BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(1) (2014), 165–171

Conditional Multiplication Operators

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Abstract. In this paper we provide a necessary and sufficient condition for the conditional multiplication operators to have closed range. Also, some other properties of these type of operators will be investigated.

2010 Mathematics Subject Classification: Primary 47B20; Secondary 47B38

Keywords and phrases: Conditional multiplication operator, conditional expectation, normal, self-adjoint, closed range, spectrum, compact.

1. Introduction and Preliminaries

Let (X,Σ,μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathscr{A}\subseteq\Sigma$ and $1\le p\le\infty$, the L^p-space $L^p(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ is abbreviated to $L^p(\mathscr{A})$ where $\mu_{|_{\mathscr{A}}}$ is the restriction of μ to $\mathscr{A}$. Also its norm is denoted by $\|.\|_p$ on which $L^p(\mathscr{A})$ is a Banach subspace of $L^p(\Sigma)$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function f is defined by $\sigma(f)=\{x\in X: f(x)\neq 0\}$. Equalities and inequalities between measurable functions and also equality between sets can be interpreted as the almost everywhere sense, and the set of measure zero, respectively. For each nonnegative $f\in L^0(\Sigma)$ or $f\in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique measurable function $E(f)$ with the following conditions:

(i) $E(f)$ is $\mathscr{A}$-measurable and integrable,

(ii) If F is any $\mathscr{A}$-measurable set for which $\int_F f d\mu$ exists, we have the functional relation

$$\int_F f d\mu=\int_F E(f)d\mu.$$

Now associated with every complete σ-finite subalgebra $\mathscr{A}\subseteq\Sigma$, the mapping $E: L^p(\Sigma)\to L^p(\mathscr{A})$, $1\le p\le\infty$, uniquely defined by the assignment $f\mapsto E(f)$, is called the *conditional expectation operator with respect to* $\mathscr{A}$. The mapping E is a linear operator and, in particular, it is a contraction operator. In case $p=2$, it is the orthogonal projection of $L^2(\Sigma)$ onto

Communicated by Mohammad Sal Moslehian.
Received: October 28, 2011; *Revised:* January 16, 2012.

$L^2(\mathscr{A})$. The role of this operator is important in this note and we list here some of its useful properties:

- If f is an $\mathscr{A}$-measurable function, then $E(fg) = fE(g)$.
- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$.
- $\sigma(f) \subseteq \sigma(E(f))$, for each nonnegative $f \in L^p(\Sigma)$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^0(\mathscr{A})$.

For more details on the properties of E see [11]. Recall that an $\mathscr{A}$-atom of the measure μ is an element $A \in \mathscr{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space $(X, \mathscr{A}, \mu_{|_{\mathscr{A}}})$ can be partitioned uniquely as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathscr{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [13]).

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. The set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ is denoted by $\mathscr{B}(\mathscr{H}, \mathscr{K})$. We denote $\mathscr{B}(\mathscr{H}, \mathscr{H})$ simply by $\mathscr{B}(\mathscr{H})$. For $T \in \mathscr{B}(\mathscr{H})$, the null-space, the range of T and its spectrum are denoted by $\mathscr{N}(T)$, $\mathscr{R}(T)$ and $Sp(T)$, respectively. A subspace $M \subseteq \mathscr{H}$ is said to be *invariant* for $T \in \mathscr{B}(\mathscr{H})$ whenever $T(M) \subseteq M$. Also M *reduces* T if M is invariant for both T and its adjoint T^*. Given a $B \in \Sigma$, we shall abbreviate the subspace $L^p(B, \Sigma_B, \mu_{|_{\Sigma_B}})$ to $L^p(B)$ on which $\Sigma_B = \{B \cap \Delta : \Delta \in \Sigma\}$.

Consider $L^\infty(\mathscr{A})$ as a subring of the ring $L^\infty(\Sigma)$. Define

$$\mathscr{K}_\infty := L^\infty(\mathscr{A}) \star \mathscr{N}(E) = \left\{ \begin{bmatrix} M_u & 0 \\ M_v & M_u \end{bmatrix} : u \in L^\infty(\mathscr{A}),\ v \in \mathscr{N}(E) \right\},$$

where M_w represents the linear transformation of multiplication by w. This matrix form for $\mathscr{K}_\infty$ suggest the viewing $\mathscr{K}_\infty$ as a ring of $L^\infty(\mathscr{A})$-linear operators on the $L^\infty(\mathscr{A})$-module $L^\infty(\mathscr{A}) \oplus \mathscr{N}(E)$. Note that $L^\infty(\mathscr{A}) = \{Ef : f \in L^\infty(\Sigma)\}$ and $\mathscr{N}(E) = \{f - Ef : f \in L^\infty(\Sigma)\}$. Since for each $g \in L^\infty(\mathscr{A})$, $f - Ef = (f+g) - E(f+g)$, thus the representation of the members of the $\mathscr{N}(E)$ is not unique. By these observations, $[u] \in \mathscr{K}_\infty$ if and only if

$$[u] = \begin{bmatrix} M_{E(u)} & 0 \\ M_{u-E(u)} & M_{E(u)} \end{bmatrix},$$

for some $u \in L^\infty(\Sigma)$. Let $u \in L^0(\Sigma) \cap \mathscr{D}(E)$, where $\mathscr{D}(E)$ denotes the domain of E. The mapping $T_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p \leq \infty$, defined by $T_u(f) = uE(f) + fE(u) - E(u)E(f)$ is called the conditional multiplication operator induced by a weight function u. Note that if u is an $\mathscr{A}$-measurable function, then $T_u = M_u$. Define $\mathscr{K}_p = \{T_u : u \in L^0(\Sigma) \cap \mathscr{D}(E)$ and $T_u \in \mathscr{B}(L^p(\Sigma))\}$. For $1 \leq p < \infty$, although the $L^p(\Sigma)$ spaces are not rings, however every member of $\mathscr{K}_\infty$ has the form $T_u \in \mathscr{B}(L^\infty(\Sigma))$, $u \in L^\infty(\Sigma)$.

Conditional multiplication operators are closely related to the integral operators (see Example 2.1 (b)), averaging operators on order ideals in Banach lattices and to the operators called conditional expectation-type operators which have been introduced in [1]. Also in [3], operators that are representable as products involving multiplications and conditional expectations are studied. Some properties and the applications of conditional multiplication operators are also studied in [10] and [7]. In this paper we investigate other operator properties of the members of $\mathscr{K}_p$ such as reducibility, closedness of range and compactness. Meanwhile, their kernel and their spectrum are characterized. Finally we close this note off by presenting some examples to illustrate the utility of the results in part. For a beautiful exposition of the study of weighted conditional expectation operators on L^p-spaces, see [4, 5] and the references therein.

2. Characterization of conditional multiplication operators

First note that since $\mathscr{R}(E) = L^2(\mathscr{A})$ and E is a projection, the Hilbert space $L^2(\Sigma)$ can be decomposed into a direct sum of the subspaces $L^2(\mathscr{A})$ and $\mathscr{N}(E)$, such that the assignment

$$f \mapsto \begin{bmatrix} E(f) \\ f - E(f) \end{bmatrix}$$

is an isometric isomorphism from $L^2(\Sigma)$ onto $L^2(\mathscr{A}) \oplus \mathscr{N}(E)$. Also note that the matrices of the operators T_u and T_u^* with respect the above decomposition, denoted by $[T_u]$ and $[T_u^*]$, are the following forms

$$(2.1) \qquad [T_u] = \begin{bmatrix} M_{E(u)} & 0 \\ M_{u-E(u)} & M_{E(u)} \end{bmatrix} \quad \text{and} \quad [T_u^*] = \begin{bmatrix} M_{\overline{E(u)}} & EM_{\bar{u}-\overline{E(u)}} \\ 0 & M_{\overline{E(u)}} \end{bmatrix}.$$

In sense of matrix theory, it should be considered that $[T_u]$ is bounded if for each $f \in L^2(\Sigma)$, the assignment

$$f \mapsto [T_u] \begin{bmatrix} E(f) \\ f - E(f) \end{bmatrix}$$

defines a bounded operator on $L^2(\Sigma)$. A moment's consideration of T_u's matrix in (2.1) shows that T_u is a bounded operator if and only if $M_{E(u)} : L^2(\mathscr{A}) \to L^2(\mathscr{A})$, $M_{E(u)} : \mathscr{N}(E) \to \mathscr{N}(E)$ and $M_{u-E(u)} : L^2(\mathscr{A}) \to \mathscr{N}(E)$ are bounded operators. It is known that the boundedness of $M_{E(u)}$ and $M_{u-E(u)}$ implies that $E(u) \in L^\infty(\mathscr{A})$ and $E(|u-E(u)|^2) \in L^\infty(\mathscr{A})$ respectively (see [9]). Since $E(|u|^2) = E(|u-E(u)|^2) + |E(u)|^2$, it follows that $E(|u|^2) \in L^\infty(\Sigma)$.

On the other hand, if $E(|u|^2) \in L^\infty(\mathscr{A})$, then $E(u) \in L^\infty(\mathscr{A})$, because $|E(u)|^2 \leq E(|u|^2)$. Thus the multiplication operator $M_{E(u)}$ is bounded on the subspaces $L^2(\mathscr{A})$ and $\mathscr{N}(E)$. Moreover, in this case, we claim that $M_{u-E(u)}$ is also bounded. Let $f \in L^\infty(\mathscr{A})$ be an arbitrary. Then we have

$$\begin{aligned} \|M_{u-E(u)}f\|_2^2 &= \int_X |u-E(u)|^2 |f|^2 d\mu = \int_X E(|u-E(u)|^2)|f|^2 d\mu \\ &= \int_X (E(|u|^2) - |E(u)|^2)|f|^2 d\mu \leq \|E(|u|^2)\|_\infty \|f\|_2^2. \end{aligned}$$

Then $\|M_{u-E(u)}\| \leq \sqrt{\|E(|u|^2)\|_\infty}$. Consequently, an operator $T_u : L^2(\Sigma) \to L^2(\Sigma)$ is bounded if and only if $E(|u|^2) \in L^\infty(\mathscr{A})$.

Now, let T_u be a bounded linear operator on $L^2(\Sigma)$. Then it is not difficult to verify that

$$(2.2) \qquad [T_u^* T_u] = \begin{bmatrix} M_{|E(u)|^2} + EM_{|u-E(u)|^2} & EM_{(\bar{u}-\overline{E(u)})E(u)} \\ M_{\overline{E(u)}(u-E(u))} & M_{|E(u)|^2} \end{bmatrix}$$

and

$$(2.3) \qquad [T_u T_u^*] = \begin{bmatrix} M_{|E(u)|^2} & M_{E(u)}EM_{\bar{u}-\overline{E(u)}} \\ M_{\overline{E(u)}(u-E(u))} & M_{u-E(u)}EM_{\bar{u}-\overline{E(u)}} + M_{|E(u)|^2} \end{bmatrix}.$$

At first glance it can be readily inferred that the matrices (2.2) and (2.3) are equal (entrywise equality) whenever $u = E(u)$, i.e., $u \in L^0(\mathscr{A})$. Therefore, by the preceding argument we get that $\|u\|_\infty \leq \sqrt{\|E(|u|^2)\|_\infty}$. Hence, T_u is normal if and only if $u \in L^\infty(\mathscr{A})$. Moreover, by the equality of the T_u and T_u^* matrices represented in (2.1), T_u is self-adjoint operator if and only if $u \in L^\infty(\mathscr{A})$ is real-valued.

It is well known that if a subspace, taking part in a underlying space decomposition, is invariant for an operator then its corresponding block must be zero in its matrix representation. These observations establish the following proposition.

Proposition 2.1. *For a conditional multiplication operator T_u on $L^2(\Sigma)$, the following assertions hold.*

(a) *T_u is a bounded operator if and only if $E(|u|^2) \in L^\infty(\mathscr{A})$.*
(b) *The bounded operator T_u is normal if and only if $u \in L^\infty(\mathscr{A})$.*
(c) *The bounded operator T_u is self-adjoint operator if and only if $u \in L^\infty(\mathscr{A})$ is real-valued.*
(d) *$\mathscr{N}(E)$ is invariant subspace for T_u.*
(e) *$L^2(\mathscr{A})$ reduces T_u if and only if u is an $\mathscr{A}$-measurable function, i.e., T_u is a multiplication operator.*

Lemma 2.1. *Let $T_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p < \infty$, be a bounded linear operator. Then $L^p(X \setminus \sigma(E(u)) \cap \mathscr{N}(E)_{|_{\sigma(u)}} \subseteq \mathscr{N}(T_u) \subseteq L^p(X \setminus \sigma(E(u))$ where $\mathscr{N}(E)_{|_{\sigma(u)}} = \{f \in L^p(\Sigma) : E(f) = 0 \text{ on } \sigma(u)\}$.*

Proof. For given a non-zero function $f \in L^p(\Sigma)$, let $f \in \mathscr{N}(T_u)$. Then $uE(f) + fE(u) - E(u)E(f) = 0$. Taking the conditional expectation E of both sides equation, gives $E(u)E(f) = 0$. Thus $\sigma(E(u)) \cap \sigma(E(f)) = \emptyset$ and so $E(f) \notin L^p(\sigma(E(u))$. Since E is a contraction map we have $f \notin L^p(\sigma(E(u))$. Therefore, $f \in L^p(X \setminus \sigma(E(u))$ because $L^p(\Sigma) = L^p(X \setminus \sigma(E(u)) \oplus L^p(\sigma(E(u))$.

In other hand, let $f \in L^p(X \setminus \sigma(E(u)) \cap \mathscr{N}(E)_{|_{\sigma(u)}}$ be an arbitrary nonzero function. Then $uE(f) = 0$ and $fE(u) = 0$. Hence $E(f)E(u) = 0$ which means that $f \in \mathscr{N}(T_u)$. ∎

Remark 2.1. If the weight function u is nonnegative then $\mathscr{N}(T_u) = L^p(X \setminus \sigma(E(u))$, since in this case $\sigma(u) \subseteq \sigma(E(u))$, and so $L^p(X \setminus \sigma(E(u)) \cap \mathscr{N}(E)_{|_{\sigma(u)}} = L^p(X \setminus \sigma(E(u))$.

In the following theorem we give a necessary and sufficient condition for an operator $T_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p < \infty$, for which it has closed range.

Theorem 2.1. *Suppose $T_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p < \infty$, is a bounded linear operator. Then T_u has closed range if and only if $E(|u|^p) \geq \delta$ a.e on $\sigma(E(u))$ for some $\delta > 0$.*

Proof. By Lemma 2.1, if T_u has closed range then T_u is bounded below on $L^p(\sigma(E(u))$. Then there exists a constant $k > 0$ such that

$$\|T_u f\|_{\sigma(E(u))} \geq k\|f\|_{\sigma(E(u))}, \qquad f \in L^p(\sigma(E(u))).$$

Let $\delta = k/2$ and put $U = \{x \in \sigma(E(u)) : E(|u|^p)(x) < \delta\}$. Suppose on contrary $\mu(U) > 0$. Since $(X, \mathscr{A}, \mu_{|\mathscr{A}})$ is σ-finite measure space, we can find a set $B \in \mathscr{A}$ such that $Q := B \cap \sigma(E(u)) \subseteq U$ with $0 < \mu(Q) < \infty$. Then the $\mathscr{A}$-measurable characteristic function χ_Q lies in $L^p(\sigma(E(u)))$ and satisfies

$$\|T_u \chi_Q\|_p^p = \int_{\sigma(E(u))} |u\chi_Q|^p d\mu = \int_{\sigma(E(u))} |u|^p \chi_Q d\mu = \int_{\sigma(E(u))} E(|u|^p)\chi_Q d\mu$$

$$\leq \delta^p \int_{\sigma(E(u))} \chi_Q d\mu = \delta^p \|\chi_Q\|^p_{L^p(\sigma(E(u))}.$$

This is contrary to the choice of k. Therefore, $\mu(U) = 0$ i.e., $E(|u|^p) \geq \delta$ a.e on $\sigma(E(u))$.

Conversely, suppose $E(|u|^p) \geq \delta$ a.e on $\sigma(E(u))$ and $\{T_u f_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $\mathscr{R}(T_u)$ such that $\|T_u f_n - g\|_p \to 0$ for some $g \in L^p(\Sigma)$. Hence

$$E(T_u f_n) = E(u)E(f_n) \xrightarrow{L^p} E(g).$$

Since, by Proposition 2.9 in [8], $E(1/|u|^p)\chi_{\sigma(E(u))} = (1/E(|u|^p))\chi_{\sigma(E(u))}$, then we have $(E(1/|u|))^p \chi_{\sigma(E(u))} \leq 1/\delta$, and so we get that $E(1/|u|) \leq 1/\delta^{1/p}$ a.e. on $\sigma(E(u))$. Therefore, we have

$$\left|\frac{E(g)}{E(u)}\chi_{\sigma(E(u))}\right| = \left|E(g)E(\frac{1}{u})\chi_{\sigma(E(u))}\right| \leq |E(g)|E(\frac{1}{|u|})\chi_{\sigma(E(u))} \leq \frac{|E(g)|}{\delta^{\frac{1}{p}}}\chi_{\sigma(E(u))}.$$

This follows that $E(g)/(E(u))\chi_{\sigma(E(u))} \in L^p(\sigma(E(u)))$. Consequently,

$$E(f_n) \xrightarrow{L^p} \frac{E(g)}{E(u)}\chi_{\sigma(E(u))}$$

and so

$$f_n \xrightarrow{L^p} \left\{g + E(g) - \frac{uE(g)}{E(u)}\right\}\frac{\chi_{\sigma(E(u))}}{E(u)} := f.$$

Thus $T_u f_n \xrightarrow{L^p} T_u f$ and hence $g = T_u f$, which implies that T_u has closed range. ∎

Lemma 2.2. *Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces. Suppose that $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{K})$ and $C \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. If A and B are normal operators, then*

$$Sp\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = Sp(A) \cup Sp(B).$$

Proof. See [2]. ∎

Theorem 2.2. *Let u be a nonnegative weight function. Then for a bounded conditional multiplication operator $T_u : L^2(\Sigma) \to L^2(\Sigma)$ we have*

$$Sp(T_u) \cup \{0\} = ess\ rang\{E(u)\} \cup \{0\}.$$

Proof. Let $Z = X \backslash \sigma(E(u)) \neq \emptyset$. Since $\sigma(u) \subseteq \sigma(E(u))$ so by making use of T_u^*'s matrix, represented in (2.1), we get that $L^2(Z, \mathscr{A}_{|Z}, \mu_{|Z}) \subseteq \mathscr{N}(T_u^*)$. Then $\overline{\mathscr{R}(T_u)} = \mathscr{N}(T_u^*)^{\perp} \subseteq L^2(Z^c, \mathscr{A}_{|Z^c}, \mu_{|Z^c})$ which means that T_u is not onto and so $0 \in \mathrm{Sp}(T_u)$. On the other hand, a moment's consideration of the matrix of T_u^* and Lemma 2.5 show that $\mathrm{Sp}(T_u) = \overline{\mathrm{Sp}(T_u^*)} = \overline{\mathrm{Sp}(M_{\overline{E(u)}})} = ess\ rangE(u)$. ∎

Theorem 2.3. *Suppose $(X, \mathscr{A}, \mu_{|\mathscr{A}})$ can be partitioned as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$. Then the bounded linear operator T_u on $L^p(\Sigma)$ is compact if and only if $u(x) = 0$ for μ-almost all $x \in B$ and for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \mu(A_n \cap D_\varepsilon(u)) > 0\}$ is finite, where $D_\varepsilon(u) = \{x \in X : E(|u|)(x) \geq \varepsilon\}$.*

Proof. Recall that T_u is compact if and only if $T_u^* = EM_{\bar{u}} + M_{\overline{E(u)}}(I - E)$ is compact. Since $ET_u^* = EM_{\bar{u}}$, so if T_u is compact then EM_u is compact. In other hand, if EM_u is compact then $M_{E(u)}$ is compact one since $EM_{u|L^2(\mathscr{A})} = M_u$. Thus $M_{\overline{E(u)}}$ is compact operator([12]).

Eventually T_u^* and in turn T_u are compact operators. Consequently, the compactness of T_u is equivalent to the compactness of EM_u, which is in turn equivalent to the asserted statement by Theorem 2.2 in [6]. ∎

Corollary 2.1. *Suppose that (X,Σ,μ) is a non-atomic measure space. Then the bounded linear operator T_u on $L^2(\Sigma)$ is compact if and only if it is a zero operator.*

Define $\mathfrak{L}_p = \{u \in L^0(\Sigma)\cap \mathscr{D}(E) : T_u \in \mathscr{K}_p\}$. Then for $1 \le p < \infty$, $u \in \mathfrak{L}_p$ if and only if $E(|u|^p) \in L^\infty(\mathscr{A})$ (see [7]). Hence for $u \in \mathfrak{L}_p$ one can define its norm by $\|u\|_{\mathfrak{L}_p} := \|E(|u|^p)\|_\infty^{1/p}$ such that $(\mathfrak{L}_p, \|\cdot\|_{\mathfrak{L}_p})$ is respected as a normed algebra.

Theorem 2.4. *$(\mathfrak{L}_p, \|\cdot\|_{\mathfrak{L}_p})$ is a Banach space and for each $u \in \mathfrak{L}_p$ the inequality $\|u\|_{\mathfrak{L}_p} \le \|T_u\| \le 3\|u\|_{\mathfrak{L}_p}$ holds.*

Proof. First we show that the inequality $\|u\|_{\mathfrak{L}_p} \le \|T_u\| \le 3\|u\|_{\mathfrak{L}_p}$ holds. Suppose $u \in \mathfrak{L}_p$ and $f \in L^1(\mathscr{A})$. Then

$$\int_X E(|u|^p)|f|d\mu = \int_X |T_u(|f|^{\frac{1}{p}})|^p d\mu = \|T_u(|f|^{\frac{1}{p}})\|_p^p \le \|T_u\|^p\|f\|_1.$$

It follows that $\|u\|_{\mathfrak{L}_p} \le \|T_u\|$. On the other hand, by the properties of conditional expectation operators, it is easy to see that for each $f \in L^p(\Sigma)$ with $\|f\|_p \le 1$,

$$\max\{\|E(u)f\|_p, \|uE(f)\|_p, \|E(u)E(f)\|_p\} \le \|E(|u|^p)\|^{\frac{1}{p}},$$

and so $\|T_u\| \le 3\|u\|_{\mathfrak{L}_p}$.

Now, let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathfrak{L}_p}$ and let $f \in L^p(\Sigma), g \in L^q(\Sigma)$ with $1/p + 1/q = 1$ be arbitrary elements. Since

$$|\int_X T_{u_n - u_m}(f)\bar{g}d\mu| \le 3\|u_n - u_m\|_{\mathfrak{L}_p}\|f\|_p\|g\|_q, \tag{2.4}$$

$\{T_{u_n}\}_{n=1}^\infty$ is a Cauchy sequence in the weak operator topology. By Theorem 4.1 in [10] the subalgebra $\{T_u : u \in \mathfrak{L}_p\}$ is a maximal abelian and so it is weakly closed. Therefore, $\{T_{u_n - u_0}\}_{n=1}^\infty$ is weakly convergent to 0 for some $u_0 \in \mathfrak{L}_p$. By the dominated convergence theorem we have

$$\int_X \lim_{n\to\infty}(u_n - u_0)fgd\mu = \lim_{n\to\infty}\int_X T_{u_n-u_0}(f)gd\mu = 0.$$

Thus $\lim_{n\to\infty}(u_n - u_0) = 0$ a.e. on X and hereby $\lim_{n\to\infty} E(|u_n - u_0|^p) = 0$ a.e. on X since E is a contraction map. Eventually, $\|u_n - u_0\|_{\mathfrak{L}_p} \to 0$ as $n \to \infty$. ∎

Example 2.1. (a) Let $X = [-\pi/2, \pi/2]$, μ be the Lebesgue measure on the σ-algebra Σ consisting of all Lebesgue measurable subsets of X. Moreover, suppose $\mathscr{A}$ is the σ-subalgebra generated by the all symmetric intervals about origin. For given a $f \in L^p(\Sigma)$, then under the above hypotheses, $E(f)$ is just equal to the even part of f i.e., $(f(x)+f(-x))/2$. Define $u(x) = x^2 + \sin x + 1$. Then we have

$$T_u f = (\frac{1}{2}\sin x + x^2 + 1)f(x) + \frac{1}{2}\sin x f(-x).$$

It is easy to see that $1 \le E(|u|^p) \in L^\infty(\mathscr{A})$ on X. Hence by Theorem 2.4 and Theorem 2.10, T_u is bounded and has closed range.

(b) Let $X = [0,1]\times[0,1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathscr{A} = \{A \times [0,1] : A$ is a Lebesgue set in $[0,1]\}$. Then, for each f in $L^p(\Sigma)$, $(Ef)(x,y) = \int_0^1 f(x,t)dt$, which is independent of the second coordinate. Now, if we take $u(x,y) = y^{-x/8}$ with $p \geq 1$, then for all $f \in L^p(\Sigma)$ we have

$$(T_u f)(x,y) = (y^{\frac{-x}{8}} - \frac{8}{8-x})\int_0^1 f(x,t)dt + \frac{8}{8-x} f(x,y).$$

Since $E(|u|^p)(x,y) = 8/|8-px|$, then

$$\|E(|u|^p)\|_\infty = \begin{cases} \frac{8}{8-p} & 1 \leq p < 8 \\ \infty & p \geq 8. \end{cases}$$

Thus, the operator integral T_u is bounded operator on $L^p(\Sigma)$ if and only if $1 \leq p < 8$. In this case $\|T_u\| = (8/(8-p))^{1/p}$. Also, X is nonatomic and $1 \leq E(|u|^p) \in L^\infty(\mathscr{A})$ on X, then by Theorem 2.1 and Corollary 2.1, the bounded operator T_u has closed range but not compact operator on $L^p(\Sigma)$. Moreover, when $p = 2$ then $Sp(T_u) \cup \{0\} = [1, 8/7] \cup \{0\}$ by Theorem 2.2.

Acknowledgment. The authors would like to thank the referee(s) for very helpful comments and valuable suggestions.

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