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COMPACT WEIGHTED FROBENIUS-PERRON OPERATORS AND THEIR SPECTRA

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Abstract. In this paper we characterize the compact weighted Frobenius - Perron operator $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$ and determine its spectra. Also, it is shown that every weakly compact weighted Frobenius-Perron operator on $L^1(\Sigma)$ is compact.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space and let $\varphi : X \to X$ be a non-singular transformation, i.e. φ is Σ-measurable function and $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. This assumption about φ just says that the measure $\mu\circ\varphi^{-1}$ is absolutely continuous with respect to the measure μ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual), where $\mu \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A))$ for $A \in \Sigma$. We shall assume that the restriction of μ to σ-subalgebra $\varphi^{-1}(\Sigma)$ of Σ, is σ-finite, and we denote by $(X, \varphi^{-1}(\Sigma), \mu)$ the completion of $(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$. We denote by h the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$. We will write $L^1(\varphi^{-1}(\Sigma))$ for $L^1(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$. $L^1(\varphi^{-1}(\Sigma))$ may then be viewed as a subspace of $L^1(\Sigma)$ and denote its norm by $\|.\|_1$. Support of a measurable function f will be denoted by $\mathrm{supp}(f) = \{x \in X; f(x) \neq 0\}$. Relationships

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between functions f and between sets are interpreted in the almost every where sense. For any non-negative Σ-measurable functions f as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\varphi^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A Ef d\mu = \int_A f d\mu, \qquad \text{for all } A \in \varphi^{-1}(\Sigma).$$

Hence we obtain an operator E from $L^1(\Sigma)$ onto $L^1(\varphi^{-1}(\Sigma))$ which is called conditional expectation operator associated with the σ-algebra $\varphi^{-1}(\Sigma)$. It is easy to show that for each $f \in L^1(\Sigma)$, there exists a Σ-measurable function g such that $E(f) = g \circ \varphi$. To obtain a unique g with this property we may assume and do that $\mathrm{supp}(g) \subseteq \mathrm{supp}(h)$. We therefore write $g = E(f) \circ \varphi^{-1}$, though we make no assumptions regarding the invertibility of φ (see [9]). It is easy to check that $E(f) \circ \varphi^{-1} - E(g) \circ \varphi^{-1} = E(f-g) \circ \varphi^{-1}$ and $|E(f) \circ \varphi^{-1}| = |E(f)| \circ \varphi^{-1}$ for all $f, g \in L^1(\Sigma)$. We list here some of its useful properties:

- $E(fg) = E(f)g$ whenever g is $\varphi^{-1}(\Sigma)$-measurable and both conditional expectations are defined.
- $|E(f)|^p \leq E(|f|^p)$, for each $p \geq 1$.
- If $f \geq 0$ then $E(f) \geq 0$; if $E(|f|) = 0$ then $f = 0$.

Let f be a real-valued measurable function. Consider the set $B_f = \{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$. The function f is said to be conditionable with respect to $\varphi^{-1}(\Sigma)$, if $\mu(B_f) = 0$. If f is complex-valued, then f is conditionable if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. For more details on the properties of E see [9, 10].

The aim of this paper is to carry some of the results obtained for the weighted composition operators and (classic) Frobenius-Perron operators in [4, 8, 11] to the weighted Frobenius-Perron operators. In the paper, first we give a necessary and sufficient condition for compactness of the weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$. Then, by making use of this condition we determine the spectrum of the compact operator $\mathcal{P}_\varphi^u$. One should note that the illustration of spectrum of the Frobenius-Perron operators, in general case, is still an open problem (see [3]). We also show that every weakly compact weighted Frobenius-Perron operator on $L^1(\Sigma)$ is compact.

2. **Main Results**

Suppose $\varphi : X \to X$ is a non-singular transformation and let $u : X \to \mathbb{C}$ be a conditionable measurable function. If A is any Σ-measurable set for which $\int_{\varphi^{-1}(A)} uf d\mu$ exists, the linear operator $\mathcal{P}_\varphi^u : L^1(\Sigma) \to L^1(\Sigma)$ defined by $\int_A \mathcal{P}_\varphi^u f d\mu = \int_{\varphi^{-1}(A)} uf d\mu$ is called a weighted Frobenius-Perron operator associated with the pair (u, φ). Note that the operator $\mathcal{P}_\varphi^u$ is a bounded operator on $L^1(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$ and its norm is given by $\|\mathcal{P}_\varphi^u\| = \|u\|_\infty$ (see [7]).

Take a set $A \in \Sigma$ with $\mu(A) > 0$. We say that A is an atom if, for any $C \in \Sigma$ with $C \subseteq A$, we have either $\mu(C) = 0$ or $\mu(A \backslash C) = 0$. Let A be an atom. Since μ is σ-finite, it follows that $\mu(A) < \infty$. Also, every Σ-measurable function f on X is constant almost everywhere on A. As is well known that, a σ-finite measure space (X, Σ, μ) is uniquely decomposed as

$$X = B \cup \{A_i : i \in \mathbb{N}\}, \tag{2.1}$$

where B is a non-atomic set and $\{A_i\}_{i\in\mathbb{N}}$ is a countable collection of disjoint atoms (see [12]).

Lemma 2.1. *Let B_0 be a non-atomic set in Σ with $0 < \mu(B_0) < \infty$ and let $\varphi : X \to X$ be a non-singular measurable transformation. Then $\varphi^{-1}(\Sigma \cap B_0)$ has no atoms.*

Proof. See ([6], Lemma 1). □

Theorem 2.2. *Let $\mathcal{P}_\varphi^u$ be a bounded Frobenius-Perron operator on $L^1(\Sigma)$ and suppose (X, Σ, μ) can be partitioned as (2.1). Then $\mathcal{P}_\varphi^u$ is a compact operator on $L^1(\Sigma)$ if and only if $u(\varphi^{-1}(B)) = 0$ ($u(x) = 0$ for μ-almost all $x \in \varphi^{-1}(B)$), and for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \mu(\varphi^{-1}(A_n) \cap D_\varepsilon(u)) > 0\}$ is finite, where $D_\varepsilon(u) = \{x \in X : |u(x)| \geq \varepsilon\}$.*

Proof. Suppose that $\mathcal{P}_\varphi^u$ is a compact operator. First we show that $u(\varphi^{-1}(B)) = 0$. Suppose the contrary. Since $D_\varepsilon(u) \subseteq D_\varepsilon(E(|u|)) := \{x \in X : E(|u|)(x) \geq \varepsilon\}$, then there exists $\delta > 0$ such that $\mu(\varphi^{-1}(B) \cap D_\delta(E(|u|))) \geq \mu(\varphi^{-1}(B) \cap D_\delta(u)) > 0$. Since $\varphi^{-1}(\Sigma)$ is a σ-finite, there is a $B_0 \in \Sigma \cap B$ with $0 < \mu(\varphi^{-1}(B_0) \cap D_\delta(E(|u|))) < \infty$. Hence $J_0 := \varphi^{-1}(B_0) \cap D_\delta(E(|u|)) \in \varphi^{-1}(\Sigma \cap B) \cap \Sigma = \varphi^{-1}(\Sigma \cap B)$. By Lemma 2.1, J_0 has no atoms. Choose a sequence $\{B_n\}_{n=1}^\infty \subseteq \Sigma \cap B_0$, such that $J_{n+1} \subseteq$

$J_n \subseteq J_0$, $0 < \mu(J_{n+1}) = \mu(J_n)/2$, where $J_n := \varphi^{-1}(B_n) \cap D_\delta(E(|u|)) \in \varphi^{-1}(\Sigma)$. For all $n \in \mathbb{N}$, define $f_n = \bar{u}\chi_{J_n}/(\|u\|_\infty \mu(J_n))$. Then $\|f_n\|_1 \leq 1$. Now by using the change of variable formula ($\int_X hf d\mu = \int_X f \circ \varphi d\mu$, for any non-negative measurable function f), for any $m, n \in \mathbb{N}$ with $m > n$ we get that

$$\|\mathcal{P}^u_\varphi f_n - \mathcal{P}^u_\varphi f_m\|_1 = \int_X h|E(u(f_n - f_m))| \circ \varphi^{-1} d\mu$$

$$= \int_X |E(u(f_n - f_m))| d\mu = \int_X \frac{E(|u|^2)}{\|u\|_\infty} \left| \frac{\chi_{J_n}}{\mu(J_n)} - \frac{\chi_{J_m}}{\mu(J_m)} \right| d\mu$$

$$\geq \int_{J_n \setminus J_m} \frac{(E(|u|))^2 d\mu}{\|u\|_\infty \mu(J_n)} \geq \frac{\delta^2}{\|u\|_\infty} \int_{J_n \setminus J_m} \frac{d\mu}{\mu(J_n)}$$

$$= \frac{\delta^2}{\|u\|_\infty} \frac{\mu(J_n \setminus J_m)}{\mu(J_n)} = \frac{\delta^2}{\|u\|_\infty} \left(1 - \frac{\mu(J_m}{\mu(J_n)}\right) > \frac{\delta^2}{2\|u\|_\infty},$$

which shows that the sequence $\{\mathcal{P}^u_\varphi f_n\}_{n \in \mathbb{N}}$ dose not contain a convergent subsequence. But this is a contradiction. □

Now, we show that for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \mu(\varphi^{-1}(A_n) \cap D_\varepsilon(u)) > 0\}$ is finite. Suppose the contrary again. Then, for some $\varepsilon > 0$, there is a subsequence $\{A_k\}_{k \in \mathbb{N}}$ of disjoint atoms in Σ such that $\mu(\varphi^{-1}(A_k) \cap D_\varepsilon(E(|u|))) > 0$, for all $k \in \mathbb{N}$. Put $G_k = \varphi^{-1}(A_k) \cap D_\varepsilon(E(|u|))$. Hence we obtain a sequence of pairwise disjoint sets $\{G_k\}_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $G_k \in \varphi^{-1}(\Sigma)$ and $\mu(G_k) > 0$. Moreover, since $\varphi^{-1}(\Sigma)$ is σ-finite, then h is finite valued and for each $k \in \mathbb{N}$, $\mu(A_k) < \infty$. Hence $\mu(G_k) \leq \mu(\varphi^{-1}(A_k)) = \int_{A_k} h d\mu = h(A_k)\mu(A_k) < \infty$. For any $k \in \mathbb{N}$, take $f_k = \bar{u}\chi_{G_k}/(\|u\|_\infty \mu(G_k))$. Then $\|f_k\|_1 \leq 1$. Since for each $i \neq j$, $G_i \cap G_j = \emptyset$, it follows that

$$\|\mathcal{P}^u_\varphi f_i - \mathcal{P}^u_\varphi f_j\|_1 = \int_X \frac{E(|u|^2)}{\|u\|_\infty} \left| \frac{\chi_{G_i}}{\mu(G_i)} - \frac{\chi_{G_j}}{\mu(G_j)} \right| d\mu$$

$$= \int_X \left(\frac{E(|u|^2)\chi_{G_i}}{\|u\|_\infty \mu(G_i)} \right) d\mu + \int_X \left(\frac{(E(|u|^2)\chi_{G_j}}{\|u\|_\infty \mu(G_j)} \right) d\mu$$

$$\geq \int_X \left(\frac{(E(|u|))^2 \chi_{G_i}}{\|u\|_\infty \mu(G_i)} \right) d\mu + \int_X \left(\frac{(E(|u|))^2 \chi_{G_j}}{\|u\|_\infty \mu(G_j)} \right) d\mu \geq \frac{2\varepsilon^2}{\|u\|_\infty}.$$

This contradicts the compactness of $\mathcal{P}^u_\varphi$.

The proof of the sufficient part is the same as for Theorem 2.9 in [7].

Corollary 2.3. *Suppose that μ is nonatomic, i.e. $X = B$. Then a weighted Frobenius-Perron operator on $L^1(\Sigma)$ is compact if and only if it is a zero operator. In particular, no classic Frobenius-Perron operator on $L^1(\Sigma)$ is compact.*

Our next task is about the spectra. For the classic Frobenius-Perron operator P_φ on $L^1(\Sigma)$, some basic properties of its spectra were described by Jiu Ding [2, 3, 4, 5] and some other mathematicians. In this sequel, we determine the spectrum, $\sigma(\mathcal{P}_\varphi^u)$, of a compact weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$.

The kth iterate φ^k of the non-singular measurable transformation $\varphi : X \to X$ is defined by $\varphi^0(x) = x$ and $\varphi^k(x) = \varphi(\varphi^{k-1}(x))$ for all $x \in X$ and $k \in \mathbb{N}$. From now on, we assume that the sequence $h_n := \frac{d\mu\circ\varphi^{-n}}{d\mu}$ is uniformly bounded.

Definition 2.4. *An atom A is called an invariant atom with respect to φ, if for all $n \in \mathbb{Z}$, $\varphi^n(A)$ is an atom. An invariant atom A with respect to φ is called a fixed atom of φ of order one, if $u(A) \neq 0$ and $\varphi(A) = A = \varphi^{-1}(A)$. Also, it is called of order $2 \leq k \in \mathbb{N}$, if $\prod_{i=0}^{k-1} u(\varphi^i(A)) \neq 0$, $\varphi^{-k}(A) = A = \varphi^k(A)$ and $\varphi^i(A) \neq A$ for $i = \pm 1, \ldots, \pm(k-1)$.*

Theorem 2.5. *Let $\mathcal{P}_\varphi^u$ be a compact weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$. If we set*

$$\Lambda = \{\lambda \in \mathbb{C} : \lambda^k = \prod_{i=0}^{k-1} u(\varphi^i(A)), \text{ for some fixed atom } A \text{ of } \varphi \text{ of order } k\},$$

then we have $\sigma(\mathcal{P}_\varphi^u) \cup \{0\} = \Lambda \cup \{0\}$.

Proof. To prove the theorem, we adopt the method used by Kamowitz [8] and Takagi [11]. Let A be an invariant atom and $u(\varphi^m(A)) = 0$ for some $m \in \mathbb{N}$. We claim that $\mathcal{P}_\varphi^u$ is not onto. If is not, then there exists $f \in L^1(\Sigma)$ such that $\mathcal{P}_\varphi^u f = \chi_{\varphi^{m+1}(A)}$. This implies that

$$0 = \int_{\varphi^m(A)} uf d\mu = \int_{\varphi^{m+1}(A)} \mathcal{P}_\varphi^u f d\mu = \mu(\varphi^{m+1}(A)) > 0,$$

which is a contradiction. Thus in this case $0 \in \sigma(\mathcal{P}_\varphi^u)$. Now, let A be a fixed atom of φ of order one and suppose $\lambda = u(A)$. We claim that the equation $\lambda f - \mathcal{P}_\varphi^u f = \chi_A$ is not solvable for a non-zero $f \in L^1(\Sigma)$.

Indeed, since $\varphi^{-1}(A) = A$, we have

$$(\mathcal{P}_\varphi^u f)(A) = \frac{1}{\mu(A)} \int_A \mathcal{P}_\varphi^u f d\mu = \frac{1}{\mu(A)} \int_{\varphi^{-1}(A)} uf d\mu$$
$$= \frac{1}{\mu(A)} \int_A uf d\mu = u(A)f(A) = (\lambda f)(A).$$

Hence, we get that $(\lambda f - \mathcal{P}_\varphi^u f)(A) = 0$ while $\chi_A(A) = 1$. Therefore $\lambda \in \sigma(\mathcal{P}_\varphi^u)$. Now, suppose that A is a fixed atom of φ of order $k \geq 2$ and $\lambda^k = \prod_{i=0}^{k-1} u(\varphi^i(A))$. By induction, we can easily show that

$$\lambda^k f(A) - ((\mathcal{P}_\varphi^u)^k(f))(A) = \lambda^{k-1} + \sum_{i=1}^{k-1} \lambda^{k-i-1}((\mathcal{P}_\varphi^u)^i(\chi_A))(A). \tag{2.2}$$

Put $U_k = \prod_{i=0}^{k-1}(u \circ \varphi^i)$. Then $(\mathcal{P}_\varphi^u)^k = P_{\varphi^k} M_{U_k}$, where M_{U_k} is a multiplication operator (see [7]). Since $\varphi^{-k}(A) = A$ and $\varphi^{-i}(A) \neq A$ for $i = \pm 1, \ldots \pm (k-1)$, then we have

$$((\mathcal{P}_\varphi^u)^k(f))(A) = \frac{1}{\mu(A)} \int_A (\mathcal{P}_\varphi^u)^k(f) d\mu = \frac{1}{\mu(A)} \int_A P_{\varphi^k}(U_k f) d\mu$$
$$= \frac{1}{\mu(A)} \int_{\varphi^{-k}(A)} U_k f d\mu = \frac{1}{\mu(A)} \int_A U_k f d\mu = U_k(A) f(A)$$

and

$$((\mathcal{P}_\varphi^u)^i(\chi_A))(A) = (P_{\varphi^i}(U_i \chi_A))(A)$$
$$= \frac{1}{\mu(A)} U_i(\varphi^{-i}(A)) \chi_A(\varphi^{-i}(A)) \mu(\varphi^{-i}(A)) = 0.$$

It follows that, the left hand side of (2.2) equals 0, while the right hand side of (2.2) equals λ^{k-1}. This contradiction shows that $\lambda \in \sigma(\mathcal{P}_\varphi^u)$. Therefore $\Lambda \cup \{0\} \subseteq \sigma(\mathcal{P}_\varphi^u) \cup \{0\}$.

Now, we show the opposite inclusion. Let $\lambda \notin \Lambda \cup \{0\}$, and suppose that $\mathcal{P}_\varphi^u f = \lambda f$, for some $f \in L^1(\Sigma)$. Since every non-zero spectral value λ of $\mathcal{P}_\varphi^u$ is an eigenvalue of $\mathcal{P}_\varphi^u$, we must show that f is zero μ-almost everywhere on X.We first show that $f(A) = 0$ for every invariant atom A. Let A be a fixed atom of φ of order k. Since $\mathcal{P}_\varphi^u f = \lambda f$, by induction, we get $(P_{\varphi^k} U_k) f = \lambda^k f$, and so $U_k(A) f(A) = \lambda^k f(A)$. Since $U_k(A) \neq \lambda^k$, we have $f(A) = 0$.

By the first part of the proof, we can assume that for all $k \in \mathbb{N} \cup \{0\}$, $u(\varphi^k(A)) \neq 0$. Put $\mathcal{K}(A) = \{\varphi^i(A) : i \in \mathbb{N} \cup \{0\}\}$. If $\mathcal{K}(A)$ is finite, then for some $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$, $\varphi^m(A) = \varphi^n(A)$. It follows that $\varphi^{m-n}(\varphi^{-m}(A)) = \varphi^{-n}(A) = \varphi^{-m}(A)$ and $\varphi^{n-m}(\varphi^{-m}(A)) = \varphi^{n-m}(\varphi^{m-n}(\varphi^{-m}(A)) = \varphi^{-m}(A)$. Thus $\varphi^{-m}(A)$ is a fixed atom of φ of order $m-n$ and hence $f(\varphi^{-m}(A)) = 0$. On the other hand, since $\lambda^m f = (\mathcal{P}^u_\varphi)^m f$ and

$$((\mathcal{P}^u_\varphi)^m(f))(A) = \frac{1}{\mu(A)} U_m(\varphi^{-m}(A)) f(\varphi^{-m}(A)) \mu(\varphi^{-m}(A)) = 0,$$

then, $f(A) = 0$.

Now, suppose that $\mathcal{K}(A)$ is infinite. We claim that the set $\{n \in \mathbb{Z} : |u(\varphi^n(A)| > \varepsilon\}$ is finite for some $\varepsilon > 0$. Suppose this dose not hold. Then the set $\{n \in \mathbb{Z} : \mu(\{x \in \varphi^{-1}(\varphi^{n+1}(A)) : |u(x)| \geq \varepsilon\}) > 0\}$ is infinite. But this contradicts the compactness of $\mathcal{P}^u_\varphi$. Put $\mathrm{N} = \max\{|m| \in \mathbb{N} : |u(\varphi^m(A))| \geq \varepsilon\}$. Choose $\varepsilon = |\lambda|/2$. Then, for each $n > \mathrm{N}$, $|u(\varphi^n(A))| < |\lambda|/2$. It follows that

$$\begin{aligned} |\lambda^n f(A)| &= h_n |u(\varphi^{-n}(A)) \dots u(\varphi^{-\mathrm{N}}(A)) \dots u(\varphi^{-1}(A)) f(\varphi^{-n}(A))| \\ &\leq h_n \|u\|^{\mathrm{N}}_\infty \, (\frac{|\lambda|}{2})^{n-\mathrm{N}} \|f\|_1. \end{aligned}$$

Thus

$$|f(A)| \leq h_n \|u\|^{\mathrm{N}}_\infty (\frac{|\lambda|}{2})^{-\mathrm{N}} (\frac{1}{2})^n \|f\|_1 \longrightarrow 0, \qquad \text{as } n \to \infty.$$

Therefore we conclude that f is zero on $\cup_{n \in \mathbb{N}} A_n$.

It remains to show that f is zero μ-almost everywhere on B. Since $L^1(\Sigma) = L^1(\cup_{n \in \mathbb{N}} A_n) \oplus L^1(B)$, hence it suffices to show that f is zero as an element of $L^1(B)$. Now, it follows from $u(\varphi^{-1}(B)) = 0$ that

$$\|\mathcal{P}^u_\varphi f\|_{L^1(B)} = \int_B |\mathcal{P}^u_\varphi f| d\mu = \int_{\varphi^{-1}(B)} u f d\mu = 0.$$

Thus $\lambda f = \mathcal{P}^u_\varphi f = 0$ and hence f is zero μ-almost everywhere on B. This completes the proof of the theorem. □

Finally, we investigate the weakly compact weighted Frobenius-Perron operators on $L^1(\Sigma)$. Recall that the operator $\mathcal{P}^u_\varphi : L^1(\Sigma) \to L^1(\Sigma)$ is said to be weakly compact if it maps bounded subsets of $L^1(\Sigma)$ into weakly sequentially compact subsets of $L^1(\Sigma)$. A classical theorem of Dunford (see [1], IV.8.9) isolates the weakly sequentially compact subsets of $L^1(\Sigma)$ as the bounded uniformly integrable subsets. We begin with the following lemma, which can be deduced from Theorem IV.8.9, and its Corollaries 8.10, 8.11 in [1].

Lemma 2.6. *Let H be a weakly sequentially compact set in $L^1(\Sigma)$. Then for each decreasing sequence $\{E_n\}$ in Σ such that $\lim_{n\to\infty}\mu(E_n)=0$ or $\cap_{n=1}^{\infty}E_n=\emptyset$, the sequence of integrals $\{\int_{E_n}|h|d\mu\}$ converges to zero uniformly for h in H.*

Theorem 2.7. *Let $\mathcal{P}_\varphi^u$ be a bounded Frobenius-Perron operator on $L^1(\Sigma)$ and suppose that (X,Σ,μ) can be partitioned as (2.1). Then $\mathcal{P}_\varphi^u$ is a weakly compact operator on $L^1(\Sigma)$ if and only if it is compact.*

Proof. It suffices to show the " only if " part. The inspiration for the proof is the method used by Takagi [11]. Let $\mathcal{P}_\varphi^u$ be a weakly compact operator on $L^1(\Sigma)$. We first show that $u(\varphi^{-1}(B))=0$. Suppose the contrary. By the same argument as in the proof of Theorem 2.2, we assume that for some $\delta>0$ and $B_0\subseteq B$, $0<\mu(\varphi^{-1}(B_0)\cap D_\delta(u))<\infty$. Now, as B_0 is non-atomic, we can find a decreasing sequence $\{B_n\}\subseteq B_0\cap\Sigma$ with $0<\mu(B_n)<\frac{1}{n}$ and $0<J_n:=\mu(\varphi^{-1}(B_n)\cap D_\delta(u))<\infty$. Let U be the closed unit ball of $L^1(\Sigma)$. Since $\mathcal{P}_\varphi^u U$ is weakly sequentially compact, Lemma 2.6 can be applied with $H=\mathcal{P}_\varphi^u U$ and $E_n=B_n$. Choose $\varepsilon=\delta^2/\|u\|_\infty$. Then there exists an $n_o\in\mathbb{N}$ such that

$$\int_{B_{n_o}}|\mathcal{P}_\varphi^u f|d\mu<\frac{\delta^2}{\|u\|_\infty},\qquad f\in U. \tag{2.3}$$

On the other hand if we take $f_{n_o}=\bar{u}\chi_{J_{n_o}}/(\|u\|_\infty\mu(J_{n_o}))$, we have

$$\int_{B_{n_o}}|\mathcal{P}_\varphi^u f|d\mu=\int_{B_{n_o}}hE\left(\frac{u\bar{u}\chi_{J_{n_o}}}{\|u\|_\infty\mu(J_{n_o})}\right)\circ\varphi^{-1}d\mu$$

$$=\int_{\varphi^{-1}(B_{n_o})}E\left(\frac{|u|^2\chi_{J_{n_o}}}{\|u\|_\infty\mu(J_{n_o})}\right)d\mu=\frac{1}{\|u\|_\infty\mu(J_{n_o})}\int_{\varphi^{-1}(B_{n_o})}|u|^2\chi_{J_{n_o}}d\mu$$

$$=\frac{1}{\|u\|_\infty\mu(J_{n_o})}\int_{J_{n_o}}|u|^2d\mu\geq\frac{\delta^2}{\|u\|_\infty}.$$

Since $f_{n_o}\in U$, this contradicts (2.3). According to Theorem 2.2, it remains to show that for any $\varepsilon>0$, the set $\mathrm{A}:=\{n\in\mathbb{N}:\mu(\varphi^{-1}(A_n)\cap D_\varepsilon(u))>0\}$ is finite. To end this, without loss of generality, we can assume that $\mathrm{A}=\mathbb{N}$ for some $\varepsilon>0$. Put $\mathrm{K}_n=\{A_k:k\geq n\}$ and $G_n=\varphi^{-1}(\mathrm{K}_n)\cap D_\varepsilon(u)$. Then we have $\cap_{n=1}^{\infty}\mathrm{K}_n=0$ and $\mu(G_n)>0$ for each $n\in\mathbb{N}$. Also, since h is essentially bounded, there is a constant $M>0$ such that $\mu(G_n)\leq\mu(\varphi^{-1}(\mathrm{K}_n))\leq M\mu(\mathrm{K}_n)\to 0$, as $n\to\infty$. So we can assume that $\mu(G_n) < \infty$ for each $n \in \mathbb{N}$. Applying Lemma 2.6 once more, there exists an $\mathrm{N} \in \mathbb{N}$ such that

$$\int_{\mathrm{K_N}} |\mathcal{P}_\varphi^u f| d\mu < \frac{\varepsilon^2}{\|u\|_\infty}, \qquad f \in U.$$

Now, for any n with $n \geq \mathrm{N}$, let $g_n = \bar{u}\chi_{G_n}/(\|u\|_\infty \mu(G_n))$. Then we have

$$\int_{\mathrm{K}_N} |\mathcal{P}_\varphi^u g_n| d\mu = \int_{\varphi^{-1}(\mathrm{K}_N)} E\left(\frac{|u|^2\chi_{G_n}}{\|u\|_\infty \mu(G_n)}\right) d\mu$$

$$= \frac{1}{\|u\|_\infty \mu(G_n)} \int_{G_n} |u|^2 \chi_{G_n} d\mu \geq \frac{\varepsilon^2}{\|u\|_\infty}.$$

Since $g_n \in U$, this contradicts (2.3). This completes the proof of the theorem. □

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References

[1] N. Dunford and J. T. Schwartz, Linear Operators, Part I, General Theory, Interscience, New York, 1958.

[2] J. Ding and A. Zhou, On the spectrum of Frobenius-Perron operators, *J. Math. Anal. Appl.* **250** (2000), no. 2, 610–620.

[3] J. Ding, The point spectrum of Frobenius-Perron and Koopman operators, *Proc. Amer. Math. Soc.* **126** (1998), no. 5, 1355–1361.

[4] J. Ding and W. E. Hornor, A new approach to Frobenius-Perron operators, *J. Math. Anal. Appl.* **187** (1994), no. 3, 1047–1058.

[5] J. Ding, Q. Du and T. Y. Li, The spectral analysis of Frobenius-Perron operators, *J. Math. Anal. Appl.* **184** (1994), no. 2, 285–301.

[6] D. J. Harrington, Co-rank of a composition operator, *Canad. Math. Bull.* **29** (1986), no. 1, 33–36.

[7] M. R. Jabbarzadeh, Weighted Frobenius-Perron and Koopman operators, *Bull. Iranian Math. Soc.* **35** (2009), no. 2, 85–96.

[8] H. Kamowitz, Compact weighted endomorphisms of $C(X)$, *Proc. Amer. Math. Soc.* **83** (1981), no. 3, 517–521.

[9] A. Lambert, Localising sets for sigma-algebras and related point transformations, *Proc. Roy. Soc. Edinburgh Sect. A* **188** (1991), no. 1-2, 111–118.

[10] M. M. Rao, Conditional Measure and Applications, Marcel Dekker, Inc., New York, 1993.
[11] H. Takagi, Compact weighted composition operators on L^p, *Proc. Amer. Math. Soc.* **116** (1992), no. 2, 505–511.
[12] A. C. Zaanen, Integration, 2nd ed., North-Holland Publishing Co., Amsterdam, 1967.

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