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ESSENTIAL NORM OF SUBSTITUTION OPERATORS ON L^p SPACES

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In this paper we determine the lower and upper estimates for the essential norm of finite sum of weighted Frobenius-Perron and weighted composition operators on L^p spaces under certain conditions.

Key words : Frobenius-perron operator, weighted composition operator, conditional expectation, essential norm.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a σ-finite measure space. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \rightarrow E^{\mathcal{A}}f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \le p \le \infty$, where $E^{\mathcal{A}}f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. For more details on the properties of $E^{\mathcal{A}}$ see [12] and [15].

Suppose $\varphi : X \to X$ is a non-singular transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$, and let $u \in L^0(\Sigma)$. Let for each $A \in \Sigma$ and $f \in L^1(\Sigma)$, $\int_{\varphi^{-1}(A)} uf d\mu$ exists. Define the measure $\mu^u_{\varphi,f}(A) = \int_{\varphi^{-1}(A)} uf d\mu$, $A \in \Sigma$. The assumption $\mu \circ \varphi^{-1} \ll \mu$ implies that $\mu^u_{\varphi,f} \ll \mu$. By the Radon-Nikodym theorem there exists a μ-unique function $\widetilde{f}^u_\varphi \in L^1_{loc}(\Sigma)$ such that $\mu^u_{\varphi,f}(A) = \int_A \widetilde{f}^u_\varphi d\mu$, for any $A \in \Sigma$. This may be expressed alternatively as

$$\int_A \widetilde{f}^u_\varphi d\mu = \int_{\varphi^{-1}(A)} uf d\mu, \qquad A \in \Sigma.$$

It follows that the weighted Frobenius-Perron operator associated with the pair (u, φ) defined as $\mathcal{P}^u_\varphi(f) = \widetilde{f}^u_\varphi$ is well defined on $L^1(\Sigma)$. Take $h_\varphi = d\mu \circ \varphi^{-1}/d\mu$. As an application of the properties of the conditional expectation and using the change of variable formula we have

$$\begin{aligned} \int_A \mathcal{P}^u_\varphi f d\mu &= \int_{\varphi^{-1}(A)} uf d\mu = \int_{\varphi^{-1}(A)} E^{\varphi^{-1}(\Sigma)}(uf) d\mu \\ &= \int_A E^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1} d\mu \circ \varphi^{-1} = \int_A h_\varphi E^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1} d\mu, \end{aligned}$$

for all $f \in L^1(\Sigma)$ and $A \in \Sigma$ with $0 < \mu(A) < \infty$. It follows that $\mathcal{P}^u_\varphi f = h_\varphi E^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1}$. Note that $(\mathcal{P}^u_\varphi)^* = uC_\varphi$, where uC_φ is a weighted composition operator defined on $L^\infty(\Sigma)$ as $uC_\varphi(f) = u.f \circ \varphi$, see [9, Proposition 2.3(vi)].

In this paper we consider finite sum of weighted Frobenius-Perron operators and weighted composition operators defined on $L^1(\Sigma)$ and $L^p(\Sigma)$ respectively of the form

$$\mathcal{P} = \sum_{i=1}^{n} \mathcal{P}_{\varphi_i}^{u_i}, \qquad W = \sum_{i=1}^{n} u_i C_{\varphi_i},$$

where for each $1 \leq i \leq n$, $u_i : X \to \mathbb{C}$ is measurable function and $\varphi_i : X \to X$ is nonsingular transformation. Also we assume that for each $1 \leq i \leq n$, $\phi_i^{-1}(\Sigma)$ is a sub-σ-finite algebra of Σ. Put $h_i = d\mu \circ \phi_i^{-1}/d\mu$ and $E_i = E^{\phi_i^{-1}(\Sigma)}$. It follows that $\mathcal{P}^* = W$ and for all $f \in L^1(\Sigma)$, $\mathcal{P}(f) = \sum_{i=1}^{n} h_i E_i(u_i f) \circ \varphi_i^{-1}$. Note that the set of all these kind bounded operators is an operator algebra.

The basic properties of weighted composition operators on measurable function spaces are studied by Lambert [12, 13], Singh and Manhas [17], Takagi [18], Hudzik and Krbec [3], Cui, Hudzik, Kumar and Maligranda [4], Arora [1] and some other mathematicians. Also for a beautiful exposition of the study of classic Frobenius-Perron operators on $L^1(\Sigma)$, see [5, 6] and the references therein.

In this paper, first we give some sufficient and necessary conditions for boundedness and compactness of finite sum of weighted Frobenius-Perron operator $\mathcal{P}$ on $L^1(\Sigma)$ and finite sum of weighted composition operator W on $L^p(\Sigma)$. Then, by making use of these conditions we determine the lower and upper estimates for the essential norm of these type operators.

2. Boundedness of W and $\mathcal{P}$

Recall that the weighted Frobenius-Perron operator $\mathcal{P}_{\varphi_i}^{u_i}$ is a bounded operator on $L^1(\Sigma)$ if and only if $u_i \in L^\infty(\Sigma)$ and its norm is given by $\|\mathcal{P}_{\varphi_i}^{u_i}\| = \|u_i\|_\infty$ (see [9]). Thus, if u_i's are nonnegative and $u_i \in L^\infty(\Sigma)$, then $\|\mathcal{P}\| \leq \|\sum_{i=1}^{n} u_i\|_\infty$. Now, suppose that $\mathcal{P}$ is bounded. Then for each $f \in L^1(\Sigma)$, there exists complex valued function $w : X \to \mathbb{C}$ such that $\bar{w}uf = |uf|$, where $u = \sum_{i=1}^{n} u_i$. Thus

$$\begin{aligned}
\|M_u f\|_1 &= \int_X \bar{w}uf\,d\mu = \sum_{i=1}^{n} \int_X \bar{w}u_i f\,d\mu = \sum_{i=1}^{n} \int_X h_i E_i(\bar{w}u_i f) \circ \varphi_i^{-1} d\mu \\
&= \sum_{i=1}^{n} \int_X \mathcal{P}_{\varphi_i}^{u_i}(\bar{w}f)\,d\mu = \int_X \mathcal{P}(\bar{w}f)\,d\mu \leq \|\mathcal{P}(\bar{w}f)\|_1 \leq \|\mathcal{P}\|\,\|f\|_1.
\end{aligned}$$

Hence $\|\sum_{i=1}^{n} u_i\|_\infty = \|u\|_\infty = \|M_u\| \leq \|\mathcal{P}\|$ and so $\|\sum_{i=1}^{n} u_i\|_\infty \leq \|\mathcal{P}\|$. These observations establish the following proposition.

Proposition 2.1 — Let $\mathcal{P}$ be a finite sum of weighted Frobenius-Perron operators on $L^1(\Sigma)$. If $\mathcal{P}$ is bounded, then $\sum_{i=1}^{n} u_i \in L^\infty(\Sigma)$ and $\|\sum_{i=1}^{n} u_i\|_\infty \leq \|\mathcal{P}\|$. Moreover, if u_i's are nonnegative, then $\mathcal{P}$ is a bounded operator if and only if for each $1 \leq i \leq n$, $\mathcal{P}_{\varphi_i}^{u_i}$ is bounded and in this case its norm is given by $\|\mathcal{P}\| = \|\sum_{i=1}^{n} u_i\|_\infty$.

Remark 2.2 : (a) Let $U = \{\alpha \in L^\infty(\Sigma) : \alpha(\mathcal{P}f) = |\mathcal{P}f|\}$ and $\|W(\alpha)\|_\infty \leq \|W(1)\|_\infty$ for each $\alpha \in U$. Then $\mathcal{P}$ is bounded if and only if $W(1) \in L^\infty(\Sigma)$ and $\|\mathcal{P}\| = \|\sum_{i=0}^{n} u_i\|_\infty$. For, if $W(1) \in L^\infty(\Sigma)$ and $f \in L^1(\Sigma)$, then $\|\mathcal{P}f\|_1 = \int_X W(\alpha) f d\mu \leq \|W(1)\|_\infty \|f\|_1$, and so $\|\mathcal{P}\| \leq \|W(1)\|_\infty$.

(b) Let $ca(X, \Sigma, \mu)$ be the set of all complex measures absolutely continuous with respect to σ-finite measure μ. Define a mapping $\Psi : L^1(X, \Sigma\mu) \longrightarrow ca(X, \Sigma, \mu)$ by $\Psi(f) = \mu_f$ with inverse $\Psi^{-1}(\nu) = \frac{d\nu}{d\mu}$, where $\mu_f(A) := \int_X f d\mu$ for all $A \in \Sigma$. Then Ψ is bounded and $\Psi^{-1} W^* \Psi = \Sigma_{i=1}^{n} \Psi^{-1}(u_i C_{\varphi_i})^* \Psi = \Sigma_{i=1}^{n} \mathcal{P}_{\phi_j}^{u_j} = \mathcal{P}$ (see [5, 9]). Thus, $\mathcal{P}$ is bounded (compact) on $L^1(\Sigma)$ if and only if W is bounded (compact) on $L^\infty(\Sigma)$.

Recall that the weighted composition operator uC_φ on $L^p(\Sigma)$ $(1 \leq p < \infty)$ is bounded if and only if $J_\varphi := h_\varphi E^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$ (see [8]). It is easy to see that $J_\varphi = \frac{d\mu_{u,\varphi}}{d\mu}$, where $\mu_{u,\varphi}(A) := \int_{\varphi^{-1}(A)} |u|^p d\mu$, for every $A \in \Sigma$. Put $J_i = J_{\varphi_i}$. In the following we determine the lower and upper estimates for the norm of W. If $J_i \in L^\infty(\Sigma)$, then $\|W\| \leq n^{\frac{p-1}{p}} \|\sum_{i=1}^{n} J_i\|_\infty^{\frac{1}{p}} \leq \sum_{i=1}^{n} \|J_i\|_\infty^{1/p}$. Indeed, for each $f \in L^p(\Sigma)$ we have

$$\begin{aligned} \|Wf\|_p^p = \int_X |\sum_{i=1}^{n} u_i f \circ \varphi_i|^p d\mu &\leq n^{p-1} \int_X (\sum_{i=1}^{n} |u_i|^p |f|^p \circ \varphi_i) d\mu \\ &= n^{p-1} \int_X (\sum_{i=1}^{n} J_i) |f|^p d\mu \leq n^{p-1} \|\sum_{i=1}^{n} J_i\|_\infty \|f\|_p^p \end{aligned}$$

and so $\|W\| \leq n^{\frac{p-1}{p}} \|\sum_{i=1}^{n} J_i\|_\infty^{\frac{1}{p}}$. Now, suppose that u_i's are nonnegative and W is bounded. It follows that $u_i C_{\varphi_i}$'s are bounded, because $0 \leq u_i C_{\varphi_i} \leq W$. Let

$A \in \Sigma$ with $0 < \mu(A) < \infty$. Then

$$\begin{aligned}
\int_A (\sum_{i=1}^n J_i) d\mu &= \sum_{i=1}^n \int_A h_i E_i(u_i^p) \circ \varphi_i^{-1} d\mu \\
&= \sum_{i=1}^n \int_X E_i(u_i^p) \circ \varphi_i^{-1} \chi_A d\mu \circ \varphi_i^{-1} \\
&= \sum_{i=1}^n \int_X E_i(u_i^p)(\chi_A \circ \varphi_i) d\mu \\
&= \int_X (\sum_{i=1}^n u_i^p \chi_{\varphi_i^{-1}(A)}) d\mu \leq \int_X (\sum_{i=1}^n u_i \chi_{\varphi_i^{-1}(A)})^p d\mu \\
&= \int_X |\sum_{i=1}^n u_i \chi_A \circ \varphi_i|^p d\mu = \|W\chi_A\|_p^p \leq \|W\|^p \|\chi_A\|_p^p \\
&= \int_A \|W\|^p d\mu.
\end{aligned}$$

It follows that $\| \sum_{i=1}^n J_i\|_\infty^{1/p} \leq \|W\|$.

Proposition 2.3 — Let $1 \leq p < \infty$. Then the following assertions hold.

(a) If $J_i \in L^\infty(\Sigma)$, then $\|W\| \leq n^{\frac{p-1}{p}} \| \sum_{i=1}^n J_i\|_\infty^{\frac{1}{p}}$.

(b) If u_i's are nonnegative, then W is bounded if and only if $J_i \in L^\infty(\Sigma)$ and

$$\| \sum_{i=1}^n J_i\|_\infty^{\frac{1}{p}} \leq \|W\| \leq n^{\frac{p-1}{p}} \| \sum_{i=1}^n J_i\|_\infty^{\frac{1}{p}}.$$

Remark 2.4 : Let $T = uC_\varphi$. Without lose of generality, we can assume that $u \geq 0$. In fact, write $u = s|u|$ where $s \in L^\infty(\Sigma)$ satisfies $|s| = 1$. Since $M_s : L^p(\Sigma) \to L^p(\Sigma)$ is an isometric isomorphism, T is bounded (compact) if and only if $|u|C_\varphi$ has this property. However, in general, in the setting of $\sum_{i=1}^n T_i$ we can not assume that the u_i's are nonnegative. Note that if $T_1 = -T_2$ is unbounded, then $T_1 + T_2 = 0$ is bounded. In general, it may be happen for $A \subset \{1, \cdots, n\}$, $J_{i \in A} \notin$

$L^\infty(\Sigma)$ but $\sum_{i=1}^n T_i$ is bounded. By the similar argument, we conclude that the converse of Proposition 2.1(a) and Proposition 2.3(a) are not true in general.

Example 2.5 : (a) Let $X = [0,1]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformations $\varphi_i : X \to X$ by

$$\varphi_1(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}]; \\ 2x-1 & x \in (\frac{1}{2}, 1], \end{cases} \qquad \varphi_2(x) = \begin{cases} 1-2x & x \in [0, \frac{1}{2}]; \\ 2x-1 & x \in (\frac{1}{2}, 1]. \end{cases}$$

Note that $h_1(x) = h_2(x) = 1$. Then for each $0 \le a < b \le 1$ and $f \in L^1(\Sigma)$ we have

$$\begin{aligned} \int_{\varphi_1^{-1}(a,b)} f(x)dx &= \int_{\frac{a}{2}}^{\frac{b}{2}} f(x)dx + \int_{\frac{a+1}{2}}^{\frac{b+1}{2}} f(x)dx \\ &= \int_{(a,b)} \frac{1}{2}\{f(\frac{x}{2}) + f(\frac{1+x}{2})\}dx. \end{aligned}$$

Similarly, we get that

$$\begin{aligned} \int_{\varphi_2^{-1}(a,b)} f(x)dx &= \int_{\frac{1-b}{2}}^{\frac{1-a}{2}} f(x)dx + \int_{\frac{a+1}{2}}^{\frac{b+1}{2}} f(x)dx \\ &= \int_{(a,b)} \frac{1}{2}\{f(\frac{1-x}{2}) + f(\frac{1+x}{2})\}dx. \end{aligned}$$

Hence

$$\begin{aligned} (E_1(f)\circ\varphi_1^{-1})(x) &= \frac{1}{2}\{f(\frac{x}{2}) + f(\frac{1+x}{2})\}, \\ (E_2(f)\circ\varphi_2^{-1})(x) &= \frac{1}{2}\{f(\frac{1-x}{2}) + f(\frac{1+x}{2})\}. \end{aligned}$$

It follows that

$$E_1(f)(x) = \frac{1}{2}\{f(x) + f(\frac{1+2x}{2})\}\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\{f(\frac{2x-1}{2}) + f(x)\}\chi_{(\frac{1}{2},1]},$$

$$E_2(f)(x) = \frac{1}{2}\{f(x) + f(1-x)\}\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\{f(-x) + f(x)\}\chi_{(\frac{1}{2},1]}.$$

Put $u_1(x) = 8x^2$ and $u_2(x) = 4(x + \frac{1}{2})$. Then we have

$$(\mathcal{P}^{\varphi_1}_{u_1} f)(x) = x^2 f(\frac{x}{2}) + (x^2 + 2x + 1) f(\frac{1+x}{2}),$$

$$(\mathcal{P}^{\varphi_2}_{u_2} f)(x) = (2 - x) f(\frac{1-x}{2}) + (2 + x) f(\frac{1+x}{2}).$$

Since u_1 and u_2 are nonnegative, by Proposition 2.1 we have $\|\mathcal{P}\| = \|u_1 + u_2\|_\infty = 14 = \|\mathcal{P}^{\varphi_1}_{u_1}\| + \|\mathcal{P}^{\varphi_2}_{u_2}\|$, where

$$(\mathcal{P}f)(x) = x^2 f(\frac{x}{2}) + (x^2 + 3x + 3) f(\frac{1+x}{2}) + (2 - x) f(\frac{1-x}{2}).$$

Direct computations show that

$$J_1(x) = 2^{p-1}\{x^{2p} + (1 + 2x + x^2)^p\},$$

$$J_2(x) = 2^{p-1}\{(2 + x)^p + (2 - x)^p\}.$$

Thus, $\|u_1 C_{\varphi_1}\|^p = \|J_1\|_\infty = 2^{p-1}(1 + 4^p)$, and $\|u_2 C_{\varphi_2}\|^p = \|J_2\|_\infty = 2^{p-1}(1 + 3^p)$. Hence by Proposition 2.3 we have

$$2^{\frac{p-1}{p}}(2 + 3^p + 4^p)^{1/p} \le \|W\| \le 4^{\frac{p-1}{p}}(2 + 3^p + 4^p)^{1/p}.$$

(b) Let $X = [0, 1]$, $d\mu = dx$ and Σ be the Lebesgue sets. Take $u_1(x) = x^2$, $u_2(x) = (2 - x)^2$, $u_3(x) = \frac{1}{\sqrt{x}}$ and $\varphi_1(x) = \varphi_2(x) = x^2$, $\varphi_3(x) = x$. Let $T_i := u_i C_{\varphi_i} : L^1(\Sigma) \to L^1(\Sigma)$. It follows that

$$J_1(x) = \frac{\sqrt{x}}{2}, J_2(x) = \frac{2}{\sqrt{x}} - \frac{3\sqrt{x}}{2}, J_3(x) = \frac{1}{\sqrt{x}}.$$

Then T_1 is bounded but T_2, T_3 and $T_1 - T_2 = (2x - 4)C_{\varphi_1}$ are not bounded operators on $L^1(\Sigma)$. Since $\|(T_2 + T_3)f\|_1 \ge \|M_{J_2 - J_3} f\|_1$, $T_2 + T_3$ and $T_1 - T_2 + T_3 = (2x - 4)C_{\varphi_1} + \frac{1}{\sqrt{x}} C_{\varphi_3}$ are not bounded, because $J_2(x) - J_3(x) = \frac{1}{\sqrt{x}} - \frac{3\sqrt{x}}{2}$ and $J_1(x) - J_2(x) + J_3(x) = 2\sqrt{x} - \frac{1}{\sqrt{x}}$ are not in $L^\infty(\Sigma)$. By the similar computations we get that

$$\mathcal{P}^{u_1}_{\varphi_1} = M_{\frac{\sqrt{x}}{2}} C_{\sqrt{x}}, \quad \mathcal{P}^{u_2}_{\varphi_2} = M_{\frac{(2-\sqrt{x})^2}{2\sqrt{x}}} C_{\sqrt{x}}, \quad \mathcal{P}^{u_3}_{\varphi_3} = M_{\frac{1}{\sqrt{x}}},$$

and so $\|\mathcal{P}^{u_1}_{\varphi_1}\| = 1$, $\|\mathcal{P}^{u_2}_{\varphi_2}\| = 4$, $\|\mathcal{P}^{u_3}_{\varphi_3}\| = \infty$. It follows that $\mathcal{P} := \mathcal{P}^{u_1}_{\varphi_1} \pm \mathcal{P}^{u_2}_{\varphi_2} \pm \mathcal{P}^{u_3}_{\varphi_3}$ is not bounded on $L^1(\Sigma)$ and $\sum_{i=1}^{3} u_i phi_i^{-1} otin L^\infty(\Sigma)$.

3. Compactness of W and $\mathcal{P}$

In this section we shall give necessary and sufficient conditions for $\mathcal{P}$ and W to be compact on $L^1(\Sigma)$ and $L^p(\Sigma)$ respectively. Let T be a linear operator on a Banach space $\mathfrak{B}$. Then T is said to be compact if, for every bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{B}$, the sequence $\{Tf_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathfrak{B}$. When $1 \leq p < \infty$, a characterization for compact weighted composition operator $L^p(\Sigma)$ spaces was obtained by Takagi [18] and independently around the same time by Chan [2]. Chan has showed that uC_φ is compact on $L^p(\Sigma)$ if and only if

$$\text{for each } \varepsilon > 0, \{x \in X : J_\varphi(x) \geq \varepsilon\} \text{ consists of finitely many atoms.} \tag{1}$$

Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as $X = B \cup \{A_j : j \in \mathbb{N}\}$, where $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection of pairwise disjoint atoms and $B \in \Sigma$, being disjoint from each A_j, is non-atomic (see [19]). Since Σ is σ-finite, so $a_j := \mu(A_j) < \infty$, for all $j \in \mathbb{N}$.

Let (X, Σ, μ) be a non-atomic σ-finite measure space. Then no bounded weighted Frobenius-Perron operator on $L^1(\Sigma)$ is compact unless it is the zero operator (see [9]). Also, in [11] has been showed that the bounded operator $\mathcal{P}^{u_i}_{\varphi_i}$ is a compact operator on $L^1(\Sigma)$ if and only if $u_i(\varphi_i^{-1}(B)) = 0$ and for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \mu(\varphi_i^{-1}(A_n) \cap N_\varepsilon(u)) > 0\}$ is finite, where $N_\varepsilon(u) = \{x \in X : |u_i(x)| \geq \varepsilon\}$. In the following we give an another sufficient condition and, under certain conditions, a necessary condition for the operator $\mathcal{P}$ to be compact.

***Theorem* 3.1** — *Let $u = \sum_{i=1}^n |u_i|$, $a_j = \mu(A_j) < \infty$ and let $N_\varepsilon(u) = \{x \in X : |u(x)| \geq \varepsilon\}$.*

(a) If for each $\varepsilon > 0$, $N_\varepsilon(u)$ consists of finitely many atoms, then $\mathcal{P}$ is compact.

(b) Let (X, Σ, μ) be a purely atomic measure space and let the sequence $\{a_j\}_{j\in\mathbb{N}}$

has no subsequence that converges to zero. If u_i's are nonnegative, then $\mathcal{P}$ is compact if and only if for each $\varepsilon > 0$, $N_\varepsilon(u)$ consists of finitely many atoms.

PROOF : (a) Let $\varepsilon > 0$ and $A = N_\varepsilon(u) = \cup_{j=1}^k A_\varepsilon^j$, where A_ε^j's are disjoint atoms. Suppose that $\mathcal{P}' = \mathcal{P}M_{\chi_A}$, where $M_{\chi_A}f = \chi_A f = \sum_{j=1}^k f(A_\varepsilon^j)\chi_{A_\varepsilon^j}$. It follows that $\mathcal{P}'$ is a finite rank operator on $L^1(\Sigma)$. Since for every $f, g \in L^1(\Sigma)$, $E_i(f) \circ \varphi_i^{-1} - E_i(g) \circ \varphi_i^{-1} = E_i(f - g) \circ \varphi_i^{-1}$, we have

$$\begin{aligned}
\|\mathcal{P}f - \mathcal{P}'f\|_1 &= \int_X |\sum_{i=1}^n h_i E_i(u_i f \chi_{X\setminus A}) \circ \varphi_i^{-1}| d\mu \\
&\leq \int_X \sum_{i=1}^n h_i E_i(|u_i f|\chi_{X\setminus A}) \circ \varphi_i^{-1} d\mu \\
&= \sum_{i=1}^n \int_X E_i(|u_i f|\chi_{X\setminus A}) \circ \varphi_i^{-1} d\mu \circ \varphi_i^{-1} \\
&= \sum_{i=1}^n \int_X E_i(|u_i f|\chi_{X\setminus A}) d\mu \\
&= \sum_{i=1}^n \int_X |u_i f|\chi_{X\setminus A} d\mu = \int_X \sum_{i=1}^n |u_i f|\chi_{X\setminus A} d\mu \\
&= \int_{X\setminus A} u|f| d\mu \leq \varepsilon \|f\|_1.
\end{aligned}$$

Hence $\|\mathcal{P} - \mathcal{P}'\| \leq \varepsilon$, and so $\mathcal{P}$ is compact.

(b) By part (a) we only show that if $\mathcal{P}$ is compact, then for each $\varepsilon > 0$, $N_\varepsilon(u)$ consists of finitely many atoms. Suppose on the contrary. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(u)$ contains infinitely many atoms. Let $\{A_j\}_{j\in\mathbb{N}}$ be disjoint atoms in $N_\varepsilon(u)$. Put $g_j = \frac{\chi_{A_j}}{\mu(A_j)}$. Let $A \in \Sigma$ with $0 < \mu(A) < \infty$. Since the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero, $\{j \in \mathbb{N} : A_j \subseteq A\}$ is finite, and so $\mu(A_j \cap A) = 0$ for sufficiently large j. Since characteristic functions are dense in $L^\infty(\Sigma)$ and

$$\left| \int_X \frac{\chi_{A_j}}{\mu(A_j)} \chi_A d\mu \right| = \frac{\mu(A_j \cap A)}{\mu(A_j)} \to 0,$$

as $j \to \infty$, it follows that $g_j \to 0$ weakly. Now, Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|\mathcal{P}g_j\|_1 \to 0$. On the other hand we have

$$\begin{aligned}\|\mathcal{P}g_j\|_1 &= \int_X |\sum_{i=1}^n h_i E_i(u_i g_j)\circ\varphi_i^{-1}|d\mu = \sum_{i=1}^n \int_X h_i E_i(u_i g_j)\circ\varphi_i^{-1} d\mu \\ &= \sum_{i=1}^n \int_X u_i g_j d\mu = \int_{A_j}\sum_{i=1}^n u_i \frac{\chi_{A_j}}{\mu(A_j)} d\mu = \int_{A_j} u \frac{\chi_{A_j}}{\mu(A_j)} d\mu \geq \varepsilon.\end{aligned}$$

But this is a contradiction.

It is known that if (X, Σ, μ) is a non-atomic and $1 \leq p < \infty$, then uC_φ is a compact operator on $L^p(\Sigma)$ if and only if it is the zero operator (see [18]). For $\varepsilon > 0$, take $N_\varepsilon(\sqrt[p]{J_i}) = \{x : \sqrt[p]{J_i(x)} \geq \varepsilon\}$. So if $N_\varepsilon(\sqrt[p]{J})$ with $J = \sum_{i=1}^n J_i$ consists of finitely many atoms, then W is compact. Now, suppose that u_i's are nonnegative and W is compact. Since for each $1 \leq i \leq n$, $0 \leq u_i C_{\varphi_i} \leq W$, then by Dodds-Fremlin theorem ([14]) $u_i C_{\varphi_i}$'s are also compact.

***Theorem* 3.2** — *Let $1 < p < \infty$, u_i's are nonnegative and the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero. If W is a compact operator on $L^p(\Sigma)$, then for each $\varepsilon > 0$, $N_\varepsilon(\sqrt[p]{J}) = \{x : \sqrt[p]{J(x)} \geq \varepsilon\}$ consists of finitely many atoms, where $J = \sum_{i=1}^n J_i$.*

PROOF : Suppose, on the contrary, there exists $\varepsilon > 0$ such that $N_\varepsilon(\sqrt[p]{J})$ either contains a non-atomic subset or has infinitely many atoms. If $N_\varepsilon(\sqrt[p]{J})$ contains a non-atomic subset B, then we can choose $\{B_j\}_{j\in\mathbb{N}} \subseteq \Sigma$ such that $B_{j+1} \subseteq B_j \subseteq B$ with $0 < \mu(B_j) < \frac{1}{j}$. For each $j \in \mathbb{N}$, define $f_j = \frac{\chi_{B_j}}{\mu(B_j)^{1/p}}$. Obviously, $\|f_j\|_p = 1$. We show that $f_j \to 0$ weakly. Let $A \subseteq \Sigma$ with $0 < \mu(A) < \infty$. Since $p > 1$, we get that

$$\left|\int_X f_j \chi_A d\mu\right| = \frac{\mu(B_j \cap A)}{\mu(B_j)^{\frac{1}{p}}} \leq \mu(B_j)^{1-\frac{1}{p}} \leq (\frac{1}{j})^{1-\frac{1}{p}} \to 0,$$

as $j \to \infty$, and so $f_j \to 0$ weakly. Since W is compact, by assumption, it follows $\|Wf_j\|_p \to 0$. Now, assume that $N_\varepsilon(\sqrt[p]{J})$ contains infinitely many atoms.

Let $\{A_j\}_{j\in\mathbb{N}}$ be disjoint atoms in $N_\varepsilon(\sqrt[p]{J})$. The same method used in the proof of Theorem 3.1(b) yields $g_j \to 0$ weakly, where $g_j := \frac{\chi_{A_j}}{\mu(A_j)^{1/p}}$. Therefore, $\|Wg_j\|_p \to 0$. On the other hand

$$\begin{aligned}\|Wf_j\|_p^p &= \int_X |\sum_{i=1}^n u_i \frac{\chi_{B_j}}{\mu(B_j)^{\frac{1}{p}}} \circ \varphi_i|^p d\mu \geq \sum_{i=1}^n \int_X u_i^p \frac{\chi_{B_j} \circ \varphi_i}{\mu(B_j)} d\mu \\ &= \frac{1}{\mu(B_j)} \sum_{i=1}^n \int_X J_i \chi_{B_j} d\mu = \frac{1}{\mu(B_j)} \int_{B_j} J d\mu \geq \varepsilon^p.\end{aligned}$$

Similarly, $\|Wg_j\|_p \geq \varepsilon$. But this is a contradiction.

4. Essential Norm of W and $\mathcal{P}$

Let $\mathfrak{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathfrak{B}$. For $T \in L(\mathfrak{B})$, the Banach algebra of all bounded linear operators on $\mathfrak{B}$ into itself, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely $\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}$. Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen in [16], the essential norm plays an interesting role in the compact problem of concrete operators. Many people have computed the essential norm of (weighted) composition operators on various function spaces. In [10], the essential norm of uC_φ on $L^p(\Sigma)$ with $1 < p < \infty$ have been computed by one of the authors as follows:

$$\|uC_\varphi\|_e = \inf\{r > 0 : G_r \text{ consists of finitely many atoms}\}, \tag{4.1}$$

where $G_r = \{x \in X : \sqrt[p]{J(x)} \geq r\}$. The question of actually calculating the essential norm of $\mathcal{P}$ and W on $L^p(\Sigma)$ spaces are not trivial. In this section, in spite of the difficulties associated with computing the essential norm exactly, we will try to determine the lower and upper estimates for the essential norm of $\mathcal{P}$ and W defined on $L^1(\Sigma)$ and $L^p(\Sigma)$ respectively.

***Theorem* 4.1** — *(a) Let $\mathcal{P} = \sum_{i=1}^n \mathcal{P}_{\varphi_i}^{u_i}$ be a bounded operator on $L^1(\Sigma)$. Put $u = \sum_{i=1}^n |u_i|$ and $\beta = \inf\{r > 0 : N_r(u)$ consists of finitely many atoms$\}$. Then $\|\mathcal{P}\|_e \leq \beta$.*

(b) Let (X, Σ, μ) be a purely atomic measure space. If the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero and u_i's are nonnegative, then $\|\mathcal{P}\|_e \geq \beta$.

PROOF : (a) Take $\varepsilon > 0$ arbitrary. Put $K = N_{\beta+\varepsilon}(u)$. The definition of β implies that K consists of finitely many atoms. Put $\mathcal{P}' = \mathcal{P}M_{\chi_K}$. It is easy to see that $\mathcal{P}'$ is a finite rank operator on $L^1(\Sigma)$. Hence, for every $f \in L^1(\Sigma)$, we have

$$\begin{aligned}
\|\mathcal{P}f - \mathcal{P}'f\| &= \int_X |\sum_{i=1}^{n} h_i E_i(u_i f \chi_{X\setminus K}) \circ \varphi_i^{-1}| d\mu \\
&\leq \int_X \sum_{i=1}^{n} h_i E_i(|u_i f| \chi_{X\setminus K}) \circ \varphi_i^{-1} d\mu \\
&= \sum_{i=1}^{n} \int_X E_i(|u_i f| \chi_{X\setminus K}) \circ \varphi_i^{-1} d\mu \circ \varphi_i^{-1} \\
&= \sum_{i=1}^{n} \int_X E_i(|u_i f| \chi_{X\setminus K}) d\mu \\
&= \sum_{i=1}^{n} \int_X |u_i||f| \chi_{X\setminus K} d\mu = \int_X \left(\sum_{i=1}^{n} |u_i|\right) |f| \chi_{X\setminus K} d\mu \\
&= \int_{X\setminus K} u|f| d\mu \leq (\varepsilon + \beta)\|f\|_1.
\end{aligned}$$

Thus, $\|\mathcal{P} - \mathcal{P}'\| \leq \varepsilon + \beta$. On the other hand, compactness of $\mathcal{P}'$ implies that $\|\mathcal{P}\|_e \leq \|\mathcal{P} - \mathcal{P}'\| \leq \varepsilon + \beta$, and so $\|\mathcal{P}\|_e \leq \varepsilon + \beta$. Consequently, $\|\mathcal{P}\|_e \leq \beta$.

(b) Let $0 < \varepsilon < \beta$. Then by definition, $N_{\beta-\varepsilon}(u)$ contains infinitely many atoms such as $\{A_j\}_{j\in N}$. Put $g_j = \frac{\chi_{A_j}}{\mu(A_j)}$. The same method used in the proof of Theorem 3.1(b) yields $g_j \to 0$ weakly and $\|\mathcal{P}g_j\|_1 \geq \beta - \varepsilon$. Now, take a compact operator T on $L^p(\Sigma)$ such that $\|\mathcal{P} - T\| < \|\mathcal{P}\|_e + \varepsilon$. Since T is compact, $\|Tg_j\|_1 < \varepsilon$ for sufficiently large j. Therefore, we get that

$$\|\mathcal{P}\|_e \geq \|\mathcal{P} - T\| - \varepsilon \geq \|\mathcal{P}g_j - Tg_j\|_1 - \varepsilon \geq \|\mathcal{P}g_j\|_1 - \|Tg_j\|_1 - \varepsilon \geq \beta - \varepsilon - \varepsilon.$$

Thus, $\|\mathcal{P}\|_e \geq \beta - 2\varepsilon$, and so $\|\mathcal{P}\|_e \geq \beta$.

Corollary 4.2 — Let (X, Σ, μ) be a purely atomic. If the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero and u_i's are nonnegative, then $\|\mathcal{P}\|_e = \beta$.

***Theorem* 4.3** — *Let $1 < p < \infty$ and let $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator on $L^p(\Sigma)$. Put $\alpha = \inf\{r > 0 : N_r(\sqrt[p]{J})$ consists of finitely many atoms$\}$. Then the followings hold.*

(a) $\|W\|_e \leq n^{\frac{1}{q}}\alpha$, where q is conjugate component of p.

(b) If the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero and u_i's are nonnegative, then $\|W\|_e \geq \alpha$.

PROOF : (a) Take $\varepsilon > 0$ arbitrary. Put $K = N_{\alpha+\varepsilon}(\sqrt[p]{J})$, $u'_i = u_i\chi_{\varphi_i^{-1}(K)}$ and $W' = \sum_{i=0}^{n} u'_i C_{\varphi_i}$. Since by definition of α, K consists of finitely many atoms, $W' = \sum_{i=0}^{n} u_i C_{\varphi_i} M_{\chi K}$ is a finite rank operator on $L^p(\mu)$. Hence for every $f \in L^p(\Sigma)$ we get that

$$
\begin{aligned}
\|Wf - W'f\|^p &\leq n^{p-1}\sum_{i=1}^{n}\int_X |u_i|^p \chi_{X\setminus K}\circ\varphi_i |f|^p\circ\varphi_i d\mu \\
&= n^{p-1}\sum_{i=1}^{n}\int_X h_i E_i(|u_i|^p)\circ\varphi_i^{-1}\chi_{X\setminus K}|f|^p d\mu \\
&= n^{p-1}\sum_{i=1}^{n}\int_{X\setminus K} J_i|f|^p d\mu \\
&= n^{p-1}\int_{X\setminus K}\sum_{i=1}^{n} J_i|f|^p d\mu \leq n^{p-1}(\alpha+\varepsilon)^p\|f\|^p.
\end{aligned}
$$

Thus, $\|W - W'\| \leq n^{\frac{1}{q}}(\alpha+\varepsilon)$. Since W' is compact, it follows that $\|W\|_e \leq \|W - W'\| \leq n^{\frac{1}{q}}(\alpha+\varepsilon)$, and so $\|W\|_e \leq n^{\frac{1}{q}}\alpha$.

(b) If $\alpha = 0$ then by Proposition 3.2, W is compact, and so $\|W\|_e = 0$. Let $\alpha > 0$. Then for every $0 < \varepsilon < \alpha$ the set $N_{\alpha-\varepsilon}(\sqrt[p]{J})$ either contains a non-atomic subset or has infinitely many atoms. The same method used in the proof of Proposition 3.1(b) and 3.2 yields $f_j,\ g_j \to 0$ weakly, as $j \to \infty$. Now, take a compact operator T on $L^p(\Sigma)$ such that $\|W-T\|<\|W\|_e+\varepsilon$. Then we have

$$\begin{aligned}\|Wf_j\|_p^p &= \int_X |\sum_{i=1}^n u_i \frac{\chi_{B_j}}{\mu(B_j)^{\frac{1}{p}}}\circ\varphi_i|^p d\mu \geq \sum_{i=1}^n \int_X u_i^p \frac{\chi_{B_j}\circ\varphi_i}{\mu(B_j)} d\mu \\ &= \frac{1}{\mu(B_j)}\sum_{i=1}^n \int_X J_i \chi_{B_j} d\mu = \frac{1}{\mu(B_j)}\int_{B_j} J d\mu \geq (\alpha-\varepsilon)^p.\end{aligned}$$

Similarly, $\|Wg_j\|_p \geq \alpha-\varepsilon$. It follows that

$$\begin{aligned}\|W\|_e &> \|W-T\|-\varepsilon \\ &\geq \begin{cases}\|Wf_j-Tf_j\|_p-\varepsilon \geq \|Wf_j\|_p-\|Tf_j\|_p-\varepsilon \geq \alpha-2\varepsilon \\ \\ \|Wg_j-Tg_j\|_p-\varepsilon \geq \|Wg_j\|_p-\|Tg_j\|_p-\varepsilon \geq \alpha-2\varepsilon.\end{cases}\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we obtain $\|W\|_e \geq \alpha$.

Corollary 4.4 — If the sequence $\{a_j\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero and u_i's are nonnegative then

$$\alpha \leq \|W\|_e \leq n^{\frac{1}{q}}\alpha.$$

Note that if $\{A_j : j\in\mathbb{N}\}=\emptyset$, i.e. $X=B$, then $L^p(\Sigma)$ dose not admit a non-zero compact weighted Frobenius-Perron (composition operator). Thus, in this case $\mathcal{K}=\{0\}$ and hence $\|\mathcal{P}\|_e=\|\mathcal{P}\|$ and $\|W\|_e=\|W\|$.

Example 4.5 : Let $X=(-\infty,0]\cup\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Let μ be the Lebesque measure on $(-\infty,0]$ and $\mu(\{n\})=\frac{1}{2^n}$, if $n\in\mathbb{N}$. Define $\varphi_i:\mathbb{N}\to\mathbb{N}$ as:

$$\varphi_1=\chi_{\{1,2,3\}}+2\chi_{\{4\}}+3\chi_{\{5,6\}}+5\chi_{\{2n+1:n\geq3\}}+(2n-2)\chi_{\{2n:n\geq4\}}+5x\chi_{(-\infty,0]},$$

$$\varphi_2=2\chi_{\{1\}}+3\chi_{\{2,3\}}+4\chi_{\{4,5,6\}}+n\chi_{\{n:n\geq7\}}+\frac{2}{3}x\chi_{(-\infty,0]}.$$

Direct computations show that

$$h_1 = \frac{7}{4}\chi_{\{1\}} + \frac{1}{4}\chi_{\{2\}} + \frac{3}{8}\chi_{\{3\}} + \frac{1}{3}\chi_{\{5\}} + \frac{1}{4}\chi_{\{2n:n\geq 3\}} + \frac{1}{5}\chi_{(-\infty,0]},$$

$$h_2 = 2\chi_{\{2\}} + 3\chi_{\{3\}} + \frac{7}{4}\chi_{\{4\}} + \chi_{\{n:n\geq 7\}} + \frac{3}{2}\chi_{(-\infty,0]}.$$

It follows that $h_1 + h_2 = \frac{7}{4}\chi_{\{1,4\}} + \frac{9}{4}\chi_{\{2\}} + \frac{27}{8}\chi_{\{3\}} + \frac{1}{3}\chi_{\{5\}} + \frac{1}{4}\chi_{\{6\}} + \chi_{\{2n+1:n\geq 3\}} + \frac{5}{4}\chi_{\{2n:n\geq 4\}} + \frac{17}{10}\chi_{(-\infty,0]}$. Put $u_1 = u_2 = 1$ and $p > 1$. By using (4.1) and Theorem 4.3(b), we have

$$\|C_{\varphi_1}\| = (\frac{7}{4})^{1/p}, \qquad \|C_{\varphi_1}\|_e = (\frac{1}{4})^{1/p},$$

$$\|C_{\varphi_2}\| = (3)^{1/p}, \qquad \|C_{\varphi_2}\|_e = 1,$$

$$(\frac{27}{8})^{1/p} \leq \|C_{\varphi_1} + C_{\varphi_2}\| \leq 2^{\frac{p-1}{p}}(\frac{27}{8})^{1/p}, \qquad \|C_{\varphi_1} + C_{\varphi_2}\|_e \leq 2^{\frac{p-1}{p}}(\frac{5}{4})^{1/p}.$$

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