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Weighted Composition Lambert-Type Operators on L^p Spaces

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Abstract. In this paper boundedness of a weighted composition Lambert-type operator $T = M_w E M_u C_\varphi$ acting between two different $L^p(\Sigma)$ spaces is characterized using some properties of conditional expectation operator. Moreover, we establish criteria for hyponormality for these types of operators on $L^2(\Sigma)$.

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1. Introduction and Preliminaries

For $1 \leq p \leq \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. The support of a measurable function f is defined as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. Suppose that φ is a measurable transformation from X into X such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, that is, φ is non-singular. Let h be the Radon–Nikodym derivative $\mathrm{d}\mu \circ \varphi^{-1}/\mathrm{d}\mu$ and we always assume that h is almost everywhere finite valued or, equivalently $\varphi^{-1}(\Sigma)$ is a sub-sigma finite algebra. We denote the vector space of all equivalence classes of almost everywhere finite-valued measurable functions on X by $L^0(\Sigma)$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. Let $u \in L^0(\Sigma)$. Then u is said to be conditionable with respect to E if $u \in \mathcal{D}(E) := \{f \in L^0(\Sigma) : E(|f|) \in L^0(\mathcal{A})\}$. An $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well known that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned uniquely as $X = (\cup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and $B \in \mathcal{A}$, being disjoint from each A_n, is non-atomic (see [7]).

For a sigma-finite algebra $\mathcal{A} \subseteq \Sigma$, we denote by E the conditional expectation $E^{\mathcal{A}}$ considered as a bounded linear idempotent transformation from

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$L^p(\Sigma)$ on to $L^p(\mathcal{A})$. Those properties of E used in our discussion are summarized below. In all cases, f and g are conditionable functions.

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- (Conditional Jensen's inequality) If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $\phi(f)$ is conditionable, then $E(\phi(f)) \geq \phi(E(f))$.
- If p and p' are conjugate exponents and $f \in L^p(\Sigma)$ and $g \in L^{p'}(\Sigma)$, then $|E(fg)| \leq E(|f|^p)|^{\frac{1}{p}} E(|g|^{p'})|^{\frac{1}{p'}}$.
- For each $f \geq 0$, $\sigma(f) \subseteq \sigma(E(f))$.

A detailed discussion and verification of these properties may be found in [6]. Now, take u and w in $\mathcal{D}(E)$. Then the triple (u, w, φ) induces a weighted composition Lambert-type operator T from $L^p(\Sigma)$ into $L^0(\Sigma)$ defined by $T = M_w E M_u C_\varphi$, where M_u and M_w are multiplication operators and C_φ is a composition operator. Throughout this paper we assume that $u\mathcal{R}(C_\varphi) \subset \mathcal{D}(E)$, $w \in \mathcal{D}(E)$, $E = E^{\mathcal{A}}$, $E_\varphi = E^{\varphi^{-1}(\Sigma)}$, φ is non-singular and $T = M_w E M_u C_\varphi$, where $\mathcal{R}(C_\varphi)$ denotes the range of C_φ.

A combination of conditional expectation operators and multiplication operators appears more often in the service of the study of other operators, such as operators generated by random measures [2], Markov operators, and Reynolds and averaging operators [6].

Some results of this article are a generalization of the work done in [4,3,1]. In the next section, the boundedness of T acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator. In Sect. 2, we discuss the measure theoretic characterizations for hyponormality of T on $L^2(\Sigma)$.

2. Weighted Composition Lambert-Type Operators

In this section, we give some sufficient and necessary conditions for boundedness of $T = M_w E M_u C_\varphi$ acting between two $L^p(\Sigma)$ spaces.

Theorem 2.1. *Let $1 < p < \infty$ and let $T : L^p(\Sigma) \to L^p(\Sigma)$ be a weighted composition Lambert-type operator $T = M_w E M_u C_\varphi$.*

(i) *If $J_1 := hE_\varphi(E(|v|^{p'}))^{p/p'} \circ \varphi^{-1} \in L^\infty(\Sigma)$, then the operator T is bounded, where $v := u(E(|w|^p)^{1/p}$ and p' is the conjugate exponents to p.*

(ii) *If $\varphi^{-1}(\Sigma) \subseteq \mathcal{A}$ and T is bounded, then $J_2 := hE_\varphi(|E(v)|)^p \circ \varphi^{-1} \in L^\infty(\Sigma)$, where $v := u(E(|w|^p)^{1/p}$.*

Proof. (i) Let $f \in L^p(\Sigma)$ and $v = u(E(|w|^p))^{1/p}$. Then we have

$$\|Tf\|_p^p = \int_X |w|^p |E(uf \circ \varphi)|^p \mathrm{d}\mu = \int_X E(|w|^p)|E(uf \circ \varphi)|^p \mathrm{d}\mu$$
$$= \int_X |E(u(E(|w|^p))^{\frac{1}{p}} f \circ \varphi)|^p \mathrm{d}\mu = \|EM_v C_\varphi f|_p^p.$$

It follows that T is bounded if and only if the operator $EM_vC_\varphi$from $L^p(\Sigma)$ into $L^p(\mathcal{A})$ is bounded. Now, let $f \in L^p(\Sigma)$. Then, by conditional type Hölder inequality, we have

$$\|EM_vC_\varphi f\|_p^p = \int_X |E(vf\circ\varphi)|^p \mathrm{d}\mu \le \int_X (E(|v|^{p'}))^{\frac{p}{p'}} E(|f|^p\circ\varphi)\mathrm{d}\mu$$
$$=\int_X (E(|v|^{p'}))^{\frac{p}{p'}} |f|^p\circ\varphi \mathrm{d}\mu = \int_X hE_\varphi((E(|v|^{p'}))^{\frac{p}{p'}})\circ\varphi^{-1}|f|^p\mathrm{d}\mu = \int_X J_1|f|^p\mathrm{d}\mu.$$

This implies that $\|T\| \le \|J_1\|_\infty^{1/p}$, and so T is bounded.

(ii) Let $A\in\Sigma$ with $0<\mu(A)<\infty$. Then

$$\int_A J_2\mathrm{d}\mu = \int_X |E(v)|^p\chi_A\circ\varphi\mathrm{d}\mu = \int_X |E(v\chi_A\circ\varphi)|^p\mathrm{d}\mu$$
$$= \|T(\chi_A)\|_p^p \le \|T\|^p\int_X \chi_A\mathrm{d}\mu = \int_A \|T\|^p\mathrm{d}\mu,$$

where $v = u(E(|w|^p))^{1/p}$. It follows that $J_2 \le \|T\|^p$, and so $J_2\in L^\infty(\Sigma)$. □

Note that, $\mathcal{A}=\Sigma$ if and only if $E=I$, the identity operator, and if φ is the identity map, then $E_\varphi = I$. Then we have the following corollary.

Corollary 2.2. (i) *If $hE_\varphi(|w|^p)\circ\varphi^{-1}(E_\varphi(|u|^{p'}))^{p/p'}\circ\varphi^{-1}\in L^\infty(\Sigma)$, then the operator $T_\varphi := M_wE_\varphi M_uC_\varphi$ is bounded on $L^p(\Sigma)$ with $1<p<\infty$. Conversely, if T_φ is bounded, then $h(|E_\varphi(u)|^p\circ\varphi^{-1})(E_\varphi(|w|^p)\circ\varphi^{-1})\in L^\infty(\Sigma)$.*

(ii) *The weighted Lambert-type operator M_wEM_u from $L^p(\Sigma)$ into $L^p(\Sigma)$ is bounded if and only if $(E|w|^p)^{1/p}(E|u|^{p'})^{1/p'}\in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|M_wEM_u\| = \|(E(|w|^p))^{1/p}(E(|u|^{p'}))^{1/p'}\|_\infty$.*

(iii) *The weighted composition operator $uC_\varphi = M_uC_\varphi$ from $L^p(\Sigma)$ into $L^p(\Sigma)$ is bounded if and only if $J_\varphi := hE_\varphi(|u|^p)\circ\varphi^{-1}\in L^\infty(\Sigma)$.*

Theorem 2.3. *Let $1<q<p<\infty$ and let p', q' be the conjugate exponents to p and q, respectively. Then the following assertions hold.*

(i) *If $S_1 := hE_\varphi(E(|v|^{q'}))^{q/q'}\circ\varphi^{-1}$ and $S_1^{1/q}\in L^r(\Sigma)$, then the weighted composition Lambert-type operator T from $L^p(\Sigma)$ to $L^q(\Sigma)$ is bounded, where $v := u(E(|w|^q)^{1/p}$ and $1/p+1/r=1/q$.*

(ii) *If $\varphi^{-1}(\mathcal{A})\subset\mathcal{A}$ and T from $L^p(\Sigma)$ to $L^q(\Sigma)$ is bounded, then $S_2^{1/q}\in L^r(\Sigma)$, where $S_2 := hE_\varphi(|E(v)|)^q\circ\varphi^{-1}$.*

Proof. (i) Let $f\in L^p(\Sigma)$ and $S_1^{1/q}\in L^r(\Sigma)$. Then, by Hölder and conditional Jensen's inequalities, we have

$$\|Tf\|_q^q = \|EM_vC_\varphi f\|_q^q = \int_X |E(vf\circ\varphi)|^p\mathrm{d}\mu \le \int_X (E(|v|^{q'}))^{\frac{q}{q'}}E(|f|^q\circ\varphi)\mathrm{d}\mu$$
$$= \int_X (E(|v|^{q'}))^{\frac{q}{q'}}|f|^q\circ\varphi\mathrm{d}\mu = \int_X hE_\varphi((E(|v|^{q'}))^{\frac{q}{q'}})\circ\varphi^{-1}|f|^q\mathrm{d}\mu$$

$$\leq \left(\int\limits_X S_1^{\frac{r}{q}} \mathrm{d}\mu\right)^{\frac{q}{r}} \left(\int\limits_X |f|^p \mathrm{d}\mu\right)^{\frac{q}{p}} = \|S_1^{\frac{1}{q}}\|_r^q \, \|f\|_p^q.$$

This implies that $\|T\| \leq \|S_1^{1/q}\|_r$, and so T is bounded.

(ii) Define linear functional Λ on $L^{\frac{p}{q}}(\mathcal{A})$ by

$$\Lambda(f) = \int\limits_X S_2 f \mathrm{d}\mu.$$

Since T is bounded, for each $f \in L^{\frac{p}{q}}(\mathcal{A})$ we get that

$$|\Lambda(f)| \leq \int\limits_X E\left(hE_\varphi(|E(v)|)^q \circ \varphi^{-1}\right)|f| \mathrm{d}\mu = \int\limits_X |E(v)|^q |f| \circ \varphi \mathrm{d}\mu$$

$$= \int\limits_X |E(v|f|^{\frac{1}{q}} \circ \varphi)|^q \mathrm{d}\mu = \|EM_v C_\varphi(|f|^{\frac{1}{p}})\|_q^q = \|T(|f|^{\frac{1}{q}})\|_q^q \leq \|T\|^q \|f\|_{\frac{p}{q}}.$$

Hence, Λ is bounded linear functional on $L^{\frac{p}{q}}(\mathcal{A})$. By the Hahn–Banach theorem, we can suppose that Λ is a bounded linear functional on $L^{\frac{p}{q}}(\Sigma)$ and $\|\Lambda\| \leq \|T\|^q$. Thus by the Riesz representation theorem, $S_2^{1/q} \in L^r(\Sigma)$. □

Corollary 2.4. *Let $1 < q < p < \infty$ and let $1/p + 1/r = 1/q$. Then the weighted composition operator uC_φ from $L^p(\Sigma)$ to $L^q(\Sigma)$ is bounded if and only if $J_q := hE_\varphi(|u|^q) \circ \varphi^{-1} \in L^{r/q}(\Sigma)$.*

Suppose that $X = (\cup_{n\in\mathbb{N}} C_n) \cup C$, where $\{C_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint Σ-atoms and $C \in \Sigma$, being disjoint from each C_n, is non-atomic. Note that $(\cup_{n\in\mathbb{N}} C_n) \cap \mathcal{A} \subseteq \cup_{n\in\mathbb{N}} A_n$ and $B \subseteq C$.

Theorem 2.5. *Let $1 < p < q < \infty$, $v = u(E(|w|^q)^{1/p}$ and let $T : L^p(\Sigma) \to L^q(\Sigma)$ be a weighted composition Lambert-type operator.*

(i) *Let $K_1 := hE_\varphi(E(|v|^{q'}))^{q/q'} \circ \varphi^{-1}$ and let $M := \sup_{n\in\mathbb{N}} \frac{K_1(C_n)}{\mu(C_n)^{q/r}}$, where p', q' be the conjugate exponents to p and q, respectively, and $1/q + 1/r = 1/p$. If $K_1 = 0$ on C and $M < \infty$, then T is bounded.*

(ii) *Let $\varphi^{-1}(\mathcal{A}) \subset \mathcal{A}$ and let $N =: \sup_{n\in\mathbb{N}} \frac{E(K_2)^{1/q}(A_n)}{\mu(A_n)^{1/r}}$. If T is bounded, then $K_2 := hE_\varphi(|E(v)|)^q \circ \varphi^{-1} = 0$ on B and $N < \infty$.*

Proof. (i) Let $f \in L^p(\Sigma)$ with $\|f\|_p = 1$. Then, by conditional Jensen's inequality, we have

$$\|Tf\|_q^q = \|EM_vC_\varphi f\|_q^q = \int\limits_X |E(vf \circ \varphi)|^q \mathrm{d}\mu \leq \int\limits_X (E(|v|^{q'}))^{\frac{q}{q'}} E(|f|^q \circ \varphi) \mathrm{d}\mu$$

$$= \int\limits_X (E(|v|^{q'}))^{\frac{q}{q'}} |f|^q \circ \varphi \mathrm{d}\mu = \int\limits_X K_1 |f|^q \mathrm{d}\mu = \sum_{n\in N} K_1(C_n)|f(C_n)|^q \mu(C_n)$$

$$= \sum_{n\in N} \frac{K_1(C_n)}{\mu(C_n)^{\frac{q}{r}}} (|f(C_n)|^q \mu(C_n))^{\frac{q}{p}} \leq M < \infty.$$

This implies that $\|T\| \leq M^{1/q}$, and so T is bounded.

(ii) First, we show that $E(K_2)=0$ on B. Suppose on the contrary. Thus, we can find some $\delta>0$ such that $\mu(\{x\in B: E(K_2)(x)>\delta\})>0$. Take $F=\{x\in B: E(K_2)(x)>\delta\}$. Since $F\subseteq B$ is a $\mathcal{A}$-measurable set and $\mathcal{A}$ is σ-finite, then for each $n\in\mathbb{N}$ there exists $F_n\subseteq F$ with $F_n\in\mathcal{A}$ such that $\mu(F_n)=\mu(F)/2^n$. Define $f_n=\frac{\chi_{F_n}}{\mu(F_n)^{1/p}}$. Then $f_n\in L^p(\mathcal{A})$ with $\|f_n\|_p=1$. Since T is bounded and $\frac{q}{p}>1$, we get that

$$\infty>\|T\|^q=\|EM_vC_\varphi\|^q\geq\|E(vf_n\circ\varphi)\|_q^q=\frac{1}{\mu(F_n)^{\frac{q}{p}}}\int\limits_{F_n}|E(v)|^q\mathrm{d}\mu$$

$$=\frac{1}{\mu(F_n)^{\frac{q}{p}}}\int\limits_{F_n}K_2\mathrm{d}\mu\geq\frac{\delta\mu(F_n)}{\mu(F_n)^{\frac{q}{p}}}=\delta\mu(F_n)^{1-\frac{q}{p}}=\delta\left(\frac{2^n}{\mu(F)}\right)^{\frac{q}{p}-1}\longrightarrow\infty,$$

when $n\to\infty$. But, this is a contradiction. It follows that $K_2=0$ on B, since $K_2\geq 0$. It remains to prove $N<\infty$. Take $f_n=\frac{\chi_{A_n}}{\mu(A_n)^{1/p}}$. Hence $f_n\in L^p(\mathcal{A})$ with $\|f_n\|_p=1$. Then we have

$$\frac{E(K_2)(A_n)}{\mu(A_n)^{\frac{q}{r}}}=\frac{E(K_2)(A_n)\mu(A_n)}{\mu(A_n)^{\frac{q}{p}}}=\int\limits_{A_n}\frac{E(K_2)}{\mu(A_n)^{\frac{q}{p}}}\mathrm{d}\mu$$

$$=\int\limits_X\frac{E(K_2)\chi_{A_n}}{\mu(A_n)^{\frac{q}{p}}}\mathrm{d}\mu=\int\limits_X\frac{K_2\chi_{A_n}}{\mu(A_n)^{\frac{q}{p}}}\mathrm{d}\mu=\|E(vf_n\circ\varphi)\|_q^q\leq\|EM_vC_\varphi\|^q=\|T\|^q.$$

This implies that $N\leq\|EM_vC_\varphi\|<\infty$. □

Corollary 2.6. *Let $1<p<q<\infty$ and let $1/q+1/r=1/p$. Then the weighted composition operator uC_φ from $L^p(\Sigma)$ to $L^q(\Sigma)$ is bounded if and only if $J_q=0$ on C and $\sup_{n\in\mathbb{N}}J_q(C_n)/\mu(C_n)^{q/r}<\infty$.*

Theorem 2.7. *Let $1<q<\infty$, $v=u(E(|w|^q)^{1/q}$ and let $T: L^1(\Sigma)\to L^q(\Sigma)$ be a weighted composition Lambert-type operator.*

(i) *If $K_1=0$ on C and $M_1:=\sum_{n\in\mathbb{N}}\frac{K_1(C_n)}{\mu(C_n)^{q-1}}<\infty$, then T is bounded.*
(ii) *If $\varphi^{-1}(\mathcal{A})\subset\mathcal{A}$ and T is bounded, then $E(K_2)=0$ on B and $N_1:=\sup_{n\in N}\frac{E(K_2)^{1/q}(A_n)}{\mu(A_n)^{1-1/q}}<\infty$.*

Proof. (i) Let $f\in L^1(\Sigma)$ with $\|f\|_1=1$. Then, by conditional Jensen's inequality, we have

$$\|Tf\|_q^q=\|EM_vC_\varphi f\|_q^q=\int\limits_X|E(vf\circ\varphi)|^q\mathrm{d}\mu\leq\int\limits_X(E(|v|^{q'}))^{\frac{q}{q'}}E(|f|^q\circ\varphi)\mathrm{d}\mu$$

$$=\int\limits_X(E(|v|^{q'}))^{\frac{q}{q'}}|f|^q\circ\varphi\mathrm{d}\mu=\int\limits_X K_1|f|^q\mathrm{d}\mu=\sum_{n\in N}K_1(C_n)|f(C_n)|^q\mu(C_n)$$

$$\sum_{n\in N}\frac{K_1(C_n)}{\mu(C_n)^{q-1}}(|f(C_n)|\mu(C_n))^q\leq M_1$$

This implies that $\|T\|\leq M_1^{1/q}$, and thus T is bounded.

(ii) It follows by the same argument in the proof of Theorem 2.5(ii). □

Example 2.8. Let $X = [0,1]^2$, $\mathrm{d}\mu = \mathrm{d}x\mathrm{d}y$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0,1] : A \text{ is a Lebesgue set in } [0,1]\}$. Then, for each f in $L^2([0,1]^2)$, $(Ef)(x,y) = \int_0^1 f(x,t)\mathrm{d}t$, which is independent of the second coordinate. Define the Baker transformation $\varphi : [0,1]^2 \to [0,1]^2$ by

$$\varphi(x,y) = \left(2x, \frac{1}{2}y\right)\chi_{[0,\frac{1}{2})\times[0,1]} + \left(2x-1, \frac{1}{2}y + \frac{1}{2}\right)\chi_{[\frac{1}{2},1]\times[0,1]}.$$

Since

$$\varphi^{-1}([0,x]\times[0,y]) = \begin{cases} [0,\frac{x}{2}]\times[0,2y] & 0 \le y < \frac{1}{2} \\ ([0,\frac{1}{2}x]\times[0,1]) \cup ([\frac{1}{2},\frac{1}{2}+\frac{1}{2}x]\times[0,2y-1]) & \frac{1}{2} \le y \le 1, \end{cases}$$

and $E(|v|^2) = E(|u|^2)E(|w|^2)$, we get that

$$J_1(x,y) = (hE_\varphi(E|v|^2)\circ\varphi^{-1})(x,y) = \begin{cases} (E|v|^2)(\frac{1}{2}x, 2y) & 0 \le y < \frac{1}{2} \\ (E|v|^2)(\frac{1}{2}+\frac{1}{2}x, 2y-1) & \frac{1}{2} \le y \le 1 \end{cases}$$

$$= \begin{cases} (\int_0^1 |u(\frac{1}{2}x,2t)|^2\mathrm{d}t)(\int_0^1 |w(\frac{1}{2}x,2t)|^2\mathrm{d}t) & 0 \le y < \frac{1}{2} \\ (\int_0^1 |u(\frac{1}{2}+\frac{1}{2}x,2t-1)|^2\mathrm{d}t)(\int_0^1 |w(\frac{1}{2}+\frac{1}{2}x,2t-1)|^2\mathrm{d}t) & \frac{1}{2} \le y \le 1. \end{cases}$$

Now, if $J_1 \in L^\infty([0,1]^2)$, by Theorem 2.1(i), the integral operator

$$(Tf)(x,y) = w(x,y)\int_0^1 u(x,t)f(\varphi(x,t))\mathrm{d}t$$

on $L^2([0,1]^2)$ is bounded.

3. Hyponormality of Operator $M_wEM_uC_\varphi$

In this section, we characterize the hyponormal weighted composition of Lambert-type operators. Our characterization is similar in spirit and statement to Lambert's characterization of hyponormal weighted composition operators [5]. We obtain Lambert's characterization as a corollary whenever $w = 1$ and $\mathcal{A} = \Sigma$.

Let (X,Σ,m) be a complete σ-finite measure space. A bounded operator T on $L^2_m = L^2(X,\Sigma,m)$ is hyponormal if $\|T^*f\| \le \|Tf\|$ for every f in L^2_m. Throughout this section, we assume that u and w are non-negative Σ-measurable functions on X and $J = hE_\varphi(E(v^2))\circ\varphi^{-1} \in L^\infty(\Sigma)$, where $v = u(E(w^2))^{1/2}$. Note that by Theorem 2.1(i), $J \in L^\infty(\Sigma)$ implies that $T = M_wEM_uC_\varphi$ is bounded on L^2_m. Let μ be the measure defined on (X,Σ) by $\mathrm{d}\mu = J\mathrm{d}m$. Then μ is supported on $A := \sigma(J)$. It is easy to see that $L^2_m \subseteq L^2_\mu$ and so the set L^2_m is dense in the Hilbert space L^2_μ. For any $f,g \in \mathcal{D}(E)$ with $\sigma(f) \subseteq \sigma(g)$, we use the notational convention of $\frac{f}{g}$ for $(\frac{f}{g})\chi_{\sigma(f)}$.

Theorem 3.1. *Let $T : L^2(\Sigma) \to L^2(\Sigma)$ be a bounded weighted composition Lambert-type operator.*

(i) *If T is hyponormal, then $\sigma(wE(u)) \subseteq A$.*
(ii) *If T is hyponormal and $\varphi^{-1}(\Sigma) \subseteq \mathcal{A}$, then*

$$(h \circ \varphi)E_{\varphi}\left(\frac{(wE(u))^2}{J}\right) \leq \chi_{\sigma(E_{\varphi}(uE(w)))}, \qquad a.e.\ \mathrm{d}m.$$

(iii) *If $\sigma(E(u)) \subseteq A$, $(h \circ \varphi)E_{\varphi}((wE(u))^2/J) \leq 1, (a.e.\ \mathrm{d}m)$ and for each $f \in L^2_m$,*

$$|E(vf \circ \varphi)|^2 = E(|v|^2)E(|f|^2 \circ \varphi),$$

$$\int_X h^2 \circ \varphi(E_{\varphi}(uE(wf)))^2 \mathrm{d}m \leq \int_X h^2 \circ \varphi(E_{\varphi}(E(u)wf))^2 \mathrm{d}m,$$

then T is hyponormal.

Proof. (i) Let $B \subseteq X - A$ be a measurable set of finite measure in Σ. Then by hyponormality of T, we have

$$\|T^*(\chi_B)\|^2_m = \int_X h^2 E_{\varphi}(uE(w\chi_B))^2 \circ \varphi^{-1} \mathrm{d}m \leq \|T(\chi_B)\|^2_m$$

$$= \int_X |wE(u\chi_B \circ \varphi)|^2 \mathrm{d}m \leq \int_X E(v^2)E(\chi_B \circ \varphi)\mathrm{d}m = \int_X J\chi_B \mathrm{d}m = 0,$$

so $hE_{\varphi}(uE(w\chi_B)) \circ \varphi^{-1} = 0$, a.e. $\mathrm{d}m$. It follows that

$$\int_B wE(u)\mathrm{d}m = \int_X uE(w\chi_B)\mathrm{d}m = \int_X hE_{\varphi}(uE(w\chi_B)) \circ \varphi^{-1} \mathrm{d}m = 0.$$

Hence, $wE(u) = 0$, a.e. $\mathrm{d}m$ on B, and so $wE(u) = 0$, a.e. $\mathrm{d}m$ on $X - A$, that is $\sigma(wE(u)) \subseteq A$.

(ii) Suppose that $\varphi^{-1}(\Sigma) \subseteq \mathcal{A}$, thus $L^0(\varphi^{-1}(\Sigma)) \subseteq L^0(\mathcal{A}) \subseteq L^0(\Sigma)$. Now, for any $f \in L^2_m$,

$$\|T^*(f)\|^2_m = \int_X h^2 E_{\varphi}(uE(wf))^2 \circ \varphi^{-1} \mathrm{d}m = \int_A h^2 E_{\varphi}(uE(wf))^2 \circ \varphi^{-1} \mathrm{d}m$$

$$= \int_A \frac{h^2 E_{\varphi}(uE(wf))^2 \circ \varphi^{-1}}{J} \mathrm{d}\mu = \int_A \frac{hE_{\varphi}(uE(wf))^2 \circ \varphi^{-1}}{E_{\varphi}(v^2) \circ \varphi^{-1}} \mathrm{d}\mu.$$

For $f \in L^2_m$, define

$$F(f) = \left(\frac{h}{E_{\varphi}(v^2) \circ \varphi^{-1}}\right)^{\frac{1}{2}} E_{\varphi}(uE(wf)) \circ \varphi^{-1}.$$

(This is well defined, since $\sigma(hE_{\varphi}(uE(wf)) \circ \varphi^{-1}) \subseteq A$). It follows that

$$\|F(f)\|^2_{\mu} = \int_X \frac{h}{E_{\varphi}(v^2) \circ \varphi^{-1}} (E_{\varphi}(uE(wf)))^2 \circ \varphi^{-1} \mathrm{d}\mu = \|T^*(f)\|^2_m$$

$$\leq \|Tf\|^2_m = \int_X |wE(uf \circ \varphi)|^2 \mathrm{d}m \leq \int_X |f|^2 \mathrm{d}\mu = \|f\|^2_{\mu},$$

since L^2_m is dense in L^2_μ, and F extends to a contraction on L^2_μ. Now, let $f, g \in L^2_m$. Then we have

$$\begin{aligned}\langle F(f), g\rangle_\mu &= \int_X \left(\frac{h}{E_\varphi(v^2)\circ\varphi^{-1}}\right)^{\frac{1}{2}} E_\varphi(uE(wf))\circ\varphi^{-1}\bar{g}\mathrm{d}\mu \\ &= \int_X (hE_\varphi(v^2)\circ\varphi^{-1})^{\frac{1}{2}} E_\varphi(uE(wf))\circ\varphi^{-1}\bar{g}h\mathrm{d}m \\ &= \int_X (h\circ\varphi E_\varphi(v^2))^{\frac{1}{2}} E_\varphi(uE(wf))\bar{g}\circ\varphi\mathrm{d}m \\ &= \int_X wf(h\circ\varphi E_\varphi(v^2))^{\frac{1}{2}} E(u)\bar{g}\circ\varphi\mathrm{d}m = \int_X f\frac{wE(u)(J\circ\varphi)^{\frac{1}{2}}\bar{g}\circ\varphi}{J}\mathrm{d}\mu.\end{aligned}$$

Thus, $F^*(g) = \frac{wE(u)(J\circ\varphi)^{1/2}g\circ\varphi}{J}$, and so

$$\begin{aligned}\|F^*(g)\|^2_\mu &= \int_X \frac{w^2E(u)^2J\circ\varphi|g|^2\circ\varphi}{J^2}\mathrm{d}\mu = \int_X \frac{w^2E(u)^2J\circ\varphi|g|^2\circ\varphi}{J}\mathrm{d}m \\ &= \int_X E_\varphi\left(\frac{w^2E(u)^2}{J}\right)J\circ\varphi|g|^2\circ\varphi\mathrm{d}m = \int_X hE_\varphi\left(\frac{w^2E(u)^2}{J}\right)J|g|^2\mathrm{d}m \\ &= \int_X hE_\varphi\left(\frac{w^2E(u)^2}{J}\right)|g|^2\mathrm{d}\mu \le \int_X |g|^2\mathrm{d}\mu,\end{aligned}$$

since F^* is a contraction on L^2_μ. It follows that $hE_\varphi((w^2E(u))^2/J)\circ\varphi^{-1} \le 1$, a.e. $\mathrm{d}\mu$ on $\sigma(J)$ and vanishes on $X\backslash\sigma(J)$, so $hE_\varphi((w^2E(u))^2/J)\circ\varphi^{-1} \le \chi_{\sigma(J)}$. Since $\varphi^{-1}(\sigma(J)) = \sigma(J\circ\varphi) = \sigma(E_\varphi(uE(w)))$, we get that $(h\circ\varphi)E_\varphi((wE(u))^2/J) \le \chi_{\sigma(J)}\circ\varphi = \chi_{\sigma(E_\varphi(uE(w)))}$, and (ii) holds.

(iii) By assumptions of this part, we can define the operator G on L^2_m, for any $f \in L^2_m$,

$$G(f) = \frac{wE(u)(J\circ\varphi)^{\frac{1}{2}}f\circ\varphi}{J}.$$

Hence,

$$\begin{aligned}\|G(f)\|^2_\mu &= \int_X \frac{|wE(u)(J\circ\varphi)^{\frac{1}{2}}f\circ\varphi|^2}{J^2}\mathrm{d}\mu \\ &= \int_X hE_\varphi\left(\frac{(wE(u))^2}{J}\right)\circ\varphi^{-1}|f|^2\mathrm{d}\mu \\ &\le \int_X |f|^2\mathrm{d}\mu = \|f\|^2_\mu.\end{aligned}$$

It follows that G is a contraction on L^2_m. In particular, G extends to a contraction on L^2_μ. By similar computation of part (ii), for f in L^2_m

$$G^*(f) = \left(\frac{h}{E_\varphi(v^2)\circ\varphi^{-1}}\right)^{\frac{1}{2}} E_\varphi(wE(u)f)\circ\varphi^{-1}$$

and $\|T^*(f)\|_m^2 \leq \|G^*(f)\|_\mu^2$. Since G^* is a contraction, we have for all f in L_m^2

$$\|T^*(f)\|_m^2 \leq \|G^*(f)\|_\mu^2 \leq \|f\|_\mu^2 = \|T(f)\|_m^2$$

that is T is hyponormal. □

Corollary 3.2. (i) *If $T_\varphi = M_w E_\varphi M_u C_\varphi$ is hyponormal, then $\sigma(wE_\varphi(u)) \subseteq \sigma(J)$ and $(h\circ\varphi)E_\varphi(u^2)E_\varphi(w^2/J) \leq \chi_{\sigma(E_\varphi(u))\cap E_\varphi(\sigma(w))}$, where $J = hE_\varphi(v^2)\circ\varphi^{-1}$ with $v = u(E_\varphi(w^2))^{1/2}$.*

(ii) *If $\sigma(E_\varphi(u)) \subseteq \sigma(J)$ and $(h\circ\varphi)E_\varphi(u^2)E_\varphi(w^2/J) \leq 1$ with $J = hE_\varphi(v^2)\circ\varphi^{-1}$, then T_φ is hyponormal.*

(iii) *Let $J_\varphi = hE_\varphi(u^2)\circ\varphi^{-1} \in L^\infty(\Sigma)$. Then the bounded weighted composition operator uC_φ on $L^2(\Sigma)$ is hyponormal if and only if*

(a) $\sigma(u) \subseteq \sigma(J_\varphi)$,

(b) $(h\circ\varphi)E_\varphi(\frac{u^2}{J_\varphi}) \leq 1$, *a.e.* d$m$.

Example 3.3. Let $m = \{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the space $l^2(m) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^\infty$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation, i.e. $\mu\circ\varphi^{-1} \ll \mu$. Direct computation shows that

$$h(k) = \frac{1}{m_k}\sum_{j\in\varphi^{-1}(k)} m_j, \quad E_\varphi(f)(k) = \frac{\sum_{j\in\varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j\in\varphi^{-1}(\varphi(k))} m_j},$$

for all non-negative sequence $f = \{f_n\}_{n=1}^\infty$ and $k \in \mathbb{N}$. Since $\sigma(J_\varphi) = \sigma(h)\cap\sigma(E_\varphi(u))$, by Corollary 3.2(ii), uC_φ is hyponormal on $l^2(m)$ if and only if $\sigma(u) \subseteq \{k \in \varphi(\mathbb{N}) : u(\varphi^{-1}(\varphi(k))) \neq \{0\}\}$ and

$$(h\circ\varphi)(k)E_\varphi\left(\frac{u^2}{J_\varphi}\right)(k) = \frac{1}{m_k}\sum_{j\in\varphi^{-1}(\varphi(k))}\frac{u(j)^2 m_j}{J_\varphi(j)} \leq 1$$

on $\sigma(u)$, where for each $j \in \mathbb{N}$,

$$J_\varphi(j) = \frac{1}{m_j}\sum_{i\in\varphi^{-1}(j)} u(i)^2 m_i \leq M$$

for some $M \geq 0$.

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