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WEIGHTED LAMBERT TYPE OPERATORS ON L^p SPACES

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Abstract. In this paper, we investigate some classic properties of weighted Lambert type operators on L^p spaces.

1. Introduction and preliminaries

Let (X,Σ,μ) be a complete σ-finite measure space. For any sub-σ-finite algebra $\mathscr{A}\subseteq\Sigma$ with $1\leqslant p\leqslant\infty$, the L^p-space $L^p(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ is abbreviated by $L^p(\mathscr{A})$, and its norm is denoted by $\|.\|_p$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $\sigma(f)=\{x\in X; f(x)\neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

For a sub-σ-finite algebra $\mathscr{A}\subseteq\Sigma$, the conditional expectation operator associated with $\mathscr{A}$ is the mapping $f\rightarrow E^{\mathscr{A}}f$, defined for all non-negative f as well as for all $f\in L^p(\Sigma)$, $1\leqslant p\leqslant\infty$, where $E^{\mathscr{A}}f$, by the Radon-Nikodym theorem, is the unique $\mathscr{A}$-measurable function satisfying

$$\int_A f\,d\mu=\int_A E^{\mathscr{A}}f\,d\mu,\quad \forall A\in\mathscr{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathscr{A}}$ is idempotent and $E^{\mathscr{A}}(L^p(\Sigma))=L^p(\mathscr{A})$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathscr{A}}(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If g is $\mathscr{A}$-measurable, then $E(fg)=E(f)g$.
- $|E(f)|^p\leqslant E(|f|^p)$.
- If $f\geqslant 0$, then $E(f)\geqslant 0$; if $f>0$, then $E(f)>0$.
- $|E(fg)|\leqslant E(|f|^p)|^{\frac{1}{p}}E(|g|^{p'})|^{\frac{1}{p'}}$, where $\frac{1}{p}+\frac{1}{p'}=1$ (Hölder inequality).
- For each $f\geqslant 0$, $\sigma(f)\subseteq\sigma(E(f))$.

A detailed discussion and verification of most of these properties may be found in [10]. We recall that an $\mathscr{A}$-atom of the measure μ is an element $A\in\mathscr{A}$ with $\mu(A)>0$ such that for each $F\in\mathscr{A}$, if $F\subseteq A$, then either $\mu(F)=0$ or $\mu(F)=\mu(A)$. A measure

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space (X,Σ,μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space $(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ can be partitioned uniquely as $X = (\bigcup_{n\in\mathbb{N}} A_n)\cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathscr{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [12]).

Let $u\in L^0(\Sigma)$. Then u is said to be conditionable with respect to E if $u\in\mathscr{D}(E) := \{f\in L^0(\Sigma) : E(|f|)\in L^0(\mathscr{A})\}$. Take u and w in $\mathscr{D}(E)$. Then the pair (u,w) induces a linear operator T from $L^p(\Sigma)$ into $L^0(\Sigma)$ defined by $T := M_wEM_u$, where M_u and M_w are multiplication operators. Note that for all $f\in L^p(\Sigma)$, $uf\in\mathscr{D}(E)$ (see [2]). If T takes $L^p(\Sigma)$ into $L^q(\Sigma)$ ($1\leqslant p,q\leqslant\infty$), then we call T a weighted Lambert type operator from $L^p(\Sigma)$ into $L^q(\Sigma)$. An easy consequence of the closed graph theorem and the result guaranteing a pointwise convergent subsequence for each $L^p(\Sigma)$ convergent sequence assures us that every weighted Lambert type operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ is a bounded linear operator from $L^p(\Sigma)$ into $L^q(\Sigma)$. Throughout this paper we assume that u and w are in $\mathscr{D}(E)$, $E = E^{\mathscr{A}}$ and $T = M_wEM_u$.

Combination of conditional expectation operator E and multiplication operators appears more often in the service of the study of other operators such as multiplication operators and weighted composition operators. Specifically, in [9], S.-T. C. Moy has characterized all operators on L^p of the form $f\to E(fg)$ for g in L^q with $E(|g|)$ bounded. In [1], R. G. Douglas analyzed positive projections on L_1 and many of his characterizations are in terms of combinations of multiplications and conditional expectations.

Some results of this article are a generalization of the work done in [3, 4] and [7]. In the next section, boundedness of $T = M_wEM_u$ acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator. Also a necessary and sufficient condition for compactness of these type operators will be investigated. In section 3, we discuss measure theoretic characterizations for weighted Lambert type operators in some operator classes on $L^2(\Sigma)$ such as, p-hyponormal, p-quasihyponormal. Next, we shall obtain the polar decomposition and the Aluthge transformation of T.

2. Bounded and compact weighted Lambert type operators

THEOREM 2.1. *(a) Let $1<p<\infty$ and p' be the conjugate exponent to p. Then the pair (w,u) induces a weighted Lambert type operator $T = M_wEM_u$ from $L^p(\Sigma)$ into $L^p(\Sigma)$ if and only if $(E|w|^p)^{\frac{1}{p}}(E|u|^{p'})^{\frac{1}{p'}}\in L^\infty(\mathscr{A})$, and in this case its norm is given by $\|T\| = \|(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty$.*

(b) The pair (w,u) induces a weighted Lambert type operator T from $L^1(\Sigma)$ into $L^1(\Sigma)$ if and only if $uE(|w|)\in L^\infty(\Sigma)$ and $\|T\| = \|uE(|w|)\|_\infty$.

Proof. (a) Let $f\in L^p(\Sigma)$. As an application of the properties of the conditional expectation operator we have

$$\begin{aligned}\|Tf\|_p^p &= \int_X |w|^p |E(uf)|^p d\mu = \int_X E(|w|^p)|E(uf)|^p d\mu \\ &= \int_X |E(u(E(|w|^p))^{\frac{1}{p}} f)|^p d\mu = \|EM_v f\|_p^p,\end{aligned}$$

where $v := u(E(|w|^p))^{\frac{1}{p}}$. Hence T is bounded from $L^p(\Sigma)$ into $L^p(\Sigma)$ if and only if $R_v := EM_v$ from $L^p(\Sigma)$ into $L^p(\mathscr{A})$ is bounded. A straightforward calculation shows that the adjoint operator $R_v^* : L^{p'}(\mathscr{A}) \to L^{p'}(\Sigma)$ is given by $R_v^* f = \bar{u} f$. Note that we can consider $R_v^* : L^{p'}(\Sigma) \to L^{p'}(\Sigma)$ as $R_v^* = M_{\bar{v}} E$. It is proved by Alan Lambert in [7] and subsequentially proved by John David Herron in a different method in [3] that R_v^* is a bounded operator if and only if $E(|v|^{p'}) \in L^\infty(\mathscr{A})$ and $\|R_v^*\| = \|E(|v|^{p'})\|_\infty^{1/p'}$. Thus, the pair (w,u) induces a weighted Lambert type operator T from $L^p(\Sigma)$ into $L^p(\Sigma)$ if and only if $(E|w|^p)^{\frac{1}{p}}(E|u|^{p'})^{\frac{1}{p'}} = \sqrt[p']{E(|v|^{p'})} \in L^\infty(\mathscr{A})$, and in this case $\|T\| = \|(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty$.

(b) Let $f \in L^1(\Sigma)$. Then we have

$$\|Tf\|_1 = \int_X |wE(uf)|d\mu = \int_X |E(E(|w|)uf)|d\mu = \|EM_{E(|w|)u}\|_1.$$

Hence T from $L^1(\Sigma)$ into $L^1(\Sigma)$ is bounded if and only if $E(|w|)u \in L^\infty(\Sigma)$ and in this case $\|T\| = \|EM_{E(|w|)u}\| = \|uE(|w|)\|_\infty$. To check this fact, one may refer to [3, Theorem 2.1.1] or [6]. □

THEOREM 2.2. *Let $1 < q < p < \infty$ and let p' and q' be conjugate components to p and q respectively. Then the pair (w,u) induces a weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $(E|u|^{p'})^{\frac{1}{p'}}(E|w|^q)^{\frac{1}{q}} \in L^r(\mathscr{A})$, where $\frac{1}{q'} + \frac{1}{r} = \frac{1}{p'}$. In this case, its norm is given by $\|T\| = \|(E|u|^{p'})^{\frac{1}{p'}}(E|w|^q)^{\frac{1}{q}}\|_r$.*

Proof. Let $f \in L^p(\Sigma)$. Then

$$\begin{aligned}\|Tf\|_q^q &= \int_X |w|^q |E(uf)|^q d\mu = \int_X E(|w|^q)|E(uf)|^q d\mu \\ &= \int_X |E(u(E(|w|^q))^{\frac{1}{q}} f)|^q d\mu = \|EM_v f\|_q^q,\end{aligned}$$

where $v := u(E(|w|^q))^{\frac{1}{q}}$. It follows that the pair (w,u) induces a weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $R_v^* = M_{\bar{v}} E = M_{\bar{v}} : L^{q'}(\mathscr{A}) \to L^{p'}(\Sigma)$ is bounded. We claim that $M_{\bar{v}} : L^{q'}(\mathscr{A}) \to L^{p'}(\Sigma)$ is bounded if and only if $(E(|v|^{p'}))^{\frac{1}{p'}} \in L^r(\mathscr{A})$. Suppose that $M_{\bar{v}}$ is bounded. Define $\Lambda : L^{\frac{q'}{p'}}(\mathscr{A}) \to \mathbb{C}$ given by $\Lambda(f) = \int_X E(|v|^{p'}) f d\mu$. We show that the linear functional Λ is bounded. For each $f \in L^{\frac{q'}{p'}}(\mathscr{A})$ we have

$$\begin{aligned}|\Lambda(f)| &\leqslant \int_X E(|v|^{p'})|f|d\mu = \int_X |v|^{p'}|f|d\mu = \int_X (|v|f|^{\frac{1}{p'}}|)^{p'}d\mu \\ &= \|M_{\bar{v}}|f|^{\frac{1}{p'}}\|_{q'}^{p'} \leqslant \|M_{\bar{v}}\|^{p'}\| \, |f|^{\frac{1}{p'}}\|_{q'}^{p'} = \|M_{\bar{v}}\|^{p'}\|f\|_{\frac{q'}{p'}}.\end{aligned}$$

This implies that $\sup\{\int_X E(|u|^{p'})|f|d\mu : f \text{ is a unit vector in } L^{\frac{q'}{p'}}(\Sigma)\} \leqslant \|M_u\|^{p'}$. It follows that $\|\Lambda\| \leqslant \|(E(|u|^{p'}))^{\frac{1}{p'}}\|_r \leqslant \|M_{\bar{v}}\|$. By the Riesz representation theorem, there exists a unique function $g \in L^{\frac{r}{p'}}(\mathscr{A})$ such that $\Lambda(f) = \int_X gfd\mu$, for each $f \in L^{\frac{q'}{p'}}(\mathscr{A})$. Therefore, $g = E(|v|^{p'})$ on X, and so $(E(|v|^{p'}))^{\frac{1}{p'}} \in L^r(\mathscr{A})$.

Conversely, if $(E(|v|^{p'}))^{\frac{1}{p'}} \in L^r(\mathscr{A})$, by Hölder's inequality we have

$$\begin{aligned}\|M_{\bar{v}}f\|_{p'}^{p'} &= \int_X |v|^{p'}|f|^{p'}d\mu = \int_X E(|v|^{p'})|f|^{p'}d\mu \\ &\leqslant \|E(|v|^{p'})\|_{\frac{r}{p'}}\|f\|_{q'}^{p'} = \|(E(|v|^{p'}))^{\frac{1}{p'}}\|_r^{p'}\|f\|_{q'}^{p'}.\end{aligned}$$

Thus $\|M_{\bar{v}}\| \leqslant \|(E(|u|^{p'}))^{\frac{1}{p'}}\|_r$ and hence $M_{\bar{v}}$ is bounded. Therefore, the proof of theorem is completed. □

THEOREM 2.3. *Let $1 < p < q < \infty$ and let p', q' be conjugate components to p and q respectively. Then the pair (w,u) induces a weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if*

(i) $E(|u|^{p'})(E(|w|^q))^{\frac{p'}{q}} = 0$ on B;

(ii) $M := \sup_{n\in\mathbb{N}} \frac{E(|u|^{p'})(A_n)(E(|w|^q))^{\frac{p'}{q}}(A_n)}{\mu(A_n)^{\frac{p'}{r}}} < \infty$, where $\frac{1}{p'} + \frac{1}{r} = \frac{1}{q'}$.

In this case, it's norm is given by $\|T\|^{p'} = M$.

Proof. Let $f \in L^p(\Sigma)$. Then $\|Tf\|_q = \|EM_vf\|_q$, where $v := u(E(|w|^q))^{\frac{1}{q}}$. Hence $T : L^p(\Sigma) \to L^q(\Sigma)$ is bounded if and only if $EM_v : L^p(\Sigma) \to L^q(\mathscr{A})$ is bounded if and only if $(EM_v)^* = M_{\bar{v}}E = M_{\bar{v}} : L^{q'}(\mathscr{A}) \to L^{p'}(\Sigma)$. So, in order to prove theorem, it suffices to show that $M_{\bar{v}} : L^{q'}(\mathscr{A}) \to L^{p'}(\Sigma)$ is bounded if and only if

(1) $E(|v|^{p'}) = 0$ on B;

(2) $M' := \sup_{n\in\mathbb{N}} \frac{E(|v|^{p'})(A_n)}{\mu(A_n)^{\frac{p'}{r}}} < \infty$.

Suppose that (1) and (2) hold. Since $\mathscr{A}$-measurable functions are constant on each $\mathscr{A}$-atom, then for every $f \in L^{q'}(\mathscr{A})$ with $\|f\|_{q'} \leqslant 1$, we get that

$$\begin{aligned}\|M_{\bar{v}}f\|_{p'}^{p'} &= \int_X |v|^{p'}|f|^{p'}d\mu = \int_{B\cup(\cup_{n\in\mathbb{N}}A_n)} E(|v|^{p'})|f|^{p'}d\mu \\ &= \sum_{n\in\mathbb{N}}\int_{A_n} E(|v|^{p'})|f|^{p'}d\mu = \sum_{n\in\mathbb{N}} E(|v|^{p'})(A_n)|f(A_n)|^{p'}\mu(A_n) \\ &= \sum_{n\in\mathbb{N}} \frac{E(|v|^{p'})(A_n)}{\mu(A_n)^{\frac{p'}{r}}}|f(A_n)|^{p'}\mu(A_n)^{\frac{p'}{q'}} \leqslant M'\|f\|_{q'}^{p'}, \quad (\text{since } \frac{p'}{q'}-\frac{p'}{r}=1).\end{aligned}$$

This implies that $\|M_{\bar{v}}\|^{p'} \leqslant M'$, and so $M_{\bar{v}}$ is bounded.

Conversely, suppose that the multiplication operator $M_{\bar{v}}$ is bounded. First, we show that $E(|v|^{p'})=0$ on B. Suppose on the contrary. Thus we can find some $\delta>0$ such that $\mu(\{x\in B: E(|v|^{p'})(x)>\delta\})>0$. Take $F=\{x\in B: E(|v|^{p'})(x)>\delta\}$. Since $F\subseteq B$ is a $\mathscr{A}$-measurable set and $\mathscr{A}$ is σ-finite, then foe each $n\in\mathscr{N}$, there exists $F_n\subseteq F$ with $F_n\in\mathscr{A}$ such that $\mu(F_n)=\frac{\mu(F)}{2^n}$. Define $f_n=\frac{\chi_{F_n}}{\mu(F_n)^{1/q'}}$. It is clear that $f_n\in L^{q'}(\mathscr{A})$ and $\|f_n\|_{q'}=1$. Thus

$$\begin{aligned}\infty > \|M_{\bar{v}}\|^{p'} \geqslant \|\bar{v}f_n\|_{p'}^{p'} &= \frac{1}{\mu(F_n)^{\frac{p'}{q'}}}\int_{F_n}|v|^{p'}d\mu \\ &= \frac{1}{\mu(F_n)^{\frac{p'}{q'}}}\int_{F_n}E(|v|^{p'})d\mu \geqslant \frac{\delta\mu(F_n)}{\mu(F_n)^{\frac{p'}{q'}}} = \delta\mu(F_n)^{1-\frac{p'}{q'}} \\ &= \delta\left(\frac{2^n}{\mu(F)}\right)^{\frac{p'}{q'}-1} \to \infty\end{aligned}$$

when $n\to\infty$, since $\frac{p'}{q'}>1$. But this is a contradiction. It remains to prove (2). Let $f_n=\frac{\chi_{A_n}}{\mu(A_n)^{1/q'}}$. Thus $f_n\in L^{q'}(\mathscr{A})$ and $\|f_n\|_{q'}=1$. Then we get that

$$\begin{aligned}\frac{E(|v|^{p'})(A_n)}{\mu(A_n)^{\frac{p'}{r}}} &= \frac{E(|v|^{p'})(A_n)\mu(A_n)}{\mu(A_n)^{\frac{p'}{q'}}} = \int_{A_n}\frac{E(|v|^{p'})}{\mu(A_n)^{\frac{p'}{q'}}}d\mu \\ &= \int_X \frac{E(|v|^{p'})\chi_{A_n}}{\mu(A_n)^{\frac{p'}{q'}}}d\mu = \int_X \frac{|v|^{p'}\chi_{A_n}}{\mu(A_n)^{\frac{p'}{q'}}}d\mu = \|\bar{v}f_n\|_{p'}^{p'} \leqslant \|M_{\bar{v}}\|^{p'}.\end{aligned}$$

This implies that $M'\leqslant\|M_{\bar{v}}\|^{p'}<\infty$. Hence the proof is completed. □

Suppose that $X=(\cup_{n\in\mathbb{N}}C_n)\cup C$, where $\{C_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint Σ-atoms and $C\in\Sigma$, being disjoint from each C_n, is non-atomic. Note that $(\cup_{n\in\mathbb{N}}C_n)\cap\mathscr{A}\subseteq\cup_{n\in\mathbb{N}}A_n$ and $B\subseteq C$.

THEOREM 2.4. *(a) Let $1<p<\infty$ and let p' be conjugate component to p. Then the pair (w,u) induces a weighted Lambert type operator T from $L^p(\Sigma)$ into $L^1(\Sigma)$ if* *and only if* $uE(|w|) \in L^{p'}(\Sigma)$. *In this case, its norm is given by* $\|T\| = \|uE(|w|)\|_{p'}$.

(b) Let $1 < q < \infty$ *and let* q' *be conjugate component to* q. *If* T *from* $L^1(\Sigma)$ *into* $L^q(\Sigma)$ *is bounded, then* $E(|u|^{q'})(E(|w|^q)^{q'/q} = 0$ *on* B *and*

$$M_1 := \sup_{n\in\mathbb{N}} \frac{E(|u|^{q'})(A_n)(E(|w|^q))^{q'/q}(A_n)}{\mu(A_n)} < \infty,$$

In this case, $\|T\|^{q'} \geqslant M_1$.

(c) Let $1 < q < \infty$ *and let* q' *be conjugate component to* q. *If* $u(E(|w|^q)^{1/q} = 0$ *on* C *and* $M_2 := \sup_{n\in\mathbb{N}} \frac{|u(C_n)|^{q'}(E(|w|^q))^{q'/q}(C_n)}{\mu(C_n)} < \infty$, *then the pair* (w,u) *induces a weighted Lambert type operator* T *from* $L^1(\Sigma)$ *into* $L^q(\Sigma)$. *In this case,* $\|T\|^{q'} \leqslant M_2$.

Proof. (a) Let $f \in L^p(\Sigma)$. Then $\|Tf\|_1 = \|EM_v f\|_1$, where $v := uE(|w|)$. Thus the boundedness of $T : L^p(\Sigma) \to L^1(\Sigma)$ implies that R_v and so $R_v^* = M_{\bar{v}} : L^\infty(\mathscr{A}) \to L^{p'}(\Sigma)$ is bounded. It follows that

$$\|v\|_{p'}^{p'} = \|\bar{v}\|_{p'}^{p'} = \|M_{\bar{v}}(\chi_X)\|_{p'}^{p'} \leqslant \|M_{\bar{v}}\|^{p'}\|\chi_X\|_\infty^{p'} = \|M_{\bar{v}}\|^{p'} < \infty,$$

and so $v \in L^{p'}(\Sigma)$ and $\|v\|_{p'} \leqslant \|M_{\bar{v}}\| = \|T\|$. Conversely, suppose that $v \in L^{p'}(\Sigma)$ and $f \in L^\infty(\mathscr{A})$. Then by Hölder's inequality, we have

$$\|Tf\|_1 = \int_X |E(vf)|d\mu = \int_X E(|vf|)d\mu = \int_X |vf|d\mu \leqslant \|v\|_{p'}\|f\|_p.$$

Thus, $\|T\| \leqslant \|v\|_{p'}$ and hence T is bounded.

(b) Let $f \in L^1(\Sigma)$. Then $\|Tf\|_q = \|R_v f\|_q$, where $v := u(E(|w|^q))^{\frac{1}{q}}$. So boundedness of T implies that $M_{\bar{v}} : L^{q'}(\mathscr{A}) \to L^\infty(\Sigma)$ is also bounded. Then for all $f \in L^{q'}(\mathscr{A})$ we have

$$\begin{aligned}
\|M_{\sqrt[q']{E(|v|^{q'})}}(f)\|_{L^\infty(\mathscr{A})}^{q'} &= \sup_{A\in\mathscr{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)}\int_A E(|v|^{q'})|f|^{q'}d\mu \\
&= \sup_{A\in\mathscr{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)}\int_A |vf|^{q'}d\mu \\
&\leqslant \sup_{A\in\Sigma,\ 0<\mu(A)<\infty} \frac{1}{\mu(A)}\int_A |vf|^{q'}d\mu = \|M_{\bar{v}}f\|_{L^\infty(\Sigma)}^{q'}.
\end{aligned}$$

It follows that $M_{\sqrt[q']{E(|v|^{q'})}}(L^{q'}(\mathscr{A})) \subseteq L^\infty(\mathscr{A}) \subseteq L^\infty(\Sigma)$, and hence $M_{\sqrt[q']{E(|v|^{q'})}} : L^{q'}(\mathscr{A}) \to L^\infty(\Sigma)$ is also bounded. Now, by the same argument in the proof of the Theorem 2.3, if we put $f_n = \chi_{F_n}/\sqrt[q']{\mu(F_n)}$, we have then

$$\begin{aligned}
\infty &> \|M_{\sqrt[q']{E(|v|^{q'})}}\|^{q'} \geqslant \|M_{\sqrt[q']{E(|v|^{q'})}}(f_n)\|_{L^\infty(\Sigma)}^{q'} \\
&= \frac{\sup_{A\in\Sigma,\ 0<\mu(A)<\infty} \frac{1}{\mu(A)}\int_A E(|v|^{q'})\chi_{F_n}d\mu}{\mu(F_n)} \geqslant \frac{\delta}{\mu(F_n)} = \frac{\delta 2^n}{\mu(F)} \longrightarrow \infty, \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction. Hence we conclude that $E(|v|^{q'}) = 0$ on B. Also, for any $m \in \mathbb{N}$ we have

$$\begin{aligned}
\infty > \|M_{\bar{v}}\|^{q'} &\geqslant \|M_{\bar{v}} \frac{\chi_{A_m}}{\sqrt[q']{\mu(A_m)}}\|^{q'}_{L^\infty(\Sigma)} \\
&\geqslant \|M_{\bar{v}} \frac{\chi_{A_m}}{\sqrt[q']{\mu(A_m)}}\|^{q'}_{L^\infty(\mathscr{A})} = \sup_{A \in \mathscr{A},\, 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A \frac{|v|^{q'} \chi_{A_m}}{\mu(A_m)} d\mu \\
&= \sup_{n \in \mathbb{N}} \frac{1}{\mu(A_n)} \int_{A_n} \frac{E(|v|^{q'}) \chi_{A_m}}{\mu(A_m)} d\mu \geqslant \frac{E(|v|^{q'})(A_m)}{\mu(A_m)}.
\end{aligned}$$

It follows that $\sup_{n \in \mathbb{N}} \frac{E(|v|^{q'})(A_n)}{\mu(A_n)} = M_1 \leqslant \|M_{\bar{v}}\|^{q'} = \|T\|^{q'} < \infty$.

(c) Suppose that $v = 0$ on C and $\sup_{n \in \mathbb{N}} \frac{|v(C_n)|^{q'}}{\mu(C_n)} < \infty$, where $v := u(E(|w|^q))^{\frac{1}{q}}$. Then for each $f \in L^{q'}(\mathscr{A})$ we have

$$\begin{aligned}
\|M_{\bar{v}}(f)\|^{q'}_{L^\infty(\Sigma)} &= \sup_{A \in \Sigma,\, 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |vf|^{q'} d\mu \leqslant \sup_{n \in \mathbb{N}} \frac{1}{\mu(C_n)} \int_{C_n} |vf|^{q'} d\mu \\
&= \sup_{n \in \mathbb{N}} \left(\frac{|v(C_n)|^{q'}}{\mu(C_n)} \right) |f(C_n)|^{q'} \mu(C_n) \leqslant M_2 \sum_{n \in \mathbb{N}} |f(C_n)|^{q'} \mu(C_n) \\
&\leqslant M_2 \|f\|^{q'}_{L^{q'}(\mathscr{A})},
\end{aligned}$$

where we have used the fact that $\frac{a+b}{c+d} \leqslant \max\{\frac{a}{c}, \frac{b}{d}\}$, for each $a, b, c, d \in (0, \infty)$. Hence the operator $M_{\bar{v}} E : L^{q'}(\Sigma) \to L^\infty(\Sigma)$ is bounded. It follows that $T = (M_{\bar{v}} E)^*_{|L^1(\Sigma)} : L^1(\Sigma) \to L^q(\Sigma)$ is also bounded. □

LEMMA 2.5. *Let $1 \leqslant p < \infty$. The bounded operator $M_v : L^p(\mathscr{A}) \to L^p(\Sigma)$ is compact if and only if for each $\varepsilon > 0$ the set $\{x \in X : (E(|v|^p))^{1/p}(x) \geqslant \varepsilon\}$ consists of finitely many $\mathscr{A}$-atoms.*

Proof. Suppose M_v is compact. We show that for each $\varepsilon > 0$ the set $K_\varepsilon := \{x \in X : (E(|v|^p))^{1/p}(x) \geqslant \varepsilon\}$ consists of finitely many $\mathscr{A}$-atoms. Assume the contrary; then for some $\varepsilon > 0$ the set K_ε either contains a subset of non-atomic part B with positive measure or has infinitely many $\mathscr{A}$-atoms. Since $\mathscr{A}$ is σ-finite, in both cases we can find a sequence of pairwise disjoint measurable subsets $\{A_n\}_{n \in \mathbb{N}}$ with $0 < \mu(A_n) < \infty$. Define $f_n = \frac{\chi_{A_n}}{\mu(A_n)^{1/p}}$. Then $\|f_n\|_p = 1$ and for $n \neq m$ we have

$$\begin{aligned}
\|M_v f_n - M_v f_m\|_p^p &= \int_X |vf_n - vf_m|^p d\mu \geqslant \int_{A_n \cup A_m} |vf_n - vf_m|^p d\mu \\
&= \int_{A_n} |vf_n|^p d\mu + \int_{A_m} |vf_m|^p d\mu \\
&= \int_{A_n} E(|v|^p)|f_n|^p d\mu + \int_{A_m} E(|v|^p)|f_m|^p d\mu \geqslant 2\varepsilon^p,
\end{aligned}$$

which shows that the sequence $\{M_v f_n\}$ dose not contain any convergent subsequence, and so T is not compact.

Conversely, suppose that for each $\varepsilon > 0$, $K_\varepsilon = \cup_{k=1}^{n} A_k^\varepsilon$. Define $M_{v_\varepsilon}(f) = M_v(f\chi_{K_\varepsilon}) = \Sigma_{k=1}^{n} v f \chi_{A_k^\varepsilon}$, for all $f \in L^p(\Sigma)$. It is clear that M_{v_ε} is a finite rank operator on $L^p(\Sigma)$ and $\|M_v - M_{v_\varepsilon}\| < \varepsilon$. Thus M_v is compact. $\square$

In the following theorem, we give a necessary and sufficient condition for the compactness of T on $L^p(\Sigma)$.

THEOREM 2.6. *Let $1 < p < \infty$ and let p' be the conjugate component to p. Then the weighted Lambert type operator $T : L^p(\Sigma) \to L^p(\Sigma)$ is compact if and only if for each $\varepsilon > 0$ the set $\{x \in X : (E(|u|^{p'}))^{\frac{1}{p'}}(x)(E(|w|^p))^{\frac{1}{p}}(x) \geqslant \varepsilon\}$ consists of finitely many $\mathscr{A}$-atoms.*

Proof. Let $f \in L^p(\Sigma)$. Then $\|Tf\|_p = \|EM_v f\|_p$, where $v := u(E(|w|^p))^{\frac{1}{p}}$. Thus $T : L^p(\Sigma) \to L^p(\Sigma)$ is compact if and only if $EM_v : L^p(\Sigma) \to L^p(\mathscr{A})$ is compact. But by Lemma 2.5, $M_{\bar{v}}E = M_{\bar{v}} : L^{p'}(\mathscr{A}) \to L^{p'}(\Sigma)$ is compact if and only if for each $\varepsilon > 0$, the set

$$\{x \in X : (E(|v|^{p'}))^{\frac{1}{p'}}(x) \geqslant \varepsilon\} = \{x \in X : (E(|u|^{p'}))^{\frac{1}{p'}}(E(|w|^p))^{\frac{1}{p}}(x) \geqslant \varepsilon\}$$

consists of finitely many $\mathscr{A}$-atoms. $\square$

THEOREM 2.7. *The weighted Lambert type operator $T : L^1(\Sigma) \to L^1(\Sigma)$ is compact if and only if for each $\varepsilon > 0$ the set $K_\varepsilon := \{x \in X : u(x)E(|w|)(x) \geqslant \varepsilon\}$ consists of finitely many Σ-atoms.*

Proof. Suppose that T is compact but for some $\varepsilon > 0$ the set K_ε either contains a subset of non-atomic part C with positive measure or has infinitely many Σ-atoms. In both cases we can find a sequence of pairwise disjoint measurable subsets $\{C_n\}_{n\in\mathbb{N}}$ with $0 < \mu(C_n) < \infty$. Define $f_n = \frac{\bar{u}E(|w|)\chi_{C_n}}{\mu(C_n)\|T\|^2}$. Then $\|f_n\|_1 \leqslant 1/\|T\|$ and $\sigma(f_n) \cap \sigma(f_m) = \emptyset$, for $n \neq m$. Then

$$\|Tf_n - Tf_m\|_1 \geqslant \int_{C_n \cup C_m} E(|w|)|E(uf_n - uf_m)|d\mu$$

$$\int_{C_n} (E(|w|))^2 E\left(\frac{|u|^2\chi_{C_n}}{\mu(C_n)\|T\|^2}\right) d\mu + \int_{C_m} (E(|w|))^2 E\left(\frac{|u|^2\chi_{C_m}}{\mu(C_m)\|T\|^2}\right) d\mu \geqslant \frac{2\varepsilon^p}{\|T\|^2},$$

which shows that the sequence $\{Tf_n\}$ dose not contain a convergent subsequence, and so T is not compact.

Conversely, suppose that for each $\varepsilon > 0$, $K_\varepsilon = \cup_{k=1}^{n} C_k^\varepsilon$. Define $T_\varepsilon(f) = T(f\chi_{K_\varepsilon}) = \Sigma_{k=1}^{n} T(f\chi_{C_k^\varepsilon})$, for all $f \in L^1(\Sigma)$. It is clear that T_ε is a finitely rank operator on $L^1(\Sigma)$ and $\|T - T_\varepsilon\| < \varepsilon$. Thus T is compact. $\square$

COROLLARY 2.8. *Let $1 \leqslant p < \infty$. If the measure space $(X, \mathscr{A}, \mu)$ is non-atomic, then the weighted Lambert type operator $T : L^p(\Sigma) \to L^p(\Sigma)$ is compact if and only if $T = 0$.*

Proof. Suppose that T is a nonzero compact weighted Lambert type operator. Then $0 \neq \|T\| = \|(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty$, and so there exists $\delta > 0$ such that the set $\{x \in X : (E(|w|^p))^{\frac{1}{p}}(x)(E(|u|^{p'}))^{\frac{1}{p'}}(x) > \delta\}$ contains an $\mathscr{A}$-measurable subset B with positive measure. Hence, we obtain a sequence of pairwise disjoint subsets $B_n \subseteq B$ such that for every $n \in \mathbb{N}$, $B_n \in \mathscr{A}$ and $0 < \mu(B_n) < \infty$. Define

$$f_n = \frac{\overline{u}|u|^{p'-2}(E(|w|^p))^{\frac{p'-1}{p}}}{\|T\|^{\frac{p'}{p}}(\mu(B_n))^{\frac{1}{p}}}\chi_{B_n}.$$

It is easy to see that $\|f_n\|_p \leqslant 1$ and

$$\begin{aligned}\|Tf_n - Tf_m\|_p^p &= \int_X |w|^p|E(uf_n - uf_m)|^p d\mu \\ &\geqslant \int_{B_n} \frac{(E(|w|^p))^{p'}(E(|u|^{p'}))^p}{\|T\|^{p'}\mu(B_n)} d\mu + \int_{B_m} \frac{(E(|w|^p))^{p'}(E(|u|^{p'}))^p}{\|T\|^{p'}\mu(B_m)} d\mu \\ &\geqslant \frac{2\delta^{pp'}}{\|T\|^{p'}}.\end{aligned}$$

But this is a contradiction. □

3. Weighted Lambert type operators on $L^2(\Sigma)$

Let $\mathscr{H}$ be the infinite dimensional complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded operators on $\mathscr{H}$. An operator $A \in \mathscr{L}(\mathscr{H})$ is a partial isometry if $\|Ah\| = \|h\|$ for h orthogonal to the kernel of A. It is known that an operator A on a Hilbert space is partial isometry if and only if $AA^*A = A$. The operator A is said to be positive operator and write $A \geqslant 0$, if $\langle Ah, h\rangle \geqslant 0$, for all $h \in \mathscr{H}$. Let $p \in (0, \infty)$. An operator $A \in \mathscr{L}(\mathscr{H})$ is p-hyponormal if $(A^*A)^p \geqslant (AA^*)^p$, A is p-quasihyponormal if $A^*(A^*A)^pA \geqslant A^*(AA^*)^pA$, and A is normaloid if $\|A\|^n = \|A^n\|$ for all $n \in \mathbb{N}$. The hierarchical relationship between the classes is as follows:

$$p\text{-hyponormal} \Rightarrow p\text{-quasihyponormal} \Rightarrow \text{normaloid}.$$

The next collection of results addresses the question as to when a Lambert type operator is of those various operator theoretic types. Also, the polar decomposition and the Aluthge transformation for these type operators are calculated.

LEMMA 3.1. *Let $g \in L^\infty(\mathscr{A})$ and let $T : L^2(\Sigma) \to L^2(\Sigma)$ be a weighted Lambert type operator. If $M_gT = 0$, then $g = 0$ on $\sigma(E(|w|^2)E(|u|^2))$.*

Proof. Let $f \in L^2(\Sigma)$. Then $gwE(uf) = M_gT(f) = 0$. Now, by Theorem 2.1,

$$0 = \|M_gT\|^2 = \||g|^2E(|w|^2)E(|u|^2)\|_\infty,$$

which implies that $|g|^2E(|w|^2)E(|u|^2)=0$, and so $g=0$ on $\sigma(E(|w|^2)E(|u|^2))$. □

THEOREM 3.2. *The weighted Lambert operator T is a partial isometry if and only if $E(|w|^2)E(|u|^2)=\chi_A$ for some $A\in\mathscr{A}$.*

Proof. Suppose T is partial isometry. Then $TT^*T=T$, that is

$$Tf=E(|w|^2)E(|u|^2)Tf,$$

and hence $(E(|w|^2)E(|u|^2)-1)Tf=0$ for all $f\in L^2(\Sigma)$. Put $S=\sigma(E(|u|^2))$ and $G=\sigma(E(|w|^2))$. By Lemma 3.1 we get that $E(|w|^2)E(|u|^2)=1$ on $S\cap G$, which implies that $E(|w|^2)E(|u|^2)=\chi_A$, where $A=S\cap G$.

Conversely, suppose that $E(|w|^2)E(|u|^2)=\chi_A$ for some $A\in\mathscr{A}$. It follows that $A=S\cap G$, and we have

$$TT^*T(f)=E(|w|^2)E(|u|^2)Tf=\chi_{S\cap G}wE(uf)=wE(uf),$$

where we have used the fact that $\sigma(Tf)=\sigma(|Tf|^2)\subseteq S\cap G$, which this is a consequence of Hölder's inequality for conditional expectation E. □

LEMMA 3.3. *Let T be a weighted Lambert type operator on $L^2(\Sigma)$ and let $p\in(0,\infty)$. Then*

$$(T^*T)^p=M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p}EM_u$$

and

$$(TT^*)^p=M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p}EM_{\bar{w}},$$

where $S=\sigma(E(|u|^2))$ and $G=\sigma(E(|w|^2))$.

Proof. Suppose that $f\in L^2(\Sigma)$. Then by induction we obtain

$$(T^*T)^{\frac{1}{n}}=M_{\bar{u}(E(|u|^2))^{\frac{1-n}{n}}\chi_S(E(|w|^2))^{\frac{1}{n}}}EM_u$$

and

$$(TT^*)^{\frac{1}{n}}=M_{w(E(|w|^2))^{\frac{1-n}{n}}\chi_G(E(|u|^2))^{\frac{1}{n}}}EM_{\bar{w}}$$

for all $n\in\mathbb{N}$. Now the reiteration of powers of operators $(T^*T)^{\frac{1}{n}}$ and $(TT^*)^{\frac{1}{n}}$, yields

$$(T^*T)^{\frac{m}{n}}=M_{\bar{u}(E(|u|^2))^{\frac{m-n}{n}}\chi_S(E(|w|^2))^{\frac{m}{n}}}EM_u$$

and

$$(TT^*)^{\frac{m}{n}}=M_{w(E(|w|^2))^{\frac{m-n}{n}}\chi_G(E(|u|^2))^{\frac{m}{n}}}EM_{\bar{w}}$$

for all $m,n\in\mathbb{N}$. Finally, by using of the functional calculus the desired formula is proved. □

THEOREM 3.4. *Let T be a weighted Lambert type operator on $L^2(\Sigma)$ and let $p \in (0,\infty)$. Then the following assertions hold.*

(a) T is hyponormal if and only if T is p-hyponormal.

(b) If $|E(uw)|^2 \geqslant E(|u|^2)E(|w|^2)$, then T is p-quasihyponormal.

(c) If T is p-quasihyponormal, then $|E(uw)|^2 \geqslant E(|u|^2)E(|w|^2)$ on $\sigma(E(u)) \cap G$.

(d) If $\sigma(w) = \sigma(u) = X$, then T is p-quasihyponormal if and only if $|E(uw)|^2 \geqslant E(|u|^2)E(|w|^2)$.

Proof. (a) Applying Lemma 3.3 we obtain that $(T^*T)^p \geqslant (TT^*)^p$ if and only if

$$M_{\chi_{S\cap G}(E(|u|^2))^{p-1}(E(|w|^2))^{p-1}}(M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}}) \geqslant 0.$$

This inequality holds if and only if

$$T^*T - TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}} \geqslant 0,$$

where we have used the fact that $T_1T_2 \geqslant 0$ if $T_1 \geqslant 0$, $T_2 \geqslant 0$ and $T_1T_2 = T_2T_1$ for all $T_i \in \mathscr{B}(\mathscr{H})$, the set of all bounded linear operators on Hilbert space $\mathscr{H}$.

(b) By Lemma 3.3, it is easy to check that

$$T^*(T^*T)^pT = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p|E(uw)|^2}EM_u;$$
$$T^*(TT^*)^pT = M_{\bar{u}(E(|w|^2))^{p+1}(E(|u|^2))^p}EM_u.$$

It follows that $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$ if

$$M_{(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p}M_{(|E(uw)|^2 - E(|w|^2)E(|u|^2))}M_{\bar{u}}EM_u \geqslant 0.$$

By the same argument in (a), this inequality holds if $M_{(|E(uw)|^2 - E(|w|^2)E(|u|^2))} \geqslant 0$; i.e. $|E(uw)|^2 - E(|w|^2)E(|u|^2) \geqslant 0$.

(c) Suppose that T is p-quasihyponormal. Then for all $f \in L^2(\mathscr{A})$, we have

$$\langle T^*(T^*T)^pT - T^*(TT^*)^pTf, f\rangle$$
$$= \int_X (E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p(|E(uw)|^2 - E(|w|^2)E(|u|^2))|E(u)|^2|f|^2 d\mu \geqslant 0.$$

Thus

$$(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p(|E(uw)|^2 - E(|w|^2)E(|u|^2))|E(u)|^2 \geqslant 0,$$

and hence we obtain $|E(uw)|^2 \geqslant E(|w|^2)E(|u|^2)$ on $\sigma(E(u)) \cap G$.

(d) It follows from (c) and (d). □

THEOREM 3.5. *Let T be a weighted Lambert type operator on $L^2(\Sigma)$. If $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then T is normaloid.*

Proof. By induction, we have

$$T^n f = (E(uw))^{n-1} wE(uf), \quad f \in L^2(\Sigma), \; n \in \mathbb{N}.$$

Now, by Theorem 2.1(a) we obtain

$$\|T^n\| = \| \, |E(uw)|^{n-1} (E(|u|^2))^{\frac{1}{2}} (E(|w|^2))^{\frac{1}{2}} \|_\infty, \; n \in \mathbb{N}.$$

Since $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, we get that

$$\|T^n\| = \|(E(|u|^2))^{\frac{n}{2}} (E(|w|^2))^{\frac{n}{2}} \|_\infty = \|T\|^n,$$

for all $n \in \mathbb{N}$. Thus the theorem is proved. □

It is well known that every operator A on a Hilbert space $\mathscr{H}$ can be decomposed into $A = U|A|$ with a partial isometry U, where $|A| = (A^*A)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathscr{N}(U) = \mathscr{N}(|A|)$. Then this decomposition is called the polar decomposition.

By the operator matrices method, we shall obtain the polar decomposition of weighted Lambert type operator $T = M_w E M_u$. Notice that $L^2(\Sigma)$ is the direct sum of the $\mathscr{R}(E) = L^2(\mathscr{A})$ and $\mathscr{N}(E) = \{f - Ef : f \in L^2(\Sigma)\}$. With respect to the direct sum decomposition, $L^2(\Sigma) = L^2(\mathscr{A}) \oplus \mathscr{N}(E)$, the matrix form of T is

$$T = \begin{bmatrix} M_{E(u)E(w)} & M_{E(w)} E M_u \\ M_{wE(u)} - M_{E(u)E(w)} & M_w E M_u - M_{E(w)} E M_u \end{bmatrix}.$$

THEOREM 3.6. *The unique polar decomposition of T is $U|T|$, where*

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f) = \left(\frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} wE(uf),$$

for all $f \in L^2(\Sigma)$.

Proof. With respect to the direct sum decomposition, $L^2(\Sigma) = L^2(\mathscr{A}) \oplus \mathscr{N}(E)$, the matrix form of T^*T is

$$T^*T = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where

$$A_1 = M_{E(|w|^2)|E(u)|^2};$$
$$A_2 = M_{E(|w|^2)\overline{E(u)}} E M_u;$$
$$A_3 = M_{E(|w|^2)\bar{u}E(u)} - M_{E(|w|^2)|E(u)|^2};$$
$$A_4 = M_{E(|w|^2)\bar{u}} E M_u - M_{E(|w|^2)\overline{E(u)}} E M_u.$$

Now, suppose that the matrix form of $|T|$ with respect to $L^2(\mathscr{A})\oplus\mathscr{N}(E)$ is

$$|T|=\begin{bmatrix} B_1 & B_2\\ B_3 & B_4\end{bmatrix}.$$

Since $|T|^2=T^*T$, then

$$\begin{cases} B_1^2+B_2B_3 &= A_1;\\ B_1B_2+B_2B_4 &= A_2;\\ B_3B_1+B_4B_3 &= A_3;\\ B_3B_2+B_4^2 &= A_4.\end{cases}$$

One of the solutions of this system is

$$\begin{aligned} B_1 &= M_{\alpha|E(u)|^2};\\ B_2 &= M_{\alpha\overline{E(u)}}EM_u;\\ B_3 &= M_{\alpha\bar{u}E(u)}-M_{\alpha|E(u)|^2};\\ B_4 &= M_{\alpha\bar{u}}EM_u-M_{\alpha\overline{E(u)}}EM_u,\end{aligned}$$

where

$$\alpha=\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S,\qquad S=\sigma(E(|u|^2).$$

Since the mapping $f\mapsto\begin{bmatrix}Ef & f-Ef\end{bmatrix}$ is an isometric isomorphism from $L^2(\Sigma)$ onto $L^2(\mathscr{A})\oplus\mathscr{N}(E)$, we get that

$$|T|(f)=(B_1+B_3)Ef+(B_2+B_4)(f-Ef)=\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S\bar{u}E(uf).$$

Define a linear operator U whose action is given by

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),\quad f\in L^2(\Sigma).$$

Then $T=U|T|$ and by Theorem 3.2, U is a partial isometry. Also, it is easy to see that $\mathscr{N}(T)=\mathscr{N}(U)$. Since for all $f\in L^2(\Sigma)$, $\|Tf\|=\|\,|T|f\|$, hence $\mathscr{N}(|T|)=\mathscr{N}(U)$ and so this decomposition is unique. □

THEOREM 3.7. *The Aluthge transformation of T is*

$$\widehat{T}(f)=\frac{\chi_S E(uw)}{E(|u|^2)}\bar{u}E(uf),\qquad f\in L^2(\Sigma).$$

Proof. Define operator V on $L^2(\Sigma)$ as

$$Vf=\left(\frac{E(|w|^2)}{(E(|u|^2))^3}\right)^{\frac{1}{4}}\chi_S\bar{u}E(uf),\quad f\in L^2(\Sigma).$$

Then we have $V^2 = |T|$ and so by direct computation we obtain

$$\widehat{T}(f) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}(f) = \frac{\chi_S E(uw)}{E(|u|^2)}\bar{u}E(uf). \quad \square$$

REMARK 3.8. By applying of the methods, which are used in the proofs of Theorem 3.6 and Theorem 3.7, we can compute the polar decomposition and Aluthge transformation of $T^* = U^*|T^*|$ as follows:

$$|T^*|(f) = \left(\frac{E(|u|^2)}{E(|w|^2)}\right)^{\frac{1}{2}} \chi_G \bar{w}E(\bar{w}f);$$

$$U^*(f) = \left(\frac{\chi_{S\cap G}}{(E(|u|^2)E(|w|^2)}\right)^{\frac{1}{2}} \bar{u}E(\bar{w}f);$$

$$\widehat{T^*}(f) = \frac{\chi_G E(\overline{uw})}{E(|w|^2)} wE(\bar{w}f),$$

for all $f \in L^2(\Sigma)$.

EXAMPLE 3.9. **(a)** Let $X = [0,1] \times [0,1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathscr{A} = \{A \times [0,1] : A$ is a Lebesgue set in $[0,1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(x,y) = \int_0^1 f(x,t)dt$, which is independent of the second coordinate. This example is due to A. Lambert and B. Weinstock [8]. Now, if we take $u(x,y) = y^{\frac{-x}{8}}$ and $w(x,y) = \sqrt{(4-x)y}$, then $\sqrt{E(|u|^2)}(x,y) = \sqrt{\frac{4}{4-x}}$ and $\sqrt{E(|w|^2)}(x,y) = \sqrt{\frac{4-x}{2}}$. Hence $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty = \sqrt{2}$, and so T is not compact, where

$$(Tf)(x,y) = \sqrt{(4-x)y}\int_0^1 t^{\frac{-x}{8}} f(x,t)dt, \quad f \in L^2(\Sigma).$$

The direct computations show that

$$|T|(x,y) = \frac{(4-x)y^{\frac{-x}{8}}}{2\sqrt{2}} \int_0^1 t^{\frac{-x}{8}} f(x,t)dt,$$

$$U(x,y) = \frac{\sqrt{(4-x)y}}{\sqrt{2}} \int_0^1 t^{\frac{-x}{8}} f(x,t)dt$$

and

$$\widehat{T}(f)(x,y) = \frac{8(4-x)^{\frac{3}{4}}y^{\frac{-x}{8}}}{\sqrt{2}(12-x)} \int_0^1 t^{\frac{-x}{8}} f(x,t)dt.$$

Also, it is easy to see that $\|R_u\|_{L^2(\Sigma)\to L^2(\Sigma)} = \|M_u\|_{L^2(\mathscr{A})\to L^2(\Sigma)} = \frac{2}{\sqrt{3}}$. However, the multiplication operator $M_u : L^2(\Sigma) \to L^2(\Sigma)$ is not bounded, because $u \notin L^\infty(\Sigma)$ (see [11]).

(b) Let $X=\mathbb{N}$, $\Sigma=2^{\mathbb{N}}$ and let μ be the counting measure. Put

$$A=\{\{2\},\{4,6\},\{8,10,12\},\{14,16,18,20\},\cdots\}\cup\{\{1\},\{3\},\{5\},\cdots\}.$$

If we let $A_1=\{2\}$, $A_2=\{4,6\}$, $A_3=\{8,10,12\}$, $\cdots$, then we see that $\mu(A_n)=n$ and for every $n\in\mathbb{N}$, there exists $k_n\in\mathbb{N}$ such that $A_n=\{2k_n,2(k_n+1),\cdots,2(k_n+n-1)\}$. Let $\mathscr{A}$ be the σ-algebra generated by the pattition A of $\mathbb{N}$. Note that, $\mathscr{A}$ is a sub-σ-finite algebra of Σ and each of element of $\mathscr{A}$ is an $\mathscr{A}$-atom. It is known that the conditional expectation of any $f\in\mathscr{D}(E)$ relative to $\mathscr{A}$ is

$$E(f)=\sum_{n=1}^{\infty}\left(\frac{1}{\mu(A_n)}\int_{A_n} f d\mu\right)\chi_{A_n}+\sum_{n=1}^{\infty} f(2n-1)\chi_{\{2n-1\}}.$$

Define $u(n)=\frac{n^2}{n+1}$ and $w(n)=\frac{1}{2^n n^4}$, for all $n\in\mathbb{N}$. For each even number $m\in\mathbb{N}$, there exists $n_m\in\mathbb{N}$ such that $m\in A_{n_m}$. Thus we get that

$$E(|w|^p)(m)=\frac{1}{2^{2pk_{n_m}}(2k_{n_m})^{4p}}+\cdots+\frac{1}{2^{2pk_{n_m}+2pn_m-2p}(2k_{n_m}+2n_m-2)^{4p}}$$

and

$$E(|u|^{p'})(m)=\frac{4k_{n_m}^{2p'}}{(2k_{n_m}+1)^{p'}}+\frac{(2k_{n_m}+2)^{2p'}}{(2k_{n_m}+3)^{p'}}+\ldots+\frac{(2k_{n_m}+2n_m-2)^{2p'}}{(2k_{n_m}+2n_m-1)^{p'}}.$$

It is easy to see that

$$E(|w|^p)(m)\leqslant\frac{n_m}{2^{2pk_{n_m}}(2k_{n_m})^{4p}}\ \text{ and }\ E(|u|^{p'})(m)\leqslant\frac{n_m(2k_{n_m}+2n_m-2)^{2p'}}{(2k_{n_m}+2n_m-1)^{p'}},$$

and so

$$(E(|w|^p))^{\frac{1}{p}}(m)(E(|u|^{p'}))^{\frac{1}{p'}}(m)\leqslant\frac{(n_m)^{\frac{1}{p}}}{2^{2k_{n_m}}(2k_{n_m})^4}\ \frac{(n_m)^{\frac{1}{p}}(2k_{n_m}+2n_m-2)^2}{(2k_{n_m}+2n_m-1)}.$$

Since $n_m\leqslant k_{n_m}$, then

$$(E(|w|^p))^{\frac{1}{p}}(m)(E(|u|^{p'}))^{\frac{1}{p'}}(m)\leqslant\frac{n_m(2k_{n_m}+2n_m-2)}{2^{2k_{n_m}}(2k_{n_m})^4}\leqslant\frac{1}{2^{2k_{n_m}}4(k_{n_m})^2}.$$

Also, for all $n\in\mathbb{N}$, we have

$$(E(|w|^p))^{\frac{1}{p}}(2n-1)(E(|u|^{p'}))^{\frac{1}{p'}}(2n-1)=\frac{1}{2^{n-1}n^3}.$$

Thus $\|(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty\leqslant 1$, and so by Theorem 2.1(a) and Theorem 2.7, the operator $T=M_wEM_u$ is a compact weighted Lambert type operator on $L^p(\Sigma)$. However, the operator R_u is not compact, because $(E(|u|^{p'}))^{1/p'}(m)\to\infty$ as $m\to\infty$.

(c) Let $dA(z)$ be the normalized Lebesgue measure on open unit disc $\mathbb{D}$. Recall that for $1 \leqslant p < \infty$ the Bergman space $L_a^p(\mathbb{D})$ is the collection of all functions $f \in H(\mathbb{D})$, holomorphic functions on $\mathbb{D}$, for which $\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty$. Let $\mathscr{A}$ be the σ-algebra generated by $\{(z^n)^{-1}(U) : U \subseteq \mathbb{C} \text{ is open}\}$. Then

$$E(u)(\xi) = \frac{1}{n} \sum_{\zeta^n = \xi} u(\zeta), \;\; u \in H(\mathbb{D}), \;\; \xi \in \mathbb{D} \setminus \{0\},$$

(see [5]). Note that $|u| \leqslant nE(|u|)$ and $E(L_a^p(\mathbb{D})) \subseteq L_a^p(\mathbb{D})$. Since E is contraction, $u \in H^\infty(\mathbb{D})$ if and only if $E(|u|^s) \in L^\infty(\mathbb{D})$, for all $s \geqslant 1$. It follows that the operator $R_u(f)(\xi) = \frac{1}{n} \sum_{\zeta^n = \xi} (uf)(\zeta)$ on $L_a^p(\mathbb{D})$ is bounded if and only if the multiplication operator $M_u : L_a^p(\mathbb{D}) \to L_a^p(\mathbb{D})$ is bounded. It is known that M_u is bounded if and only if $u \in H^\infty(\mathbb{D})$. This example shows that these operators are closely related to averaging operators.

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