A CONDITIONAL EXPECTATION TYPE OPERATOR ON L^p SPACES

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Abstract. In this paper we discuss some of the basic operator-theoretic characterizations for conditional expectation type operator $T = EM_u$ on L^p spaces.

1. Introduction and Preliminaries

Let $L(X, \Sigma, \mu)$ be a σ-finite measure space. For any complete σ-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. We understand $L^p(\mathcal{A})$ as a Banach subspace of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

For any non-negative Σ-measurable function f as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E(f)$ such that

$$\int_A Ef d\mu = \int_A f d\mu, \qquad \text{for all } A \in \mathcal{A}.$$

Hence we obtain an operator E from $L^p(\Sigma)$ onto $L^p(\mathcal{A})$ which is called conditional expectation operator associated with the σ-algebra $\mathcal{A}$. This operator will play a major role in our work, and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$.
- $|E(f)|^p \leq E(|f|^p)$.
- $\|E(f)\|_p \leq \|f\|_p$.
- If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$.

Let f be a real-valued measurable function. Consider the set $B_f = \{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$. The function f is said to be conditionable with respect to $\mathcal{A}$, if $\mu(B_f) = 0$. If f is complex-valued, then f is conditionable if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. We denote the linear space of all conditionable Σ-measurable functions on X by $L^0(\Sigma)$. It is known that $|E(f)|^2 = E(|f|^2)$ if and only if $f \in L^0(\mathcal{A})$. For more details on the properties of E see [5], [6] and [9].

Recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A

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measure with no atoms is called non-atomic. It is well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned uniquely as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [12]). Note that since $\mathcal{A}$ is σ- finite, it follows that $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$.

Combination of conditional expectation operator E and multiplication operator M_u appears more often in the service of the study of other operators such as multiplication operators, weighted composition operators and Lambert operators (see [8] and [7]). These operators are closely related to averaging operators on order ideals in Banach lattices and to operators called conditional expectation-type operators introduced in [1]. In this paper, we investigate some of the basic operator-theoretic questions for the conditional type operator $T = EM_u$ between L^p spaces. For a beautiful exposition of the study of weighted conditional expectation operators on L^p-spaces, see [6] and the references therein.

2. The Operator $\mathbf{T} = \mathbf{EM_u}$

Let $1 \leq p \leq \infty$. We shall always take $u \in L^0(\Sigma)$ for which $uf \in L^0(\Sigma)$ for all $f \in L^p(\Sigma)$. In other words, the operator $T = EM_u$ is defined on all $L^p(\Sigma)$. A straightforward calculation shows that for $1 \leq p < \infty$, the adjoint operator $T^* : L^q(\mathcal{A}) \to L^q(\Sigma)$ is given by $T^*f = \bar{u}f$, where $\frac{1}{p} + \frac{1}{q} = 1$ (note that we can consider $T^* : L^q(\Sigma) \to L^q(\Sigma)$ as $T^* = M_{\bar{u}}E$). Let $1 \leq q < \infty$. It is proved by Alan Lambert in [8] that T^* is a bounded operator if and only if $E(|u|^q) \in L^\infty(\mathcal{A})$. In this case $\|T^*\| = \|E(|u|^q)\|_\infty^{1/q}$. In the case $q = \infty$, we claim that T^* is bounded if and only if $u \in L^\infty(\Sigma)$ and its norm is given by $\|T^*\| = \|u\|_\infty$. Indeed, if $u \in L^\infty(\Sigma)$ and $f \in L^\infty(\mathcal{A})$, we have

$$\|\bar{u}f\|_{L^\infty(\mathcal{A})} = \sup_{A\in\mathcal{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |\bar{u}f| d\mu$$

$$\leq \|u\|_\infty \sup_{A\in\mathcal{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |f| d\mu = \|u\|_\infty \|f\|_{L^\infty(\mathcal{A})}.$$

It follows that $T^*(L^\infty(\mathcal{A})) \subseteq L^\infty(\mathcal{A}) \subseteq L^\infty(\Sigma)$, and $\|T^*\| \leq \|u\|_\infty$. On the other hand, if T^* is bounded, then

$$\|u\|_\infty = \|\bar{u}\chi_X\|_\infty = \|T^*\chi_X\|_\infty \leq \|T^*\| < \infty.$$

These observations establish the following proposition.

Proposition 2.1. (a) $T = EM_u$ defines a bounded linear operator from $L^1(\Sigma)$ into $L^1(\mathcal{A})$ if and only if $u \in L^\infty(\Sigma)$. In this case $\|T\| = \|u\|_\infty$.

(b) Let $1 < p < \infty$. T defines a bounded operator from $L^p(\Sigma)$ into $L^p(\mathcal{A})$ if and only if $E(|u|^q) \in L^\infty(\mathcal{A})$, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case $\|T\| = \|E(|u|^q)\|_\infty^{1/q}$.

In the following theorem we investigate a necessary and sufficient condition for T to be compact.

Theorem 2.2. Let $1 < p < \infty$. Suppose $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$. Then the bounded linear operator $T = EM_u$ from $L^p(\Sigma)$ into $L^p(\mathcal{A})$ is compact if and only if $u(B) = 0$ ($u(x) = 0$ for μ-almost all $x \in B$) and for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \mu(A_n \cap D_\varepsilon(u)) > 0\}$ is finite, where $D_\epsilon(u) = \{x \in X : E(|u|)(x) \geq \varepsilon\}$.

Proof. Suppose T is a compact operator. First we show that $u(B) = 0$. Suppose the contrary i.e., $\mu\{x \in B : u(x) \neq 0\}) > 0$. Then there is $\delta > 0$ and $B_0 \in \mathcal{A} \cap B$ such that $0 < \mu(B_0 \cap D_\delta(u)) < \infty$. Since $J_0 := B_0 \cap D_\delta(u) \in \mathcal{A} \cap B_0$ has no atoms, hence we can choose a sequence $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \cap B_0$, such that $J_{n+1} \subseteq J_n \subseteq J_0$, $0 < \mu(J_{n+1}) = \frac{\mu(J_n)}{2}$, where $J_n := B_n \cap D_\delta(u)$. Note that for all $n \in \mathbb{N}$, J_n is $\mathcal{A}$-measurable. Put

$$f_n = \frac{\bar{u}|u|^{\frac{q-p}{p}}\chi_{J_n}}{\{\|E(|u|^q)\|_\infty \mu(J_n)\}^{\frac{1}{p}}}, \qquad n \in \mathbb{N}.$$

Boundedness of T implies that $E(|u|^q) \in L^\infty(\mathcal{A})$ and hence $\|f_n\|_p \leq 1$. Now, for any $m, n \in \mathbb{N}$ with $m > n$ we have

$$\|Tf_n - Tf_m\|_p^p = \int_X |E(u(f_n - f_m))|^p d\mu$$

$$\int_X \frac{[E(|u|^{\frac{q}{p}+1})]^p}{\|E(|u|^q)\|_\infty} \left| \frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{p}}} - \frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{p}}} \right|^p d\mu \geq \frac{\delta^{(\frac{q}{p}+1)p}}{\|E(|u|^q)\|_\infty} \int_{J_n \setminus J_m} \frac{d\mu}{\mu(J_n)}$$

$$= \frac{\delta^{q+p}}{\|E(|u|^q)\|_\infty} \frac{\mu(J_n \setminus J_m)}{\mu(J_n)} = \frac{\delta^{q+p}}{\|E(|u|^q)\|_\infty} \left(1 - \frac{\mu(J_m}{\mu(J_n)}\right) > \frac{\delta^{q+p}}{2\|E(|u|^q)\|_\infty},$$

which shows that the sequence $\{Tf_n\}_{n\in\mathbb{N}}$ dose not contain a convergent subsequence. But this is a contradiction.

Now, we show that for any $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \mu(A_n \cap D_\varepsilon(u)) > 0\}$ is finite. By the way of contradiction, for some $\varepsilon > 0$, there is a subsequence $\{A_k\}_{k\in\mathbb{N}}$ of disjoint atoms in $\mathcal{A}$ such that $\mu(A_k \cap D_\varepsilon(u)) > 0$, for all $k \in \mathbb{N}$. Put $G_k = A_k \cap D_\varepsilon(u)$. Hence, we obtain a sequence of pairwise disjoint sets $\{G_k\}_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $G_k \in \mathcal{A}$ and $0 < \mu(G_k) = \mu(A_k) < \infty$. For any $k \in \mathbb{N}$, take $f_n = \bar{u}|u|^{\frac{q-p}{p}}\chi_{G_n}/(\|E(|u|^q)\|_\infty \mu(G_n))^{1/p}$. Then $\|f_n\|_p \leq 1$. Since for each $n \neq m$, $G_n \cap G_m = \emptyset$, it follows that

$$\|Tf_n - Tf_m\|_p^p \geq \int_X \frac{(E(|u|))^{q+p}\chi_{G_n}}{\|E(|u|^q)\|_\infty \mu(G_n)} d\mu + \int_X \frac{(E(|u|))^{q+p}\chi_{G_m}}{\|E(|u|^q)\|_\infty \mu(G_m)} d\mu \geq \frac{2\varepsilon^{q+p}}{|E(|u|^q)\|_\infty},$$

which contradicts the compactness of T.

Conversely, suppose that $u(B) = 0$ and for an arbitrary $\varepsilon > 0$, there exist at most finite $\mathcal{A}$-atoms $\{A_\varepsilon^k\}_{k=1}^n \subseteq \{A_n\}_{n\in\mathbb{N}}$ such that $\mu(A_\varepsilon^k \cap D_\varepsilon(u)) > 0$. Put $B_\varepsilon = \cup_{k=1}^n A_\varepsilon^k$. Then $E(|u|) < \varepsilon$ on $X \setminus B_\varepsilon$ and hence $|u| < \varepsilon$ on $X \setminus (B_\varepsilon \cup B)$. Set

$v = \chi_{B_\varepsilon} u$ and $T_1 = EM_v$. It is easy to see that $u = v = 0$ on B and $u = v$ on B_ε. Now, since $B_\varepsilon \cup B \in \mathcal{A}$, then foe each $f \in L^p(\Sigma)$ we have that

$$\|(T - T_1)f\|_p^p = \int_X |E(u-v)f|^p d\mu = \int_{X\setminus(B_\varepsilon \cup B)} |E(uf)|^p d\mu$$

$$\leq \int_{X\setminus(B_\varepsilon \cup B)} E(|uf|^p) d\mu = \int_{X\setminus(B_\varepsilon \cup B)} |uf| d\mu \leq \varepsilon^p \int_X |f|^p d\mu = \varepsilon^p \|f\|_p^p.$$

On the other hand, we have

$$T_1 f = E(\chi_{B_\varepsilon} uf) = E(\sum_{k=1}^{n} \chi_{A_\varepsilon^k} uf) = \sum_{k=1}^{n} E(\chi_{A_\varepsilon^k} uf)$$

$$= \sum_{k=1}^{n} E(uf)(A_\varepsilon^k)\chi_{A_\varepsilon^k} = \sum_{k=1}^{n} (Tf)(A_\varepsilon^k)\chi_{A_\varepsilon^k}.$$

Therefore, T_1 has finite rank and hence T is compact.

Remark 2.3. Under the same assumptions as in Theorem 2.2, if we take $f_n = \bar{u}\chi_{J_n}/(\|u\|_\infty \mu(J_n))$, then by the same method used in the proof of Theorem 2.2, $T = EM_u$ from $L^1(\Sigma)$ into $L^1(\mathcal{A})$ is compact if and only if $u(B) = 0$ and for any $\varepsilon > 0$, the set $\{x \in X : E(|u|)(x) \geq \varepsilon\}$ consists of finitely many atoms.

In the following theorem we show that if $T = EM_u$ is weakly compact on $L^1(\Sigma)$, then it is compact. Recall that the operator $T : L^1(\Sigma) \to L^1(\Sigma)$ is said to be weakly compact if it maps bounded subsets of $L^1(\Sigma)$ into weakly sequentially compact subsets of $L^1(\Sigma)$. We begin with the following lemma, which can be deduced from Theorem IV.8.9, and its Corollaries 8.10, 8.11 in [4].

Lemma 2.4. Let H be a weakly sequentially compact set in $L^1(\Sigma)$. Then for each decreasing sequence $\{E_n\}$ in Σ such that $\lim_{n\to\infty} \mu(E_n) = 0$ or $\cap_{n=1}^{\infty} E_n = \emptyset$, the sequence of integrals $\{\int_{E_n} |h| d\mu\}$ converges to zero uniformly for h in H.

Theorem 2.5. Suppose (X, Σ, μ) can be partitioned as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$. Then the bounded operator $T = EM_u$ is a weakly compact operator on $L^1(\Sigma)$ if and only if it is compact.

Proof. It suffices to show the " only if " part. To prove the theorem, we use the method which inspired by Takagi [10]. Let T be a weakly compact operator on $L^1(\Sigma)$. We first show that $u(B) = 0$. To obtain a contradiction, we may assume that for some $\delta > 0$ and $B_0 \subseteq B$, $0 < \mu(B_0 \cap D_\delta(u)) < \infty$. By the same argument in the proof of Theorem 2.2, as B_0 is non-atomic, we can find a decreasing sequence $\{B_n\} \subseteq B_0 \cap \Sigma$ with $0 < \mu(B_n) < \frac{1}{n}$ and $0 < \mu(B_n \cap D_\delta(u)) < \infty$. Let U be the closed unit ball of $L^1(\Sigma)$. Since $T(U)$ is weakly sequentially compact, we can apply Lemma 2.4, with $H = T(U)$ and $E_n = B_n$. Choose $\varepsilon = \delta^2/\|u\|_\infty$. Then there exists an $n_o \in \mathbb{N}$ such that

$$\int_{B_{n_o}} |Tf| d\mu < \frac{\delta^2}{\|u\|_\infty}, \qquad f \in U. \tag{2.1}$$

On the other hand if we take $f_{n_o} = \bar{u}\chi_{J_{n_o}}/(\|u\|_\infty \mu(J_{n_o}))$, we have

$$\int_{B_{n_o}} |Tf|d\mu = \int_{B_{n_o}} |E\left(\frac{u\bar{u}\chi_{J_{n_o}}}{\|u\|_\infty \mu(J_{n_o})}\right)|d\mu$$

$$= \int_{B_{n_o}} E\left(\frac{|u|^2\chi_{J_{n_o}}}{\|u\|_\infty \mu(J_{n_o})}\right)d\mu = \frac{1}{\|u\|_\infty \mu(J_{n_o})}\int_{B_{n_o}} |u|^2\chi_{J_{n_o}}d\mu$$

$$= \frac{1}{\|u\|_\infty \mu(J_{n_o})}\int_{J_{n_o}} |u|^2 d\mu \geq \frac{\delta^2}{\|u\|_\infty}.$$

Since $f_{n_o} \in U$, this contradicts (2.1). According to the Theorem 2.2, it remains to show that for any $\varepsilon > 0$, the set $\mathrm{A} := \{n \in \mathbb{N} : \mu(A_n \cap D_\varepsilon(u)) > 0\}$ is finite. To this end, without loss of generality, we can assume that $\mathrm{A} = \mathbb{N}$ for some $\varepsilon > 0$. Put $\mathrm{K}_n = \{A_k : k \geq n\}$. It follows that $\cap_{n=1}^\infty K_n = \emptyset$. Applying Lemma 2.4 once more, there exists an $\mathrm{N} \in \mathbb{N}$ such that

$$\int_{\mathrm{K_N}} |Tf|d\mu < \frac{\varepsilon^2}{\|u\|_\infty}, \qquad f \in U.$$

Now, for any n with $n \geq \mathrm{N}$, let $g_n = \bar{u}\chi_{A_n}/(\|u\|_\infty \mu(A_n))$. Then we have

$$\int_{\mathrm{K}_N} |Tg_n|d\mu = \int_{\mathrm{K}_N} E\left(\frac{|u|^2\chi_{A_n}}{\|u\|_\infty \mu(A_n)}\right)d\mu = \frac{1}{\|u\|_\infty \mu(A_n)}\int_{A_n} |u|^2 d\mu \geq \frac{\varepsilon^2}{\|u\|_\infty}.$$

Since $g_n \in U$, this contradicts (2.1). This completes the proof of the theorem.

Corollary 2.6. Let $1 \leq p < \infty$ and $E(|u|) > 0$ a.e. on X. If the bounded operator $T = EM_u : L^p(\Sigma) \to L^p(\mathcal{A})$ is (weakly) compact, then $\mathcal{A}$ is purely atomic.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. The set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{K},\mathcal{H})$. If $\mathcal{H} = \mathcal{K}$, $\mathcal{B}(\mathcal{H},\mathcal{H})$ will be written by $\mathcal{B}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{K},\mathcal{H})$, the range and the null-space of A are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. If $A \in \mathcal{B}(\mathcal{H})$, the spectrum of A is denoted by $\mathrm{Sp}(A)$.

Now, we consider matrix form of $T = EM_u$. Notice that $L^2(\Sigma)$ is the direct sum of the $\mathcal{R}(E) = L^2(\mathcal{A})$ with $\mathcal{N}(E) = \{f - Ef : f \in L^2(\Sigma)\}$. With respect to the direct sum decomposition, $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, the matrix form of T is

$$T = \begin{bmatrix} ETE & ET(I-E) \\ (I-E)TE & (I-E)T(I-E) \end{bmatrix} = \begin{bmatrix} M_{Eu} & EM_u \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

In this sequel, we investigate closedness of range and spectrum of T on $L^2(\Sigma)$. We begin with the following lemma, which can be deduced from Theorem 2.3 in [2] and Example 7 in [3].

Lemma 2.7. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Suppose that $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

(i) If A and B are normal operators, then $\mathrm{Sp}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \mathrm{Sp}(A) \cup \mathrm{Sp}(B)$.

(ii) If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then the range $\mathcal{R}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)$ is closed if and only if at least one of $\dim \mathcal{N}(A^*)$ or $\dim \mathcal{N}(B)$ is finite.

Theorem 2.8. Suppose that the operator $T = EM_u : L^2(\Sigma) \to L^2(\mathcal{A})$ is bounded. Then

(i) $\mathrm{Sp}(T) \cup \{0\} = \text{ess range } \{E(u)\} \cup \{0\}$.
(ii) Let $|E(u)| \geq \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$. Then T has closed range if and only if $|E(u)| > 0$ a.e. on X except at most on finitely many atoms.

Proof. (i) If $\mathcal{A} \neq \Sigma$, then $\mathcal{R}(T) \subseteq L^2(\mathcal{A}) \subset L^2(\Sigma)$. Therefore T is not surjective and so $0 \in \mathrm{Sp}(T)$. On the other hand, by Lemma 2.7 (i), since $\mathrm{Sp}(M_{Eu}) = \text{ess range } \{E(u)\}$, the result holds.

(ii) It is known that the multiplication operator M_{Eu} has closed range if and only if $|E(u)| \geq \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$. Now, by Lemma 2.7 (i) and (2.2) we have:

$$\mathcal{R}(T) \text{ is closed} \Longleftrightarrow \mathcal{R}\left(\begin{bmatrix} M_{Eu} & EM_u \\ 0 & 0 \end{bmatrix}\right) \text{ is closed} \Longleftrightarrow \dim \mathcal{N}(M_{\overline{Eu}}) < \infty$$
$$\Longleftrightarrow |E(u)| > 0 \text{ a.e. on } X \text{ except at most on finitely many atoms.}$$

It is well known that every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(T)$, then this decomposition is called the polar decomposition.

Now, by the operator matrices method we obtain the polar decomposition of $T = EM_u$. Direct computations show that

$$T^*T = \begin{bmatrix} M_{|E(u)|^2} & EM_{u\overline{Eu}} \\ M_{\bar{u}Eu} & M_{\bar{u}}EM_u \end{bmatrix} \text{ and } |T| = \begin{bmatrix} M_{\frac{|E(u)|^2}{\sqrt{E(|u|^2)}}} & EM_{\frac{u\overline{Eu}}{\sqrt{E(|u|^2)}}} \\ M_{\frac{\bar{u}E(u)-|E(u)|^2}{\sqrt{E(|u|^2)}}} & M_{\frac{\bar{u}-\overline{Eu}}{\sqrt{E(|u|^2)}}}EM_u \end{bmatrix}.$$

Then for each $f \in L^2(\Sigma)$ we have that

$$|T|\begin{bmatrix} Ef & f - Ef \end{bmatrix} = \begin{bmatrix} M_{\frac{|E(u)|^2}{\sqrt{E(|u|^2)}}} & EM_{\frac{u\overline{Eu}}{\sqrt{E(|u|^2)}}} \\ M_{\frac{\bar{u}E(u)-|E(u)|^2}{\sqrt{E(|u|^2)}}} & M_{\frac{\bar{u}-\overline{Eu}}{\sqrt{E(|u|^2)}}}EM_u \end{bmatrix} \begin{bmatrix} Ef \\ f - Ef \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\overline{E(u)}E(uf)}{\sqrt{E(|u|^2)}} & \frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}} - \frac{\overline{E(u)}E(uf)}{\sqrt{E(|u|^2)}} \end{bmatrix}.$$

Notice that, since for each conditionable function u, $E(|u|) = 0$ implies that $E(u) = 0 = u$, we used the notational convention of $\frac{u}{\sqrt{E(|u|^2)}}$ for $\frac{u}{\sqrt{E(|u|^2)}}\chi_{\sigma(u)}$.

Now, since the mapping $f \mapsto \begin{bmatrix} Ef & f - Ef \end{bmatrix}$ is an isometric isomorphism from $L^2(\Sigma)$ onto $L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, then we get that $|T|(f) = \frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}}$. Hence for any

$f \in L^2(\Sigma)$, $E(uf) = U(\frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}})$. It is easy to check that $U(f) = \frac{E(uf)}{\sqrt{E(|u|^2)}}$ and U is a partial isometry (see [6]). These calculations establish the following proposition.

Proposition 2.9. The polar decomposition of $T = EM_u$ on $L^2(\Sigma)$ is $U|T|$, where $U = M_{1/\sqrt{E(|u|^2)}}T$ and $|T| = M_{\bar{u}/\sqrt{E(|u|^2)}}T$

Let $p \in (0, \infty)$. Recall that an operator A on a Hilbert space $\mathcal{H}$ is p-hyponormal if $(A^*A)^p \geq (AA^*)^p$; A is ∞-hyponormal if A is p-hyponormal for all p; and A is p-quasihyponormal if $A^*(A^*A)^pA \geq A^*(AA^*)^pA$. For all unit vectors $x \in \mathcal{H}$, if $\||A|^pU|A|^px\| \geq \||A|^px\|^2$, then A is called a p-paranormal operator. By using the property of real quadratic forms (see [11]), A is p-paranormal if and only if

$$|A|^pU^*|A|^{2p}U|A|^p - 2k|A|^{2p} + k^2 \geq 0, \quad \text{for all} \ \ k \geq 0. \tag{2.3}$$

The following lemma is significant amount of consideration for the next computations.

Lemma 2.10. Let $f \in L^2(\Sigma)$ and $Af := \bar{u}E(uf)$. Then for all $p \in (0, \infty)$

$$A^pf = \bar{u}[E(|u|^2)]^{p-1}E(uf).$$

Proof. Suppose $f \in L^2(\Sigma)$, then by induction we obtain

$$A^{\frac{1}{n}}f = \bar{u}[E(|u|^2)]^{\frac{1-n}{n}}E(uf), \qquad n \in \mathbb{N}.$$

Now the reiteration of powers of operator $A^{\frac{1}{n}}$, yields

$$A^{\frac{m}{n}}f = \bar{u}[E(|u|^2)]^{\frac{(1-n)m}{n}}[E(|u|^2)]^{m-1}E(uf), \quad m, n \in \mathbb{N}.$$

Finally, by using of the functional calculus the desired formula is proved.

Lemma 2.11. Let $T = EM_u$ be a bounded operator on $L^2(\Sigma)$. Then T is ∞-hyponormal if and only id $u \in L^\infty(\mathcal{A})$.

Proof. By Lemma 2.10, it is easy to verify that $(T^*T)^p = M_{\bar{u}[E(|u|^2)]^{p-1}}T$ and $(TT^*)^p = M_{[E(|u|^2)]^p}$, for all $0 < p < \infty$. Then we get that $(T^*T)^p \geq (TT^*)^p$ if and only if

$$M_{[E(|u|^2)]^{p-1}}(M_{\bar{u}}T - M_{E(|u|^2)}) \geq 0 \Longleftrightarrow M_{\bar{u}}T - M_{E(|u|^2)} \geq 0,$$

where we have used the fact that $T_1T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1T_2 = T_2T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$. Thus for any $0 < f \in L^2(\mathcal{A})$ we have

$$0 \leq (M_{\bar{u}}Tf - M_{E(|u|^2)}f, f) = \int_X (\bar{u}E(uf) - E(|u|^2)f)\bar{f}d\mu$$
$$= \int_X (\bar{u}E(u) - E(|u|^2))|f|^2d\mu = \int_X (|E(u)|^2 - E(|u|^2))|f|^2d\mu.$$

Since $f > 0$, this gives $|E(u)|^2 \geq E(|u|^2)$. On the other hand we always have $|E(u)|^2 \leq E(|u|^2)$. Hence $u \in L^\infty(\mathcal{A})$. Notice that if $u \in L^\infty(\mathcal{A})$, then it is easy to see that $(T^*T)^p \geq (TT^*)^p$.

Theorem 2.12. Let $T = EM_u$ be a bounded operator on $L^2(\Sigma)$. Then the following are equivalent:

(i) T is ∞-hyponormal.
(ii) T is p-hyponormal.

(iii) T is p-quasihyponormal.
(iv) T is p-paranormal.
(v) $u \in L^\infty(\mathcal{A})$.

Proof. By Lemma 2.11, we complete the proof by showing (iii)$\Leftrightarrow$ (v) and (iv)$\Leftrightarrow$ (v) below.

(iii)$\Leftrightarrow$ (v) By Lemma 2.10, it is easy to verify that $T^*(TT^*)^pT = M_{\bar{u}[E(|u|^2)]^p}T$ and $T^*(T^*T)^pT = M_{\bar{u}|E(u)|^2[E(|u|^2)]^{p-1}}T$. Therefore, $T^*(T^*T)^p \geq T^*(TT^*)^pT$ if and only if $M_{[E(|u|^2)]^{p-1}}(M_{\bar{u}|E(u)|^2-\bar{u}E(|u|^2)}T) \geq 0$. Therefore, for any $0 < f \in L^2(\mathcal{A})$ we have

$$0 \leq \int_X (\bar{u}|E(u)|^2 - \bar{u}E(|u|^2))E(u)|f|^2 d\mu = \int_X (|E(u)|^4 - |E(u)|^2 E(|u|^2))|f|^2 d\mu.$$

It follows that $|E(u)|^2 \geq E(|u|^2)$ and hence $|E(u)|^2 = E(|u|^2)$. Thus $u \in L^\infty(\mathcal{A})$. Conversely, if $u \in L^\infty(\mathcal{A})$, then

$$T^*(T^*T)^pT = T^*(TT^*)^pT = M_{\bar{u}|u|^{2p}}T,$$

which proves the desired implication.

We now prove (iv)$\Leftrightarrow$ (v). Since $|T|(f) = \frac{\bar{u}}{\sqrt[4]{E(|u|^2)}}E(\frac{uf}{\sqrt[4]{E(|u|^2)}})$, by Lemma 2.10 we get that

$$|T|^p(f) = \bar{u}[E(|u|^2)]^{\frac{p-2}{2}}E(uf), \qquad f \in L^2(\Sigma).$$

Also since $U^*(f) = \frac{\bar{u}}{\sqrt{E(|u|^2)}}E(f)$, by a direct computation, we have

$$|T|^pU^*|T|^{2p}U|T|^pf = \bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2E(uf), \qquad f \in L^2(\Sigma).$$

By condition (2.3), T is p-paranormal if and only if

$$k^2 - 2kM_{\bar{u}[E(|u|^2)]^{p-1}}T + M_{\bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2}T \geq 0, \qquad \text{for all } \ k \geq 0$$

$$\Longleftrightarrow M_{\bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2}T \geq (M_{\bar{u}[E(|u|^2)]^{p-1}}T)^2 = M_{\bar{u}[E(|u|^2)]^{2p-2}|E(|u|^2)}T.$$

Therefore, for any $0 < f \in L^2(\mathcal{A})$ we have

$$\int_X |E(u)|^2(E(|u|^2)^{2p-2}\left(|E(u)|^2 - E(|u|^2)\right)|f|^2 d\mu \geq 0.$$

It follows that $|E(u)|^2 \geq E(|u|^2)$ and hence $u \in L^\infty(\mathcal{A})$. Conversely, if $u \in L^\infty(\mathcal{A})$, it is easy to check that condition (2.3) holds for all $k \geq 0$. Hence the proof is complete.

Example 2.13. Let $X = [-1, 1]$, $d\mu = dx$, Σ the Lebesgue sets, and $\mathcal{A}$ the σ-subalgebra generated by the symmetric sets about the origin. Now any real valued function on X can be written uniquely as a sum of an even function and an odd function, one simply uses the functions $f_e(x) = (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Put $0 < a \leq 1$. Then for each $f \in L^2(\Sigma)$ we have $\int_{-a}^a E(f)(x)dx = \int_{-a}^a f_e(x)dx$ and consequently, $Ef = f_e$. This example is due to Alan Lambert [8]. Now, if u is an even and continuous function on X, then $T = EM_u$ is ∞-hyponormal and hence is p-paranormal. Note that if $u(x) = 1 + x$, then T is not p-paranormal.

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