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Compact Lambert type operators between two L^p spaces

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ABSTRACT

We investigate compactness of weighted Lambert type operators between two L^p spaces.

1. Introduction and preliminaries

Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a sub-σ-algebra of Σ such that $(X, \mathcal{A}, \mu)$ is also σ-finite. All functional equations and set relations are taken modulo sets of measure 0. We denote the vector space of all equivalence classes of almost everywhere finite-valued measurable functions on X by $L^0(\Sigma)$. The support of a measurable function f is defined as $\sigma(f) = \{x \in X; f(x) \neq 0\}$.

Let $E = E^{\mathcal{A}}$ be the conditional expectation operator. For any $\mathcal{A}$-set A and $f \in \bigcup_{p \geq 1} L^p(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$, $\int_A f d\mu = \int_A Ef d\mu$. As an operator on $L^p(\Sigma)$, E is idempotent and $E(L^p(\Sigma)) = L^p(\mathcal{A})$. The interested reader can find more extensive list of properties of conditional expectations in [1] and [10].

We recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$, such that for each $F \in \mathcal{A}$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known that every σ-finite measure space $(X, \mathcal{A}, \mu)$ can be partitioned uniquely as $X = (\bigcup_{n \in \mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n \in \mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [11]).

If $E(f)$ exists for a function $f \in L^0(\Sigma)$, then we say f is conditionable. Let u and w be conditionable. As defined in [3], the weighted Lambert type operator T is the bounded operator $T := M_w E M_u$ from $L^p(\Sigma)$

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into $L^q(\Sigma)$, where M_u and M_w are multiplication operators. Throughout this paper, we assume that u and w are conditionable and T is a weighted Lambert type operator.

Our interest in operators of the form M_wEM_u stems from the fact that such products tend to appear often in the study of those operators related to conditional expectation. Multiplication-conditional expectation products appear in [2], where it is shown that every contractive projection on certain L^1-spaces can be decomposed into an operator of the form M_wEM_u and a nilpotent operator. In [4,5,7] operators that are representable as products involving multiplications and conditional expectations are studied. Also, in [9], S.T.C. Moy has characterized all operators on L^p of the form EM_u and M_wEM_u. Some classical properties of the operator EM_u on $L^p(\Sigma)$ spaces are characterized in [6] and [8]. The authors have characterized boundedness of T between two $L^p(\Sigma)$ spaces, polar decomposition and some other classical properties of T on $L^2(\Sigma)$ in [3]. In this paper we investigate the class of compact linear operators between different L^p-spaces having the form M_wEM_u.

2. Compact weighted Lambert type operators

In this section we discuss compactness of the weighted Lambert type operator $T = M_wEM_u$ between two different L^p-spaces. Also, some examples are provided to illustrate some concrete applications of the main results of the paper.

Theorem 2.1. *Let $1 < q < p < \infty$ and let p', q' be conjugate component to p and q, respectively. Then the weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ is compact if and only if*

(i) $(E|w|^q)^{1/q}(E|u|^{p'})^{1/p'} = 0$ *a.e. on B;*
(ii) $\sum_{n\in\mathbb{N}}(E(|w|^q))^{p'q'/(q'-p')}(A_n)(E(|u|^{p'}))^{q'/(q'-p')}(A_n)\mu(A_n) < \infty$.

Proof. Let $f \in L^p(\Sigma)$. Then

$$\|Tf\|_q^q = \int_X |w|^q\big|E(uf)\big|^q d\mu = \int_X E\big(|w|^q\big)\big|E(uf)\big|^q d\mu$$
$$= \int_X \big|E\big(u\big(E\big(|w|^q\big)\big)^{\frac{1}{q}} f\big)\big|^q d\mu = \|EM_v f\|_q^q,$$

where $v := u(E(|w|^q))^{1/q}$. It follows that the weighted Lambert type operator $T : L^p(\Sigma) \to L^q(\Sigma)$ is compact if and only if $R_v^* = M_{\bar{v}}E = M_{\bar{v}} : L^{q'}(\mathcal{A}) \to L^{p'}(\Sigma)$ is compact. Since $\|M_{\bar{v}}f\|_{p'} = \|M_{(E(|v|^{p'}))^{1/p'}}f\|_{p'}$, we have that $M_{\bar{v}}$ is compact if and only if the multiplication operator $M_{(E(|v|^{p'}))^{1/p'}}$ is compact. Now, suppose that T is compact. Firstly, we show that $E(|v|^{p'}) = 0$ a.e. on B. Assume that there exists some $\delta > 0$ such that the set $B_0 = \{x \in B : E(|v|^{p'})(x) > \delta\}$ has positive measure. We may also assume $\mu(B_0) < \infty$. Since B_0 has no $\mathcal{A}$-atoms, we can find $E_n \in \mathcal{A}$ of positive measure satisfying $E_{n+1} \subseteq E_n \subseteq B_0$ for all n and $\lim_n \mu(E_n) = 0$. Put $f_n = \chi_{E_n}/(\mu(E_n))^{1/p'}$. Then for each n, f_n is a bounded element in $L^{q'}(\mathcal{A})$ and $\lim_n f_n(x) = 0$ for all $x \in X \setminus \bigcap_n E_n$. It follows that $\lim_n E(|v|^{p'})f_n = 0$ point-wise almost everywhere, because $\mu(\bigcap_n E_n) = 0$. By compactness of $M_{(E(|v|^{p'}))^{1/p'}}$, $\{(E(|v|^{p'}))^{1/p'}f_n\}_n$ is uniformly integrable in $L^{p'}(\Sigma)$. Hence there exists $t \geq 0$ such that

$$\sup_{n\in\mathbb{N}} \int_{\{E(|v|^{p'})f_n \geq t\}} E\big(|v|^{p'}\big) f_n^{p'} d\mu < \delta,$$

and so $\int_{E_N} E(|v|^{p'}) f_n^{p'} d\mu < \delta$ for some large $N \in \mathbb{N}$. Then we infer that

$$\delta > \int\limits_{E_N} E(|v|^{p'}) f_N^{p'} d\mu = \| M_{(E(|v|^{p'}))^{1/p'}} f_N^{p'} \|^{p'} = \frac{1}{\mu(E_N)} \int\limits_{E_N} E(|v|^{p'}) \geq \delta,$$

but this is a contradiction. Next, we examine the condition (ii). Since T is compact we have that M_v is bounded and so by Theorem 2.2 in [3], $E(|v|^{p'})^{1/p'} \in L^{p'q'/(q'-p')}(\mathcal{A})$. Thus $\sum_{n\in\mathbb{N}} (E(|v|^{p'}))^{q'/(q'-p')}(A_n)\mu(A_n) < \infty$.

Conversely, assume that the conditions (i) and (ii) hold. By the same preceding discussion, it suffices to establish the compactness of $M_{(E(|v|^{p'}))^{1/p'}} : L^{q'}(\mathcal{A}) \to L^{p'}(\Sigma)$. The condition (ii) implies that for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\sum_{n>N_\varepsilon} (E(|v|^{p'}))^{q'/(q'-p')}(A_n)\mu(A_n) < \varepsilon$. Put $v_\varepsilon := \sum_{n\leq N_\varepsilon} (E(|v|^{p'}))^{1/p'}(A_n)\chi_{A_n}$. Obviously, M_{v_ε} is a bounded and finite rank operator from $L^{q'}(\mathcal{A})$ to $L^{p'}(\Sigma)$. Moreover, by the Hölder inequality, for each $f \in L^{q'}(\mathcal{A})$ we have

$$\begin{aligned}
\| M_{(E(|v|^{p'}))^{1/p'}} f - M_{v_\varepsilon} f \|_{p'}^{p'} &= \int\limits_{\bigcup_{n>N_\varepsilon} A_n} E(|v|^{p'}) |f|^{p'} d\mu \\
&= \sum_{n>N_\varepsilon} E(|v|^{p'})(A_n)\mu(A_n) |f(A_n)|^{p'} \\
&= \sum_{n>N_\varepsilon} E(|v|^{p'})(A_n)\mu(A_n)^{\frac{q'-p'}{q'}} |f(A_n)|^{p'} \mu(A_n)^{\frac{p'}{q'}} \\
&\leq \Big(\sum_{n>N_\varepsilon} E(|v|^{p'})^{\frac{q'}{q'-p'}}(A_n)\mu(A_n) \Big)^{\frac{q'-p'}{q'}} \Big(\sum_{n>N_\varepsilon} |f(A_n)|^{q'} \mu(A_n) \Big)^{\frac{p'}{q'}} \\
&\leq \varepsilon^{\frac{q'-p'}{q'}} \|f\|_{q'}^{p'}.
\end{aligned}$$

It follows that $\| M_{(E(|v|^{p'}))^{1/p'}} - M_{v_\varepsilon} \| \leq \varepsilon^{(q'-p')/q'}$. So $M_{(E(|v|^{p'}))^{1/p'}}$ is the limit of finite rank operators and is therefore compact. This completes the proof of the theorem. □

Theorem 2.2. *Let $1 < p < q < \infty$ and let p', q' be conjugate component to p and q respectively. Then the weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ is compact if and only if*

(i) $E(|u|^{p'})^{1/p'}(E(|w|^q))^{1/q} = 0$ *a.e. on B;*
(ii) $\lim_{n\to\infty} \frac{E(|u|^{p'})(A_n)(E(|w|^q))^{p'/q}(A_n)}{\mu(A_n)^{(p'-q')/q'}} = 0$, *when the number of $\mathcal{A}$-atoms is not finite.*

Proof. Let $f \in L^p(\Sigma)$. Since $\|Tf\|_q = \|EM_v f\|_q$, $v := u(E(|w|^q))^{\frac{1}{q}}$, it follows that $T : L^p(\Sigma) \to L^q(\Sigma)$ is compact if and only if $M_v : L^{q'}(\mathcal{A}) \to L^{p'}(\Sigma)$ is compact. Let T be compact. Hence M_v is compact and so is bounded. Thus by Theorem 2.3 in [3], $E(|v|^{p'}) = 0$ a.e. on B.

Now, if the condition (ii) is not satisfied, we can find a constant $\delta > 0$ such that for each $n \in \mathbb{N}$ there exists $i_n > n$ so that $E(|v|^{p'})(A_{i_n})/\mu(A_{i_n})^{(p'-q')/q'} > \delta$. For each $n \in \mathbb{N}$, define $f_n = \chi_{A_{i_n}}/\mu(A_{i_n})^{1/q'}$. Obviously, $f_n \in L^{q'}(\mathcal{A})$ and $\|f_n\|_{q'} = 1$. Now, for each $m, n \in \mathbb{N}$ with $m \neq n$ we get that

$$\begin{aligned}
\| M_{\bar{v}} f_m - M_{\bar{v}} f_n \|_{p'}^{p'} &= \int\limits_X |v|^{p'} |f_m - f_n|^{p'} d\mu = \int\limits_X E(|v|^{p'}) |f_m - f_n|^{p'} d\mu \\
&\geq \int\limits_{A_{i_m}} E(|v|^{p'}) |f_m|^{p'} d\mu + \int\limits_{A_{i_n}} E(|v|^{p'}) |f_n|^{p'} d\mu \geq 2\delta.
\end{aligned}$$

But this is a contradiction.

Conversely, assume both (i) and (ii) hold. By the same argument in the proof of Theorem 2.1, it suffices to establish the compactness of $M_{(E(|v|^{p'}))^{1/p'}}$. From (ii), for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $E(|v|^{p'})(A_n)/\mu(A_n)^{(p'-q')/q'} < \varepsilon$, for all $n > N_\varepsilon$. Take $v_\varepsilon := \sum_{i \le N_\varepsilon} E(|v|^{p'})^{1/p'}(A_i)\chi_{A_i}$. Then M_{v_ε} is a bounded and finite rank operator from $L^{q'}(\mathcal{A})$ to $L^{p'}(\Sigma)$. Moreover, for each $f \in L^{q'}(\mathcal{A})$ with $\|f\|_{q'} = 1$, we have $|f(A_n)|^{q'}\mu(A_n) \le \|f\|_{q'}^{q'} = 1$. Since $p'/q' > 1$, then $(|f(A_n)|^{q'}\mu(A_n))^{p'/q'} \le |f(A_n)|^{q'}\mu(A_n)$. Thus we have

$$\begin{aligned}\|M_{(E(|v|^{p'}))^{1/p'}}f - M_{v_\varepsilon}f\|_{p'}^{p'} &= \int\limits_{\bigcup_{n>N_\varepsilon} A_n} E\big(|v|^{p'}\big)|f|^{p'}\,d\mu \\ &= \sum_{n>N_\varepsilon} E\big(|v|^{p'}\big)(A_n)\mu(A_n)\big|f(A_n)\big|^{p'} \\ &= \sum_{n>N_\varepsilon} \frac{E(|v|^{p'})(A_n)}{\mu(A_n)^{(p'-q')/q'}}\big(\big|f(A_n)\big|^{q'}\mu(A_n)\big)^{\frac{p'}{q'}} \le \varepsilon\|f\|_{q'}^{p'}.\end{aligned}$$

Therefore $\|M_{(E(|v|^{p'}))^{1/p'}} - M_{v_\varepsilon}\| \le \varepsilon^{1/q'}$, and so $M_{(E(|v|^{p'}))^{1/p'}}$ is compact. □

Remark 2.3. (a) According to the procedure used in the proof of Theorem 2.1, the weighted Lambert type operator T from $L^p(\Sigma)$ into $L^p(\Sigma)$ is compact if and only if $\nu = 0$ a.e. on B and $\lim_{n\to\infty}\nu(A_n) = 0$, where $\nu = (E|w|^p)^{1/p}(E|u|^{p'})^{1/p'}$. See Lemma 2.5 in [3] for another characterization.

(b) If $q = 1$, then by the procedure used in the proof of Theorem 2.2, by taking $f_n = \chi_{A_{i_n}}$ we get that the weighted Lambert type operator T from $L^p(\Sigma)$ into $L^1(\Sigma)$ is compact if and only if $\sum_{n\in\mathbb{N}} E(|u|^{p'})(A_n)(E(|w|))^{p'}(A_n)\mu(A_n) < \infty$ and $E(|u|^{p'})^{1/p'}E(|w|) = 0$ a.e. on B.

(c) Let $(X, \mathcal{A}, \mu)$ be a non-atomic measure space. Since for each $\alpha > 0$ and $\beta > 0$, $\sigma(w) = \sigma(|w|^\alpha) \subseteq \sigma(E(|w|^\alpha)) = \sigma((E(|w|^\alpha))^\beta)$, then by the previous results, the weighted Lambert type operator T from $L^q(\Sigma)$ into $L^p(\Sigma)$ with $1 < p < \infty$ and $1 \le q < \infty$ is compact if and only if it is a zero operator.

Theorem 2.4. *Let $1 < q < \infty$ and let $X = (\bigcup_{n\in\mathbb{N}} C_n) \cup C$, where $\{C_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint Σ-atoms and $C \in \Sigma$, being disjoint from each C_n, is non-atomic. Then:*

(a) *If the weighted Lambert type operator T from $L^1(\Sigma)$ into $L^q(\Sigma)$ is compact, then*
  (i) *$E(|u|^{q'})^{1/q'}(E(|w|^q))^{1/q} = 0$ a.e. on B;*
  (ii) *$\lim_{n\to\infty}\frac{E(|u|^{q'})(A_n)(E(|w|^q))^{q'/q}(A_n)}{\mu(A_n)} = 0$, when the number of $\mathcal{A}$-atoms is not finite.*

(b) *The weighted Lambert type operator T from $L^1(\Sigma)$ into $L^q(\Sigma)$ is compact, if the following conditions hold:*
  (i) *$u(E(|w|^q))^{1/q} = 0$ a.e. on C, and*
  (ii) *$\lim_{n\to\infty}\frac{E(|u|^{q'})(C_n)(E(|w|^q))^{q'/q}(C_n)}{\mu(C_n)} = 0$, when the number of Σ-atoms is not finite.*

Proof. (a) Let $f \in L^1(\Sigma)$. Then $\|Tf\|_q = \|EM_vf\|_q$, where $v := u(E(|w|^q))^{\frac{1}{q}}$. It follows that $T : L^1(\Sigma) \to L^q(\Sigma)$ is compact if and only if the operator $EM_v : L^1(\Sigma) \to L^q(\Sigma)$ is compact. Suppose that EM_v is compact. Then by Theorem 2.4(b) in [3], $E(|v|^{q'}) = 0$ a.e. on B. Now, we examine the condition (ii). Assume on the contrary, thus we can find a constant $\delta > 0$ such that for each $n \in \mathbb{N}$ there exists $i_n > n$ so that $E(|v|^{p'})(A_n)/\mu(A_n) > \delta$, for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define $f_n = \chi_{A_{i_n}}/\mu(A_{i_n})^{1/q'}$. Then $f_n \in L^{q'}(\mathcal{A})$ and $\|f_n\|_{q'} = 1$. Thus, for any $m, n \in \mathbb{N}$ with $m \ne n$, we obtain

$$\|M_{\bar v}f_m - M_{\bar v}f_n\|_\infty^{q'} \ge \frac{1}{\mu(A_{i_n})^2}\int\limits_{A_{i_n}} E\big(|v|^{p'}\big)\,d\mu = \frac{E(|v|^{p'})(A_{i_n})}{\mu(A_{i_n})} \ge \delta.$$

But this is a contradiction.

(b) Assume that the conditions (i) and (ii) hold. Then for any $\varepsilon > 0$, there exists some $N_\varepsilon \in \mathbb{N}$ such that $E(|v|^{q'})(C_n)/\mu(C_n) < \varepsilon$ for all $n > N_\varepsilon$. Take $v_\varepsilon := \sum_{n \le N_\varepsilon} v(C_n)\chi_{C_n}$. It follows that EM_{v_ε} is a bounded and finite rank operator. Moreover, for any $f \in L^1(\Sigma)$ with $\|f\|_1 = 1$ we get that $|f(A_n)|\mu(A_n) \le \|f\|_1 = 1$. Since $q > 1$, then $(|f(A_n)|\mu(A_n))^q \le |f(A_n)|\mu(A_n)$. Now, by the conditional type Hölder inequality we have

$$\|EM_v f - EM_{v_\varepsilon} f\|_q^q = \int_X \left|E(v\chi_{\bigcup_{n>N_\varepsilon} C_n} f)\right|^q d\mu$$

$$\le \int_X \left(E(|v|^{q'})\right)^{\frac{q}{q'}} E\left(\chi_{\bigcup_{n>N_\varepsilon} C_n} |f|^q\right) d\mu = \int_{\bigcup_{n>N_\varepsilon} C_n} \left(E(|v|^{q'})\right)^{\frac{q}{q'}} |f|^q d\mu$$

$$\sum_{n>N_\varepsilon} \frac{(E(|v|^{q'}))^{\frac{q}{q'}}(C_n)}{\mu(C_n)^{q-1}} \left(|f(C_n)|\mu(C_n)\right)^q \le \varepsilon^{\frac{q}{q'}} \sum_{n>N_\varepsilon} |f(C_n)|\mu(C_n) \le \varepsilon^{\frac{q}{q'}}.$$

This implies that EM_v is compact. □

Example 2.5. (a) Let $X = [0,\infty) \times [0,\infty)$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0,\infty) : A \text{ is a Lebesgue set in } [0,\infty)\}$. Put $w = 1/y$ and $u = 1$. Then, for each f in $L^p(\Sigma)$ with $p > 1$,

$$(Tf)(x,y) = \frac{1}{y}E(f)(x,y) = \frac{1}{y}\int_0^\infty f(x,t)dt.$$

Since $\int_0^\infty 1/t^p dt = E(|w|^p) \notin L^\infty(\mathcal{A})$, so the averaging operator T is not compact because it is not bounded (see [3, Theorem 2.1(a)]).

(b) Let $\nu = \{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with $\mu(\{n\}) = m_n$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a measurable transformation. Take $\mathcal{A} = \varphi^{-1}(2^{\mathbb{N}})$. Then for all non-negative sequence $f = \{f_n\}_{n=1}^\infty$ and $k \in \mathbb{N}$, we have

$$E(f)(k) = \frac{\sum_{j\in\varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j\in\varphi^{-1}(\varphi(k))} m_j}.$$

Now, set $\varphi = \chi_{\{1\}} + (n-1)\chi_{\{n\in\mathbb{N}:n\ge 2\}}$ and let $\mu(\{n\}) = 1$ for all $n \in \mathbb{N}$. Then $\mathcal{A}$ is generated by the atoms

$$\{1,2\},\ \{3\},\ \{4\},\ \ldots.$$

Hence for $w = \{w_n\} \in l^0(2^{\mathbb{N}})$, $u = \{u_n\} \in \mathcal{D}(E)$ and $f = \{f_n\} \in L^p(\Sigma)$ we have

$$Tf = wE(uf) = \left(\frac{w_1(u_1f_1 + u_2f_2)}{2}, \frac{w_2(u_1f_1 + u_2f_2)}{2}, w_3u_3f_3, w_4u_4f_4, \ldots\right).$$

In particular, by Theorem 2.1, $T : L^3(\Sigma) \to L^2(\Sigma)$ is compact if and only if $\sum_{n=3}^\infty (w_nu_n)^6 < \infty$. Hence, by Theorem 2.2, $T : L^2(\Sigma) \to L^3(\Sigma)$ is compact if and only if $w_nu_n \to 0$ as $n \to \infty$.

(c) Let $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and let μ be the counting measure. Put

$$A = \left\{\{2\}, \{4,6\}, \{8,10,12\}, \{14,16,18,20\}, \cdots\right\} \cup \left\{\{1\}, \{3\}, \{5\}, \cdots\right\}.$$

If we let $A_1 = \{2\}$, $A_2 = \{4,6\}$, $A_3 = \{8,10,12\}$, $\cdots$, then we see that $\mu(A_n) = n$ and for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $A_n = \{2k_n, 2(k_n+1), \cdots, 2(k_n+n-1)\}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition A of $\mathbb{N}$. Note that, $\mathcal{A}$ is a sub-σ-finite algebra of Σ and each of element of $\mathcal{A}$ is an $\mathcal{A}$-atom. It is known that the conditional expectation of any $f \in \mathcal{D}(E)$ relative to $\mathcal{A}$ is

$$E(f) = \sum_{n=1}^{\infty}\left(\frac{1}{\mu(A_n)}\int\limits_{A_n} f\,d\mu\right)\chi_{A_n} + \sum_{n=1}^{\infty} f(2n-1)\chi_{\{2n-1\}}.$$

Put $u(n) = n$ and $w(n) = \frac{1}{n^3}$, for all $n \in \mathbb{N}$. For each even number $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that $m \in A_{n_m}$. Thus for all $1 \le p, q < \infty$ we get that

$$E\big(|w|^q\big)(m) = \frac{1}{(2k_{n_m})^{3q}} + \dots + \frac{1}{(2k_{n_m} + 2n_m - 2)^{3q}}$$

and $E(|u|^p)(m) = 2^p k_{n_m}^p + 2^p(k_{n_m} + 1)^p + \dots + 2^p(k_{n_m} + n_m - 1)^p$. Since $n_m \le k_{n_m}$, we have

$$\big(E\big(|w|^q\big)\big)^{\frac{1}{q}}(m)\big(E\big(|u|^p\big)\big)^{\frac{1}{p}}(m) \le \frac{4k_{n_m}}{2^3 k_{n_m}^3}.$$

Also, for all $n \in \mathbb{N}$

$$\big(E\big(|w|^q\big)\big)^{\frac{1}{q}}(2n-1)\big(E\big(|u|^p\big)\big)^{\frac{1}{p}}(2n-1) \le \frac{1}{(2n-1)^2}.$$

It is easy to see that

$$\lim_{n\to\infty} \frac{E(|u|^{p'})(A_n)(E(|w|^q))^{p'/q}(A_n)}{n^{(p'-q')/q'}} = 0;$$
$$\lim_{n\to\infty} E\big(|u|^{p'}\big)(2n-1)\big(E\big(|w|^q\big)\big)^{p'/q}(2n-1) = 0.$$

Thus by Theorem 2.2, the weighted Lambert type operator T from $L^p(\Sigma)$ into $L^q(\Sigma)$ is compact, when $1 < p < q < \infty$. Also, since we have

$$\lim_{n\to\infty} E\big(|u|^{q'}\big)(n)\big(E\big(|w|^q\big)\big)^{q'/q}(n) = 0,$$

by Theorem 2.4, T is a compact operator from $L^1(\Sigma)$ into $L^q(\Sigma)$, for $1 < q < \infty$.

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