\|_p^q.$$

By similar computation we obtain

$$\begin{aligned}\|uE(f)\|_q^q &\leqslant \int_X |u|^q E(|f|^q) \,\mathrm{d}\mu = \int_X E(|u|^q)E(|f|^q) \,\mathrm{d}\mu \\ &\leqslant \|(E(|u|^q))^{1/q}\|_r^q\|E(|f|^q)\|_{p/q} \leqslant \|(E(|u|^q))^{1/q}\|_r^q\|f\|_p^q\end{aligned}$$

and

$$\|E(u)E(f)\|_q^q \leqslant \int_X E(|u|^q)E(|f|^q)\,\mathrm{d}\mu$$
$$\leqslant \|(E(|u|^q))^{1/q}\|_r^q\,\|(E(|f|^q))^{1/q}\|_p^q \leqslant \|(E(|u|^q))^{1/q}\|_r^q\,\|f\|_p^q.$$

Therefore we have $\|T_u f\| \leqslant 3\|(E(|u|^q))^{1/q}\|_r\|f\|_p$ for all $f \in L^p(\Sigma)$. Consequently, T_u is bounded and hence $u \in K^{\star}_{p,q}$.

Now, suppose only that $u \in K^{\star}_{p,q}$. Define $\varphi\colon L^{p/q}(\mathcal{A}) \to \mathbb{C}$ given by $\varphi(f) = \int_X E(|u|^q)f\,\mathrm{d}\mu$. Clearly φ is a linear functional. We shall show that φ is bounded. For each $f \in L^{p/q}(\mathcal{A})$ we get that

$$|\varphi(f)| \leqslant \int_X E(|u|^q)|f|\,\mathrm{d}\mu = \int_X E((|u||f|^{1/q})^q)\,\mathrm{d}\mu = \|T_u|f|^{1/q}\|_q^q \leqslant \|T_u\|^q\|f\|_{p/q}.$$

It follows that $\|\varphi\| \leqslant \|T_u\|^q$ and hence φ is bounded. By the Riesz representation theorem, there exists a unique $g \in L^{r/q}(\mathcal{A})$ such that $\varphi(f) = \int_X gf\,\mathrm{d}\mu$ for each $f \in L^{p/q}(\mathcal{A})$. Hence $g = E(|u|^q)$ a.e. on X. That is, $(E|u|^q)^{1/q} \in L^r(\mathcal{A})$ and hence the proof is complete. □

Recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called non-atomic. It is a well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu|_{\mathcal{A}})$ can be partitioned uniquely as

$$X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B,$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [13]).

In the following theorem we characterize the members of $K^{\star}_{p,q}$ in the case $1 \leqslant p < q < \infty$.

Theorem 2.3. *Suppose $1 \leqslant p < q < \infty$ and $u \in L^0(\Sigma)$. Then $u \in K^{\star}_{p,q}$ if and only if*

(i) *$E(|u|^q) = 0$ a.e. on B;*

(ii) *$M := \sup_{n\in\mathbb{N}} \dfrac{E(|u|^q)(A_n)}{\mu(A_n)^{q/r}} < \infty$, where $\dfrac{1}{q} + \dfrac{1}{r} = \dfrac{1}{p}$.*

Proof. Suppose that both (i) and (ii) hold. Then, for each $f \in L^p(\Sigma)$ with $\|f\|_p \leqslant 1$ we have

$$\begin{aligned}\|E(u)f\|_q^q &\leqslant \int_X E(|u|^q)|f|^q \,\mathrm{d}\mu = \Big(\int_B + \int_{\bigcup A_n}\Big)(E(|u|^q)|f|^q)\,\mathrm{d}\mu \\ &= \sum_{n\in\mathbb{N}} \int_{A_n} E(|u|^q)|f|^q \,\mathrm{d}\mu = \sum_{n\in\mathbb{N}} E(|u|^q)(A_n)|f(A_n)|^q \mu(A_n) \\ &= \sum_{n\in\mathbb{N}} \frac{(E(|u|^q)(A_n)}{\mu(A_n)^{q/r}}(|f(A_n)|^p\mu(A_n))^{q/p} \leqslant M\|f\|_p^q \leqslant M,\end{aligned}$$

where we have used the fact that $E(|u|^q)$ is a constant $\mathcal{A}$-measurable function on each A_n (see [5, Theorem I.7.3]). Consequently, $\|E(u)f\|_q \leqslant M^{1/q}$. Since the conditional expectation operator E is a contraction, similar computation shows that $\|uE(f)\|_q \leqslant M^{1/q}$ and $\|E(u)E(f)\|_q \leqslant M^{1/q}$. It follows that $\|T_u\| \leqslant 3M^{1/q} < \infty$ and hence $u \in K^\star_{p,q}$.

Conversely, suppose that $u \in K^\star_{p,q}$. First we show that $E(|u|^q) = 0$ a.e. on B. Assuming the contrary, we can find some $\delta > 0$ such that $\mu(\{x \in B\colon E(|u|^q)(x) \geqslant \delta\}) > 0$. Put $F = \{x \in B\colon E(|u|^q)(x) \geqslant \delta\}$. Since $(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is a σ-finite measure space, we can suppose that $\mu(F) < \infty$. Also, since F is non-atomic so for all $n \in \mathbb{N}$ there exists $F_n \subseteq F$ such that $\mu(F_n) = \mu(F)/2^n$. For any $n \in \mathbb{N}$, put $f_n = 1/((\mu(F_n))^{1/p})\chi_{F_n}$. It is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Since $q/p > 1$, we have

$$\begin{aligned}&\infty > \|T_u\|^q \geqslant \|T_u f_n\|_q^q = \|u \star f_n\|_q^q = \|uf_n\|_q^q \\ &= \int_X |uf_n|^q \,\mathrm{d}\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} |u|^q \,\mathrm{d}\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} E(|u|^q)\,\mathrm{d}\mu \\ &\geqslant \delta\mu(F_n)/(\mu(F_n)^{q/p}) = \delta\Big(\frac{\mu(F)}{2^n}\Big)^{1-q/p} = \delta\Big(\frac{2^n}{\mu(F)}\Big)^{q/p-1} \to \infty \quad \text{as } n \to \infty,\end{aligned}$$

which is a contradiction. Hence we conclude that $\mu(\{x \in B\colon E(|u|^q)(x) \neq 0\}) = 0$. Next, we examine the supremum in (ii). For any $n \in \mathbb{N}$, put $f_n = 1/(\mu(A_n)^{1/p})\chi_{A_n}$. Then it is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Hence we have

$$\begin{aligned}\infty > \|T_u\|^q \geqslant \|T_u f_n\|_q^q &= \frac{1}{\mu(A_n)^{q/p}} \int_{A_n} E(|u|^q)\,\mathrm{d}\mu \\ &= \frac{1}{\mu(A_n)^{q/p}} E(|u|^q)(A_n)\mu(A_n) = \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/r}}.\end{aligned}$$

Since this holds for any $n \in \mathbb{N}$, we get that $M < \infty$. □

Theorem 2.4.

(i) *$u \in K^{\star}_{\infty}$ if and only if $u \in L^{\infty}(\Sigma)$.*

(ii) *If $1 \leqslant q < \infty$, then $u \in K^{\star}_{\infty,q}$ if and only if $|u| \in L^q(\Sigma)$.*

(iii) *If $1 \leqslant p < \infty$, then $u \in K^{\star}_{p,\infty}$ if and only if $u = 0$ a.e. on B and*

$$\sup_{n\in\mathbb{N}}(|u|^p(A_n)/\mu(A_n)) < \infty.$$

P r o o f. (i) Suppose that for each $f \in L^{\infty}(\Sigma)$, $u \star f \in L^{\infty}(\Sigma)$. Since the conditional expectation operator E is a contraction, we obtain

$$\|u\|_{\infty} = \|u\chi_X\|_{\infty} = \|T_u\chi_X\|_{\infty} \leqslant \|T_u\| < \infty.$$

Conversely, suppose that $u \in L^{\infty}(\Sigma)$. Then for each $f \in L^{\infty}(\Sigma)$ we have $\|T_u f\|_{\infty} \leqslant 3\|u\|_{\infty}\|f\|_{\infty}$. Thus $\|T_u\| \leqslant 3\|u\|_{\infty}$ and hence $u \in K^{\star}_{\infty}$. Consequently, we get (i).

(ii) Let $|u| \in L^q(\Sigma)$ and $f \in L^{\infty}(\Sigma)$. Then we have

$$\|uE(f)\|_q^q = \int_X |uE(f)|^q \,\mathrm{d}\mu \leqslant \|f\|_{\infty}^q \int_X |u|^q \,\mathrm{d}\mu = \|f\|_{\infty}^q \|u^q\|_q^q.$$

Hence, $\|uE(f)\|_q \leqslant \|f\|_{\infty}\|u\|_q$. Similarly, we get $\|uE(f)\|_q \leqslant \|f\|_{\infty}\|u\|_q$ and $\|E(u)E(f)\|_q \leqslant \|f\|_{\infty}\|u\|_q$. Thus $\|T_u\| \leqslant 3\|u\|_q$ and hence $u \in K^{\star}_{\infty,q}$. Conversely, suppose that $T_u(L^{\infty}(\Sigma)) \subseteq L^q(\Sigma)$. Since $T_u\chi_X \in L^q(\Sigma)$, it follows that

$$\infty > \|T_u\chi_X\|_q^q = \int_X |T_u\chi_X|^q \,\mathrm{d}\mu = \int_X |u|^q \,\mathrm{d}\mu = \|u\|_q^q.$$

Thus we get (ii).

(iii) Suppose that $u = 0$ a.e. on B and $M := \sup\limits_{n\in\mathbb{N}}(|u|^p(A_n)/\mu(A_n)) < \infty$. Then for each $f \in L^p(\Sigma)$ with $\|f\|_p \leqslant 1$ we have

$$\begin{aligned}
\|uE(f)\|_{\infty}^p &= \inf\{b \geqslant 0\colon\ |uE(f)|^p \leqslant b\} \\
&= \inf\{b \geqslant 0\colon\ |u|^p|E(f)|^p \leqslant b\} \\
&= \inf\{b \geqslant 0\colon\ |u|^p(A_n)|E(f)(A_n)|^p \leqslant b,\ n \in \mathbb{N}\} \\
&\leqslant \inf\{b \geqslant 0\colon\ |u|^p(A_n)(E|f|^p)(A_n) \leqslant b,\ n \in \mathbb{N}\} \\
&\leqslant \sup_{n\in\mathbb{N}} \frac{|u|^p(A_n)}{\mu(A_n)} = M < \infty.
\end{aligned}$$

Consequently, $\|uE(f)\|_{\infty} \leqslant M^{1/p}$. Similarly, since

$$|u(A_n)|^p = \frac{1}{\mu(A_n)} \int_{A_n} |u|^p \,\mathrm{d}\mu = \frac{1}{\mu(A_n)} \int_{A_n} E(|u|^p) \,\mathrm{d}\mu = (E(|u|^p))(A_n),$$

we get that $\|fE(u)\|_\infty \leqslant M^{1/p}$ and $\|E(u)E(f)\|_\infty \leqslant M^{1/p}$. Therefore $\|T_u\| \leqslant 3M^{1/p}$ and hence $u \in K^\star_{p,\infty}$.

Conversely, suppose that $u \in K^\star_{p,\infty}$. First we show that $u = 0$ a.e. on B. Assuming the contrary, we can find $\delta > 0$ such that $\mu(\{x \in X\colon |u(x)| \geqslant \delta\}) > 0$. Put $F = \{x \in X\colon |u(x)| \geqslant \delta\}$. Since F is non atomic, choose a number a such that $0 < a < \mu(F)$ and a sequence $F_1, F_2, \ldots \in \mathcal{A}$ of disjoint subsets of F such that $\mu(F_k) = a/2^{pk}$ for all $k \in \mathbb{N}$. We define a function f_0 on X by

$$f_0 = \sum_{k=1}^{\infty} 2^{k/2p} \chi_{F_k}.$$

It is easy to show that $f_0 \in L^p(\mathcal{A})$, but that it is not in $L^\infty(\mathcal{A})$. It follows that

$$\infty = \delta^{1/p}\|f_0\|_{L^\infty(\mathcal{A})} = \|\delta^{1/p} f_0\|_{L^\infty(\mathcal{A})} \leqslant \|T_u f_0\|_{L^\infty(\mathcal{A})} \leqslant \|T_u\|\|f_0\|_{L^p(\mathcal{A})} < \infty,$$

which is a contradiction. Hence $\mu(\{x \in X\colon |u(x)| \neq 0\}) = 0$, in other words, $u = 0$ a.e. on B.

Now, for any $n \in \mathbb{N}$, put $f_n = 1/(\mu(A_n)^{1/p})\chi_{A_n}$. It is clear that for all $n \in \mathbb{N}$, $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Then we obtain

$$\infty > \|T_u\|^p \geqslant \|T_u f_n\|_\infty^p = \|u f_n\|_\infty^p \geqslant \frac{|u|^p(A_n)}{\mu(A_n)}.$$

Therefore $M < \infty$. This complete the proof. □

3. Fredholmness of $\star$-multiplication operators

Proposition 3.1. *Let $1 \leqslant p < \infty$, $1/p + 1/q = 1$, and $u \in K^\star_p$. Then, for each $g \in L^p(\Sigma)$, $f \in L^q(\Sigma)$ and $n \in \mathbb{N}$ we have*

(i) $T_u^n g = (E(u))^{n-1}(E(u)g + nuE(g) - nE(u)E(g))$,

(ii) $T_u^{*n} f = (\overline{E(u)})^{n-1}\{nE(\bar{u}f) + \overline{E(u)}(f - nE(f))\}$.

Proof. (i) is trivial.

(ii) We will prove the result by induction. Since $E(g)f = fE(g)$ for each $g \in L^p(\Sigma)$ and $f \in L^q(\Sigma)$, we have

$$\begin{aligned}(g, T_u^* f) = (T_u g, f) &= \int (uE(g) + gE(u) - E(g)E(u))\overline{f}\,\mathrm{d}\mu \\ &= \int (gE(u\overline{f}) + E(u)g\overline{f} - gE(u)E(\overline{f}))\,\mathrm{d}\mu \\ &= \int g\Big(\overline{E(\bar{u}f) + \overline{E(u)}f - \overline{E(u)}E(f)}\Big)\,\mathrm{d}\mu \\ &= \big(g, E(\bar{u}f) + \overline{E(u)}f - \overline{E(u)}E(f)\big),\end{aligned}$$

which shows that the result holds for $n = 1$. Assume now that it holds for $n = k$ and calculate

$$\begin{aligned} T_u^{*(k+1)} f &= T_u^*\big((\overline{E(u)})^{k-1}\{kE(\bar{u}f) + \overline{E(u)}(f - kE(f))\}\big) \\ &= (\overline{E(u)})^k\{(k+1)E(\bar{u}f) - kE(f)\overline{E(u)}\} \\ &\quad + (\overline{E(u)})^k\{kE(\bar{u}f) + \overline{E(u)}(f - kE(f))\} \\ &\quad - (\overline{E(u)})^k\{kE(\bar{u}f) - (k-1)\overline{E(u)}E(f)\} \\ &= (\overline{E(u)})^k\{(k+1)E(\bar{u}f) + \overline{E(u)}\big(f - (k+1)E(f)\big)\}. \end{aligned}$$

Thus the proposition is proved. □

In what follows we use the symbols $\mathcal{N}(T_u)$ and $\mathcal{R}(T_u)$ to denote the kernel and the range of T_u, respectively. We recall that T_u is said to be a Fredholm operator if $\mathcal{R}(T_u)$ is closed, $\dim \mathcal{N}(T_u) < \infty$, and $\operatorname{codim} \mathcal{R}(T_u) < \infty$.

The next result gives a necessary and sufficient condition for a $\star$-multiplication operator T_u on $L^p(\Sigma)$ to be a Fredholm operator, thereby generalizing the result in [11] for multiplication operators.

Theorem 3.2. *Suppose that $u \in K_p^\star$ and $\mathcal{A}$ is a non-atomic measure space. Then the operator T_u is Fredholm on $L^p(\Sigma)$ $(1 \leqslant p < \infty)$ if and only if $|E(u)| \geqslant \delta$ almost everywhere on X for some $\delta > 0$.*

P r o o f. Suppose that T_u is a Fredholm operator. We first claim that T_u is onto. Suppose the contrary. Then there exists $f_0 \in L^p(\Sigma) \setminus \mathcal{R}(T_u)$. Since $\mathcal{R}(T_u)$ is closed, there exists $g_0 \in L^q(\Sigma)$, the dual space of $L^p(\Sigma)$, such that

$$(g_0, f_0) = \int \bar{f}_0 g_0 \, \mathrm{d}\mu = 1 \tag{3.1}$$

and

$$(g_0, T_u f) = \int g_0 \overline{T_u f} \, \mathrm{d}\mu = 0, \quad f \in L^p(\Sigma). \tag{3.2}$$

Now (3.1) yields that the set $B_r = \{x \in X \colon |E(\bar{f}_0 g_0)(x)| \geqslant r\}$ has positive measure for some $r > 0$. As $\mathcal{A}$ is non-atomic, we can choose a sequence $\{A_n\}$ of subsets of B_r with $0 < \mu(A_n) < \infty$ and $A_m \cap A_n = \emptyset$ for $m \neq n$. Put $g_n = \chi_{A_n} g_0$. Clearly, $g_n \in L^q(\Sigma)$ and is nonzero, because

$$\int_X |\bar{f}_0 g_n| \, \mathrm{d}\mu \geqslant \int_{A_n} |\bar{f}_0 g_n| \, \mathrm{d}\mu = \int_{A_n} E(|\bar{f}_0 g_0|) \geqslant \int_{A_n} |E(\bar{f}_0 g_0)| \, \mathrm{d}\mu \geqslant r\mu(A_n) > 0$$

for each n. Also, for each $f \in L^p(\Sigma)$, $\chi_{A_n} f \in L^p(\Sigma)$ and so (3.2) implies that

$$(T_u^* g_n, f) = (g_n, T_u f) = \int_{A_n} g_0 \overline{T_u f} \, \mathrm{d}\mu = \int_X g_0 \overline{T_u(\chi_{A_n} f)} \, \mathrm{d}\mu = (g_0, T_u(\chi_{A_n} f)),$$

which implies that $T_u^* g_n = 0$ and so $g_n \in \mathcal{N}(T_u^*)$. Since all the sets in $\{A_n\}$ are disjoint, the sequence $\{g_n\}$ forms a linearly independent subset of $\mathcal{N}(T_u^*)$. This contradicts the fact that $\dim \mathcal{N}(T^* u) = \operatorname{codim} \mathcal{R}(T_u) < \infty$. Hence T_u is onto. Let $Z(E(u)) := \sigma(E(u))^c = \{x \in X \colon E(u)(x) = 0\}$. Then $\mu(Z(E(u))) = 0$. Since $\mu(Z(E(u))) > 0$, there is an $F \subseteq Z(E(u))$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(T_u)$, then there exists $f \in L^p(\Sigma)$ such that $T_u f = \chi_F$. Then

$$\mu(F) = \int_X \chi_F \, \mathrm{d}\mu = \int_F T_u f \, \mathrm{d}\mu = \int_F E(T_u f) \, \mathrm{d}\mu = \int_F E(u) E(f) \, \mathrm{d}\mu = 0,$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma) \setminus \mathcal{R}(T_u)$, which contradicts the fact that T_u is onto. For each $n = 1, 2, \ldots$, let

$$H_n = \left\{ x \in X \colon \frac{\|E(|u|^p)\|_\infty}{(n+1)^2} < |E(u)|^p(x) \leqslant \frac{\|E(|u|^p)\|_\infty}{n^2} \right\}$$

and $H = \{n \in \mathbb{N} \colon \mu(H_n) > 0\}$. Then the H_n's are pairwise disjoint, $X = \bigcup_{n=1}^{\infty} H_n$ and $\mu(H_n) < \infty$ for each $n \geqslant 1$. Take

$$f(x) = \begin{cases} \dfrac{|E(u)|}{\mu(H_n)^{1/p}}, & x \in H_n, \ n \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_X |f|^p \, \mathrm{d}\mu = \sum_{n \in H} \int_{H_n} \frac{|E(u)|^p}{\mu(H_n)} \, \mathrm{d}\mu \leqslant \sum_{n \in H} \frac{\|E(|u|^p)\|_\infty}{n^2} < \infty.$$

Therefore $f \in L^p(\mathcal{A})$ and so there exist $g \in L^p(\Sigma)$ such that $T_u g = f$. Hence $E(u)E(g) = E(T_u g) = f$. Since $E(g) = f / E(u)$ off $Z(E(u))$ and $\mu(Z(E(u))) = 0$, it follows that

$$\begin{aligned} \int_X |g|^p \, \mathrm{d}\mu = \int_X E(|g|^p) \, \mathrm{d}\mu \geqslant \int_X |E(g)|^p \, \mathrm{d}\mu \\ = \int_X \frac{|f|^p}{|E(u)|^p} \, \mathrm{d}\mu = \sum_{n \in H} \int_{H_n} \frac{1}{\mu(H_n)} \, \mathrm{d}\mu = \sum_{n \in H} 1. \end{aligned}$$

This implies that H must be a finite set. So there is an n_0 such that $n \geqslant n_0$ implies $\mu(H_n) = 0$. Together with $\mu(Z(E(u))) = 0$, we obtain

$$\mu\left(\left\{x \in X\colon\ |E(u)|^p(x) \leqslant \frac{\|E(|u|^p)\|_\infty}{n_0^2}\right\}\right) = \mu\left(\bigcup_{n=n_0}^{\infty} H_n \cup Z(E(u))\right) = 0,$$

that is $|E(u)| \geqslant ((\|E(|u|^p)\|_\infty)/n_0^2)^{1/p} := \delta$ almost everywhere on X.

Conversely, suppose that $|E(u)| \geqslant \delta$ a.e. on X for some $\delta > 0$. Let $f \in \mathcal{N}(T_u^*)$. We have $T_u^* f = E(\bar{u}f) + \overline{E(u)}(f - E(f)) = 0$ and so $E(\bar{u}f) = E(T_u^* f) = 0$. Thus

$$\int_X \bar{u}f \,\mathrm{d}\mu = \int_X E(\bar{u}f)\,\mathrm{d}\mu = 0,$$

which implies that

$$\mathcal{N}(T_u^*) \subseteq \left\{f \in L^p(\Sigma)\colon\ \int_X \bar{u}f\,\mathrm{d}\mu = 0\right\} \subseteq L^p(Z(u), \Sigma_{Z(u)}, \mu|_{Z(u)}).$$

Also, since $E(|u|) \geqslant |E(u)| \geqslant \delta$ and X is a σ-finite measure space, we have $|u| \geqslant \delta$ and hence $\mu(Z(u)) = 0$. It follows that

$$\operatorname{codim} \mathcal{R}(T_u) = \dim \mathcal{N}(T_u^*) = 0.$$

Now, we shall show that T_u has closed range. Let $\{T_u f_n\}$ be an arbitrary sequence in $\mathcal{R}(T_u)$ and let $\|T_u f_n - g\|_p \to 0$ for some $g \in L^p(\Sigma)$. Hence we have $E(u)E(f_n) = E(T_u f_n) \xrightarrow{L^p} E(g)$. Since by hypothesis $|E(u)| \geqslant \delta$, it follows that $E(g)/E(u) \in L^p(\mathcal{A})$ and $E(f_n) \xrightarrow{L^p} E(g)/E(u)$. Consequently,

$$f_n \xrightarrow{L^p} \frac{1}{E(u)}\left\{g + E(g) - \frac{uE(g)}{E(u)}\right\} := f$$

and hence $T_u f_n \xrightarrow{L^p} T_u f$. Therefore $g = T_u f$, which implies that T_u has closed range. Thus the theorem is proved. □

Now, we consider the particular case when $p = 2$. An operator T on a Hilbert space H is normal if $TT^* = T^*T$, and T is self-adjoint if $T = T^*$.

Proposition 3.3. *Let $u \in K_2^\star$. Then the following claims are true:*

(i) *T_u is a normal operator if and only if $u \in L^\infty(\mathcal{A})$.*

(ii) *T_u is a self-adjoint operator if and only if $u \in L^\infty(\mathcal{A})$ is real valued.*

Proof. (i) Assume T_u is normal. Then for each $f \in L^2(\Sigma)$ we have $E(T_uT_u^*f) = E(u)E(\bar{u}f)$ and $E(T_u^*T_uf) = E(f)E(|u|^2) + E(u)E(\bar{u}f) - E(\bar{u})E(u)E(f)$. Therefore we obtain that $E(|u|^2) = |E(u)|^2$. Consequently $u \in L^\infty(\mathcal{A})$. Conversely, suppose that $u \in L^\infty(\mathcal{A})$ and take $f \in L^2(\Sigma)$. Then $T_u^*T_uf = T_uT_u^*f = |u|^2f$, and hence T_u is normal.

(ii) follows from (i). □

Example 3.4. Let $X = [-1, 1]$, $\mathrm{d}\mu = \mathrm{d}x$, let Σ be the Lebesgue sets, and $\mathcal{A}$ the σ-subalgebra generated by the sets symmetric about the origin. Put $0 < a \leqslant 1$. Then for each $f \in L^2(\Sigma)$ we have

$$\begin{aligned}\int_{-a}^{a} E(f)(x)\,\mathrm{d}x &= \int_{-a}^{a} f(x)\,\mathrm{d}x \\ &= \int_{-a}^{a} \Big\{\frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}\Big\}\,\mathrm{d}x \\ &= \int_{-a}^{a} \frac{f(x)+f(-x)}{2}\,\mathrm{d}x.\end{aligned}$$

Consequently, $(Ef)(x) = (f(x)+f(-x))/2$. Now, if we take $u(x) = \cos x + \sin x$, then the $\star$-multiplication operator $T_u\colon L^2(\Sigma) \to L^2(\Sigma)$ has the form

$$(T_uf)(x) = \Big(\cos x + \frac{1}{2}\sin x\Big)f(x) + \frac{1}{2}\sin x f(-x).$$

Direct computation shows that $(T_u^*f)(x) = \big(\cos x + \sin x/2\big)f(x) - \sin x/2 f(-x)$ and $|E(u)| \geqslant \cos 1$. Therefore, T_u is a Fredholm but not a normal operator. □

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