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Conditional expectation operators on the Bergman spaces

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ABSTRACT

In this note, we extend some results in Carswell and Stessin (2008) [1] to larger classes of sigma-algebras associated with the conditional expectation operators on the Bergman spaces.

1. Introduction and preliminaries

Let $(X,\mathcal{M},\mu)$ be a complete sigma-finite measure space and let $\mathcal{A}$ be a sigma-algebra of $\mathcal{M}$ such that $(X,\mathcal{A},\mu_{|\mathcal{A}})$ is also sigma-finite. The collection of (equivalence classes modulo sets of zero measure) $\mathcal{M}$-measurable complex-valued functions on X will be denoted $L^0(\mathcal{M})$, with $L^0(\mathcal{A})$ being likewise defined for $\mathcal{A}$-measurable functions. Moreover, we let $L^p(\mathcal{M})=L^p(X,\mathcal{M},\mu)$ and $L^p(\mathcal{A})=L^p(X,\mathcal{A},\mu_{|\mathcal{A}})$, for $1\leqslant p\leqslant\infty$. Also its norm is denoted by $\|.\|_p$ on which $L^p(\mathcal{A})$ is a Banach subspace of $L^p(\mathcal{M})$. A consequence of the Radon–Nikodym theorem is that to each nonnegative function $f\in L^0(\mathcal{M})$ there exists a unique nonnegative $\mathcal{E}_{\mathcal{A}}(f)\in L^0(\mathcal{A})$ such that

$$\int_{\Delta} f\,d\mu=\int_{\Delta}\mathcal{E}_{\mathcal{A}}(f)\,d\mu$$

for all $\Delta\in\mathcal{A}$. The function $\mathcal{E}_{\mathcal{A}}(f)$ is called the conditional expectation of f with respect to $\mathcal{A}$. This can be extended to real-valued and complex-valued functions by examining the conditional expectations of the positive and negative parts (in the case of real-valued functions), and the real and imaginary parts (for complex-valued functions). If $\mathcal{E}_{\mathcal{A}}(f)$ exists for a function $f\in L^0(\mathcal{M})$, then we say f is conditionable. One can show that every $L^p(\Sigma)$ function is conditionable; therefore, a linear transformation $\mathcal{E}_{\mathcal{A}}: L^p(\mathcal{M})\to L^p(\mathcal{A})$ can be defined by $f\mapsto\mathcal{E}_{\mathcal{A}}(f)$. It is clear that $\mathcal{E}_{\mathcal{A}}$ is an idempotent, and in the case of $p=2$, it is the orthogonal projection of $L^2(\mathcal{M})$ onto $L^2(\mathcal{A})$. For more details on the properties of $\mathcal{E}_{\mathcal{A}}$ on abstract measurable function spaces see [2] and [3]. It seems that in the operator theory of analytic function spaces, the operation of conditional expectation has not got the attention that it deserves.

Let $\mathcal{M}$ be the sigma-algebra of Lebesgue-measurable sets in $\mathbb{D}=\{z\in\mathbb{C}:\ |z|<1\}$ and let A be the normalized area measure on $\mathbb{D}$. Recall that the Bergman space $L_a^p(\mathbb{D})=L_a^p(\mathbb{D},\mathcal{M},A)$ consists of all analytic functions in $L^p(\mathbb{D},\mathcal{M},A)$, that is, the functions f analytic in $\mathbb{D}$ whose area integral

$$\|f\|_p^p=\int_{\mathbb{D}}\left|f(z)\right|^p\,dA(z)$$

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is finite. The analog of the Riesz projection is the Bergman projection $\mathcal{P}$ which (for $1 < p < \infty$) takes the function f in $L^p(\mathbb{D}, \mathcal{M}, A)$ to the function $\mathcal{P}(f)$ in $L^p_a(\mathbb{D})$ and is given by the formula

$$\mathcal{P}(f)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - w\bar{z})^2} \, dA(z), \quad w \in \mathbb{D}.$$

Note that if $\mathcal{E}_{\mathcal{A}}\mathcal{P} = \mathcal{P}\mathcal{E}_{\mathcal{A}}$ on $L^p(\mathbb{D}, \mathcal{M}, A)$, then $L^p_a(\mathbb{D})$ is invariant under the conditional expectation operator $\mathcal{E}_{\mathcal{A}}$; i.e., $\mathcal{E}_{\mathcal{A}}(L^p_a(\mathbb{D})) \subseteq L^p_a(\mathbb{D})$.

Let $A(\mathbb{D})$ denote the space of all functions analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. The algebra $A(\mathbb{D})$ is known as the disk algebra. Let $\varphi \in A(\mathbb{D})$. We say that φ has finite multiplicity if there exists $N \in \mathbb{N}$ such that for each $w \in \varphi(\mathbb{D})$, the level set $\varphi^{-1}(w) := \{\xi_j(w)\}_{j \geqslant 1}$ contains at most N points. By $\mathcal{A} = \mathcal{A}(\varphi)$ we denote the sigma-algebra generated by $\{\varphi^{-1}(U)\colon U \subset \mathbb{C} \text{ is open}\}$. Let m be the Lebesgue measure on $\mathbb{C}$. Since the finite measure $A \circ \varphi^{-1}$ is absolutely continuous with respect to m, we have $h := \frac{dA \circ \varphi^{-1}}{dm}$ almost everywhere finite valued.

The conditional expectation operators on the Bergman spaces were first studied by Carswell and Stessin in [1]. Our purpose in this note is partially answering to this question: which sigma-algebras of measurable subsets of $\mathbb{D}$ have the property that $\mathcal{E}_{\mathcal{A}}\mathcal{P} = \mathcal{P}\mathcal{E}_{\mathcal{A}}$. In [1] Carswell and Stessin proved that for $\varphi(z) = z^n$, this property holds and found the formula for $\mathcal{E}_{\mathcal{A}}$ by assuming $\mathcal{E}_{\mathcal{A}}\mathcal{P} = \mathcal{P}\mathcal{E}_{\mathcal{A}}$. We prove this formula without this condition. Also, we extend some results in [1] to larger classes of conditional expectation operators generated by the symbol function φ on the Bergman spaces.

2. Main results

Let $\mathbb{D}_0 = \{z \in \mathbb{D}\colon \varphi'(z) \neq 0\}$. The set $\mathbb{D} \backslash \mathbb{D}_0$ is at most countable. If $z \in \mathbb{D}_0$, then there exists some $r_z > 0$ such that φ is one-to-one on $D(z, r_z)$. The discs $D(z, r_z)$ form a cover for $\mathbb{D}_0$ and we can pick a countable subcover $\{D(z_n, r_n)\colon n \in \mathbb{N}\}$. Set $D_1 = D(z_1, r_1)$ and $D_n = D(z_n, r_n) \backslash \bigcup_{k=2}^{n} D_{k-1}$, for $n \geqslant 2$. So $\{D_n\colon n \in \mathbb{N}\}$ is a pairwise disjoint Borel cover of $\mathbb{D}_0$. Note that $\varphi_n := \varphi|_{D_n}$ maps D_n bijectively to $\varphi(D_n)$. So for any nonnegative functions $g \in L^\infty(\varphi(\mathbb{D}), \mathcal{M}_{\varphi(\mathbb{D})}, m_{|\mathcal{M}_{\varphi(\mathbb{D})}})$ and $f \in L^p(\mathbb{D}, \mathcal{M}, A)$ with $p \geqslant 1$, we have

$$\int_{D_n} (g \circ \varphi) f \, dA = \int_{D_n} g \circ \varphi \frac{f}{|\varphi'|^2} \left|\varphi'\right|^2 dA = \int_{\varphi(D_n)} g \frac{f \circ \varphi_n^{-1}}{|\varphi' \circ \varphi_n^{-1}|^2} \, dm$$

and so

$$\int_{\varphi(\mathbb{D})} gh\mathcal{E}_{\mathcal{A}}(f) \circ \varphi^{-1} \, dm = \int_{\mathbb{D}} (g \circ \varphi) f \, dA = \int_{\mathbb{D}_0} (g \circ \varphi) f \, dA = \sum_{n=1}^{\infty} \int_{D_n} (g \circ \varphi) f \, dA$$
$$= \sum_{n=1}^{\infty} \int_{\varphi(\mathbb{D}_0)} g \chi_{\varphi(D_n)} \frac{f \circ \varphi_n^{-1}}{|\varphi' \circ \varphi_n^{-1}|^2} \, dm = \int_{\varphi(\mathbb{D}_0)} g \left(\sum_{n=1}^{\infty} \chi_{\varphi(D_n)} \frac{f \circ \varphi_n^{-1}}{|\varphi' \circ \varphi_n^{-1}|^2} \right) dm.$$

For each $w \in \varphi(\mathbb{D})$, let $\{z \in \mathbb{D}\colon \varphi(z) = w\} = \{\xi_j(w)\}_{j \geqslant 1}$. Then

$$\sum_{n=1}^{\infty} \chi_{\varphi(D_n)}(w) \frac{f \circ \varphi_n^{-1}}{|\varphi' \circ \varphi_n^{-1}|^2} = \sum_{\{z \in \mathbb{D}_0\colon \varphi(z) = w\}} \frac{f(z)}{|\varphi'(z)|^2} = \sum_{j \geqslant 1} \frac{f(\xi_j(w))}{|\varphi'(\xi_j(w))|^2},$$

for any $w \notin \varphi(\mathbb{D} \backslash \mathbb{D}_0)$. Thus

$$\int_{\varphi(\mathbb{D})} gh\mathcal{E}_{\mathcal{A}}(f) \circ \varphi^{-1} \, dm = \int_{\varphi(\mathbb{D})} g(w) \left(\sum_{j \geqslant 1} \frac{f(\xi_j(w))}{|\varphi'(\xi_j(w))|^2} \right) dm(w),$$

and so

$$h(w)\mathcal{E}_{\mathcal{A}}(f) \circ \varphi^{-1}(w) = \sum_{j \geqslant 1} \frac{f(\xi_j(w))}{|\varphi'(\xi_j(w))|^2} \quad \big(w \in \varphi(\mathbb{D})\big). \tag{2.1}$$

If $f \in L^p_a(\mathbb{D})$, we may write $f = \sum_{k=0}^{3} i^k f_k$ with $0 \leqslant f_k \in L^p(\mathbb{D}, \mathcal{M}, A)$ ($f_0 = u^+$, etc.; where $u = \Re(f)$). Also, since $\mathcal{E}_{\mathcal{A}}(g_0 + g_1) \circ \varphi^{-1} = \mathcal{E}_{\mathcal{A}}(g_0) \circ \varphi^{-1} + \mathcal{E}_{\mathcal{A}}(g_1) \circ \varphi^{-1}$, for all $g_0, g_1 \in L^p(\mathbb{D}, \mathcal{M}, A)$, it follows that (2.1) holds for each $f \in L^p_a(\mathbb{D})$. Note that if h is constant, then for each $\xi_i(w)$ and $\xi_j(w)$,

$$\mathcal{E}_{\mathcal{A}}(f)\big(\xi_i(w)\big) = \mathcal{E}_{\mathcal{A}}(f)\big(\xi_j(w)\big).$$

If we take $f=1$ in (2.1), we get that

$$h(w)=\sum_{j\geqslant 1}\frac{1}{|\varphi'(\xi_j(w))|^2}. \tag{2.2}$$

Let $w=\varphi(\xi)$ with $\xi\in\{\xi_j(w)\}_{j\geqslant 1}$. Then by (2.1) and (2.2), we have

$$\mathcal{E}_{\mathcal{A}}(f)(\xi)=\frac{\sum_{j\geqslant 1}\frac{f(\xi_j(w))}{|\varphi'(\xi_j(w))|^2}}{\sum_{j\geqslant 1}\frac{1}{|\varphi'(\xi_j(w))|^2}}.$$

Let $a,b\in\mathbb{D}_0$. We say that a is equivalent with b with respect to φ if $\varphi(a)=\varphi(b)\in\{\xi_j(w)\}_{j\geqslant 1}$, for some $w\in\varphi(\mathbb{D}_0)$. The equivalent classes are denoted by $\frac{\mathbb{D}_0}{\sim}=\{\{\xi_j(w)\}_{j\geqslant 1}\colon\ w\in\varphi(\mathbb{D}_0)\}$. For suitable $g\in A(\mathbb{D})$ define the function ω_g on $\mathbb{D}$ by

$$\omega_g^j(\xi)=\frac{\frac{1}{|g(\xi_j(w))|^2}}{\sum_{j\geqslant 1}\frac{1}{|g(\xi_j(w))|^2}}.$$

Then the function ω_φ^j is constant on each element of $\frac{\mathbb{D}_0}{\sim}$; that is, for each $w\in\varphi(\mathbb{D})$, $\omega_\varphi^j|_{\{\xi_j(w)\}_{j\geqslant 1}}=c_w$, for some constant c_w. Let $\varphi^{-1}(w)=\{\xi_1(w),\dots,\xi_n(w)\}$ be distinct points in $\mathbb{D}$. Then for $i=1,\dots,n$, there exist $f_1,\dots,f_n$ in $L_a^p(\mathbb{D})$ satisfying $f_i(\xi_i(w))=1$ and $f_i(\xi_j(w))=0$ for $i\neq j$. It follows that $\omega_{\varphi'}^j=\mathcal{E}_{\mathcal{A}}(f_j)$ is constant on $\varphi^{-1}(w)$. Now, if $\mathcal{E}_{\mathcal{A}}(L_a^p(\mathbb{D}))\subseteq L_a^p(\mathbb{D})$, then the real-valued function $\mathcal{E}_{\mathcal{A}}(f_j)$ must be analytic. It follows that $\omega_{\varphi'}^j$ is constant on $\mathbb{D}_0$ (see [1, Lemma 6]). These observations establish the following theorem.

Theorem 2.1. *Suppose that $\mathcal{A}=\mathcal{A}(\varphi)$ for some $\varphi\in A(\mathbb{D})$ with finite multiplicity. For $w\in\varphi(\mathbb{D})$, let $\varphi^{-1}(w)=\{\xi_j(w)\}_{j\geqslant 1}$ be a level set. Suppose that none of the $\xi_j(w)$ belongs to $\{z\colon\ \varphi'(z)=0\}$ and that $w\notin f(\mathbb{T})$. Then for every f in $L_a^p(\mathbb{D})$ and ξ in $\varphi^{-1}(w)$,*

$$\mathcal{E}_{\mathcal{A}}(f)(\xi)=\frac{\sum_{j\geqslant 1}\frac{f(\xi_j(w))}{|\varphi'(\xi_j(w))|^2}}{\sum_{j\geqslant 1}\frac{1}{|\varphi'(\xi_j(w))|^2}}.$$

Also, the function ω defined as

$$\omega(\xi)=\frac{\frac{1}{|\varphi'(\xi_j(w))|^2}}{\sum_{j\geqslant 1}\frac{1}{|\varphi'(\xi_j(w))|^2}},$$

is constant on each level set. In particular if $\mathcal{E}_{\mathcal{A}}\mathcal{P}=\mathcal{P}\mathcal{E}_{\mathcal{A}}$, then ω is constant on $\mathbb{D}$.

Theorem 2.2. *Let $\mathcal{A}=\mathcal{A}(\varphi)$ for some $\varphi\in A(\mathbb{D})$ with finite multiplicity. If for each $w\in\mathbb{C}$, $|\varphi'|$ is constant on the set $\varphi^{-1}(w)\cap\mathbb{D}_0=\{\xi_1,\dots,\xi_{n_w}\}\cap\mathbb{D}_0$, then $\mathcal{E}_{\mathcal{A}}(L_a^p(\mathbb{D}))\subseteq L_a^p(\mathbb{D})$. Conversely, if $\mathcal{E}_{\mathcal{A}}(L_a^p(\mathbb{D}))\subseteq L_a^p(\mathbb{D})$ and $\varphi(\mathbb{T})$ is contained in the boundary of $\varphi(\mathbb{D})$, then $|\varphi'|$ is constant on the set $\mathbb{D}_0$.*

Proof. If $|\varphi'|$ is constant on the set $\{\xi_1,\dots,\xi_{n_w}\}\cap\mathbb{D}_0$, then by Theorem 2.1, we have

$$\mathcal{E}_{\mathcal{A}}(f)(\xi)=\frac{\sum_{k=1}^{n_w}f(\xi_k)\frac{1}{|\varphi'(\xi_k)|^2}}{\sum_{k=1}^{n_w}\frac{1}{|\varphi'(\xi_k)|^2}}=\frac{1}{n_w}\sum_{k=1}^{n_w}f(\xi_k)$$

for all f in $L_a^p(\mathbb{D})$ and every ξ in $f^{-1}(w)$, which implies that $\mathcal{E}_{\mathcal{A}}(f)\in L_a^p(\mathbb{D})$. Now, suppose that $\mathcal{E}_{\mathcal{A}}(L_a^p(\mathbb{D}))\subseteq L_a^p(\mathbb{D})$. Since by Theorem 2.1, ω is constant on $\mathbb{D}_0$, then for each ξ_i and ξ_j in the level set $\varphi^{-1}(w)$, $\varphi(\xi_i)=\varphi(\xi_j)$, and so $|\varphi'(\xi_i)|=|\varphi'(\xi_j)|$, which completes the proof. □

Example 2.3. (a) Let $\varphi_1(z)=z^n$ and let $\varphi_2(z)=\alpha z^2+\beta z+\gamma$, where α,β and γ are real numbers. Then $|\varphi_1'|$ and $|\varphi_2'|$ are constant on the level sets. Thus, $\mathcal{E}_{\mathcal{A}(\varphi_i)}\mathcal{P}=\mathcal{P}\mathcal{E}_{\mathcal{A}(\varphi_i)}$. Indeed, for each $f\in L_a^p(\mathbb{D})$, we have

$$\mathcal{E}_{\mathcal{A}(\varphi_1)}(f)(\xi)=\frac{1}{n}\sum_{k=1}^{n}f(\xi_k),$$

$$\mathcal{E}_{\mathcal{A}(\varphi_2)}(f)(\xi)=\frac{1}{2}f(\xi)+\frac{1}{2}f\left(-\frac{\beta+\alpha\xi}{\alpha}\right)$$

which is the same formula that we derived in the first part of Theorem 2.2.

(b) If $\varphi(z) = z^3 - 1$, then $|\varphi'|$ is not constant on the level sets. Then by Theorem 2.2, $\mathcal{E}_{\mathcal{A}}(L_a^p(\mathbb{D})) \nsubseteq L_a^p(\mathbb{D})$, and so $\mathcal{E}_{\mathcal{A}}\mathcal{P} \neq \mathcal{P}\mathcal{E}_{\mathcal{A}}$. However if we compute the formula of $\mathcal{E}_{\mathcal{A}}$, then we obtain this result, since the term $|\xi|^2$ is appeared in the formula for $\mathcal{E}_{\mathcal{A}}(f)(\xi)$ and $|\xi|^2$ is not analytic. Indeed, for each $f \in L_a^p(\mathbb{D})$, we have

$$\mathcal{E}_{\mathcal{A}(\varphi)}(f)(\xi) = \frac{1}{\sum_{k=1}^{3} \frac{1}{|\varphi'(\xi_k)|^2}} \left(f(\xi) \frac{1}{9|\xi|^4} + \cdots \right).$$

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