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A substitution vector-valued integral operator

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In this paper we introduce a substitution vector-valued integral operator T_u^φ on $L^2(X)$ associated with the pair (u, φ) and investigate some fundamental properties of T_u^φ by the language of conditional expectation operators.

1. Introduction and preliminaries

Let (X, Σ, μ) be a σ-finite measure space and $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Here the non-singularity of φ guarantees that the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is well defined as a mapping on $L^0(\Sigma)$ where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Let $h_0 = d\mu \circ \varphi^{-1}/d\mu$ be the Radon–Nikodym derivative. Recall that C_φ is bounded on $L^2(\Sigma)$ if and only if $h_0 \in L^\infty(\Sigma)$ (see [7]). We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A h_0 f d\mu, \quad A \in \Sigma, \ f \in L^1(\Sigma).$$

The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$, by Radon–Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

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$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \qquad \forall A \in \mathcal{A}.$$

We recall that $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ is an orthogonal projection. Throughout this paper, we assume that $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E^{\varphi^{-1}(\Sigma)} = E$. It is well known that for each non-negative Σ-measurable function f or for each $f \in L^2(\Sigma)$, there exists a Σ-measurable function g such that $E(f) = g \circ \varphi$. We can assume that $\sigma(g) \subseteq \sigma(h_0)$ and there exists only one g with this property. We then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ (see [1]). Let $u \in L^0(\Sigma)$. Thus u is said to be conditionable with respect to E if $u \in \mathcal{D}(E) \subseteq L^0(\Sigma)$, where $\mathcal{D}(E)$ denotes the domain of E. For more details on the properties of $E^{\mathcal{A}}$ see [4,6].

An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$, such that for each $B \in \Sigma$, if $B \subset A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure with no atoms is called non-atomic. We can easily check the following well-known facts (see[8]):

(a) Every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as

$$X = (\cup_{n\in\mathbb{N}} A_n) \cup B, \tag{1.1}$$

where $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic.

(b) Let E be a non-atomic set with $\mu(E) > 0$. Then there exists a sequence of positive disjoint Σ-measurable subsets of E, $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu(E_n) > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

For a given complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$, let $L^2(X, \mathcal{H})$ be the class of all measurable mappings $f : X \to \mathcal{H}$ such that $\|f\|_2^2 := \int_X \|f(x)\|^2 d\mu < \infty$. Let $f, g \in L^2(X, \mathcal{H})$. By using the polar identity, the mapping $x \mapsto \langle f(x), g(x)\rangle$ from X into $\mathbb{C}$ is measurable. It follows that $L^2(X, \mathcal{H})$ is a Hilbert space with inner product

$$(f, g) = \int_X \langle f(x), g(x)\rangle d\mu, \quad f, g \in L^2(X, \mathcal{H}).$$

We shall write $L^2(X)$ for $L^2(X, \mathcal{H})$ when $\mathcal{H} = \mathbb{C}$. Let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $h \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), h\rangle$ from X into $\mathbb{C}$ is measurable. We will denote this map by $\langle u, h\rangle$.

The aim of this article is to carry some of the results obtained for the weighted composition operators in [2,3,7] to a substitution vector-valued integral operator. In this paper, first we consider some basic properties of substitution vector-valued integral operators and then we give some necessary and sufficient conditions for boundedness, compactness and semi-Fredholmness of these type operators.

2. The main results

Definition 2.1. Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that u is a weakly bounded function if for some $B \geq A > 0$,

$$\sqrt{A}\|h\| \leq \|\langle u, h\rangle\|_2 \leq \sqrt{B}\|h\|, \qquad \forall h \in \mathcal{H}. \tag{2.1}$$

In the same way, u is said to be semi-weakly bounded function if u only satisfies the right hand side of the above inequalities.

Note that if $u \in L^2(X,\mathcal{H})$, then for all $h \in \mathcal{H}$, $\int_X |\langle u,h\rangle|^2 d\mu \le B\|h\|^2$, where $B = \|u\|_2^2$. So u is a semi-weakly bounded function. In addition, if u is a semi-weakly bounded function with an upper bound $B > 0$ and $\dim \mathcal{H} = n$, with the natural basis $\{e_i\}_{i\in\mathbb{N}}$, then

$$\int_X \|u(x)\|^2 d\mu = \sum_{i=1}^n \int_X |\langle u(x), e_i\rangle|^2 d\mu(x) \le nB < \infty,$$

and so $u \in L^2(X,\mathcal{H})$. Now, let $u \in L^2(X,\mathcal{H})$ be a weakly bounded function and let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis for $\mathcal{H}$. Put $h = e_\alpha$. By Parseval's identity and (2.1) we obtain

$$A\sum_{\alpha\in I} 1 \le \sum_{\alpha\in I}\int_X |\langle u(x), e_\alpha\rangle|^2 d\mu(x) = \int_X \|u(x)\|^2 d\mu(x) < \infty,$$

for some $A > 0$. Therefore, I must be a finite set and hence $\dim \mathcal{H} < \infty$. These observations establish the following proposition.

Proposition 2.2. *The following statements hold.*

(i) *If $u \in L^2(X,\mathcal{H})$, then u is a semi-weakly bounded function.*
(ii) *If u is a semi-weakly bounded function and* $\dim \mathcal{H} < \infty$, *then $u \in L^2(X,\mathcal{H})$.*
(iii) *If $u \in L^2(X,\mathcal{H})$ is a weakly bounded function, then* $\dim \mathcal{H} < \infty$.

Definition 2.3. Let $\varphi : X \to X$ be a non-singular measurable transformation and let $u : X \to \mathcal{H}$ be a weakly measurable function. Then the pair (u,φ) induces a substitution vector-valued integral operator $T_u^\varphi : L^2(X) \to \mathcal{H}$ defined by

$$\langle T_u^\varphi f, h\rangle = \int_X \langle u,h\rangle f\circ\varphi d\mu, \quad h \in \mathcal{H},\ f \in L^2(X).$$

It is easy to see that T_u^φ is well defined and linear. Moreover for each $f \in L^2(X)$,

$$\sup_{h\in\mathcal{H}_1} |\langle T_u^\varphi f, h\rangle| \le \sup_{h\in\mathcal{H}_1} \|T_u^\varphi f\|\ \|h\| = \|T_u^\varphi f\| = |\langle T_u^\varphi f, \frac{T_u^\varphi f}{\|T_u^\varphi f\|}\rangle| \le \sup_{h\in\mathcal{H}_1} |\langle T_u^\varphi f, h\rangle|,$$

where $\mathcal{H}_1$ is the closed unit ball of $\mathcal{H}$. Hence

$$\|T_u^\varphi\| = \sup_{\|f\|\le 1} \|T_u^\varphi f\| = \sup_{\|f\|\le 1}\sup_{h\in\mathcal{H}_1} |\langle T_u^\varphi f, h\rangle|.$$

Theorem 2.4. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. Then:*

(i) *If u is a semi-weakly bounded function and $h_0 \in L^\infty(\Sigma)$, then T_u^φ is bounded.*
(ii) *If for each $h \in \mathcal{H}$, the functions $\langle u,h\rangle$ are conditionable and T_u^φ is bounded, then for all $h \in \mathcal{H}_1$, $h_0E(\langle u,h\rangle)\circ\varphi^{-1} \in L^2(X)$ and $(T_u^\varphi)^* = h_0E(\langle\cdot,u\rangle)\circ\varphi^{-1}$.*

Proof. (i) Let $f \in L^2(X)$. By Hölder's inequality and change of variable formula we have

$$\|T_u^\varphi f\| = \sup_{h\in\mathcal{H}_1} |\int_X \langle u,h\rangle(f\circ\varphi)d\mu|$$

$$\begin{aligned}
&\leq (\sup_{h\in\mathcal{H}_1}\int_X |\langle u,h\rangle|^2 d\mu)^{\frac{1}{2}} (\int_X |f\circ\varphi|^2 d\mu)^{\frac{1}{2}}\\
&\leq \sup_{h\in\mathcal{H}_1}(B\|h\|)^{\frac{1}{2}} \int_X h_0 |f|^2 d\mu)^{\frac{1}{2}}\\
&\leq \sqrt{B\|h_0\|_\infty}\ \|f\|_2.
\end{aligned}$$

This shows that T_u^φ is bounded.

(ii) Since T_u^φ is a bounded operator, so there exists $M>0$ such that for each $f\in L^2(X)$, $\|T_u^\varphi f\|\leq M\|f\|_2$. For an arbitrary and fixed $h\in\mathcal{H}_1$, we define a linear functional Λ_h on $L^2(X)$ by $\Lambda_h(f)=\int_X h_0 E(\langle u,h\rangle)\circ\varphi^{-1} f d\mu$. Since

$$\begin{aligned}
|\Lambda_h(f)| &\leq \sup_{h\in\mathcal{H}_1} |\int_X h_0 E(\langle u,h\rangle)\circ\varphi^{-1} f d\mu|\\
&= \|T_u^\varphi f\|\leq M\|f\|_2,
\end{aligned}$$

hence Λ_h is a bounded linear functional on $L^2(X)$. By the Riesz representation theorem, there exists a unique function $g\in L^2(X)$ such that for each $f\in L^2(X)$, $\Lambda_h(f)=\int_X gf d\mu$. This implies that $h_0 E(\langle u,h\rangle)\circ\varphi^{-1}=g$, and so $h_0 E(\langle u,h\rangle)\circ\varphi^{-1}\in L^2(X)$. Now, let $f\in L^2(X)$ and $h\in\mathcal{H}$. Then we have

$$\begin{aligned}
(f,(T_u^\varphi)^*(h)) = \langle T_u^\varphi f,h\rangle &= \int_X \langle u,h\rangle (f\circ\varphi) d\mu\\
&= \int_X h_0 E(\langle u,h\rangle)\circ\varphi^{-1} f d\mu\\
&= (f, h_0 E(\overline{\langle u,h\rangle})\circ\varphi^{-1}).
\end{aligned}$$

Hence for all $h\in\mathcal{H}$, $(T_u^\varphi)^*(h)=h_0 E(\langle h,u\rangle)\circ\varphi^{-1}$. $\square$

Theorem 2.5. *Let $u: X\to\mathcal{H}$ be a semi-weakly bounded function with an upper bound B and let $h_0\in L^\infty(\Sigma)$. Put $S_u^\varphi = T_u^\varphi(T_u^\varphi)^*$. Then the following statements are equivalent.*

(i) *The operator $S_u^\varphi:\mathcal{H}\to\mathcal{H}$ is invertible.*
(ii) *The operator $T_u^\varphi: L^2(X)\to\mathcal{H}$ is surjective.*

Proof. (i) $\Rightarrow$ (ii) Since S_u^φ is a self-adjoint operator on $\mathcal{H}$, then by [5, Theorem 9.2.1] we have $\inf_{h\in\mathcal{H}_1}\langle S_u^\varphi h,h\rangle = \inf_{h\in\mathcal{H}_1}\|(T_u^\varphi)^*(h)\|^2\in spec(S_u^\varphi)$, the spectrum of S_u^φ. By hypothesis $0\notin spec(S_u^\varphi)$. Hence, $\inf_{h\in\mathcal{H}_1}\|(T_u^\varphi)^*(h)\|>0$. It follows that $\inf_{h\in\mathcal{H}_1}\|(T_u^\varphi)^*(h)\|\|h\|\leq\|(T_u^\varphi)^*(h)\|$, and so T_u^φ is surjective.

(ii) $\Rightarrow$ (i) Let T_u^φ be surjective. Then there exists $M>0$ such that for each $h\in\mathcal{H}$ $\|(T_u^\varphi)^*(h)\|^2\geq M\|h\|^2$. So $\langle S_u^\varphi(h),h\rangle = \langle T_u^\varphi(T_u^\varphi)^*(h),h\rangle = \|(T_u^\varphi)^*(h)\|^2\geq M\|h\|^2$. Moreover, for each $h\in\mathcal{H}$ we have

$$\begin{aligned}
\langle S_u^\varphi(h),h\rangle &= \langle T_u^\varphi(T_u^\varphi)^*(h),h\rangle\\
&= \int_X \langle u,h\rangle((T_u^\varphi)^*(h))\circ\varphi d\mu\\
&= \int_X \langle u,h\rangle h_0\circ\varphi E(\overline{\langle u,h\rangle}) d\mu
\end{aligned}$$

$$= \int_X E\left(\langle u,h\rangle h_0\circ\varphi E(\overline{\langle u,h\rangle})\right)d\mu$$

$$= \int_X h_0\circ\varphi E(\langle u,h\rangle)\overline{E(\langle u,h\rangle)}d\mu$$

$$\leq \int_X h_0\circ\varphi E(|\langle u,h\rangle|^2)d\mu$$

$$= \int_X h_0\circ\varphi|\langle u,h\rangle|^2 d\mu \leq \|h_0\|_\infty B\|h\|^2.$$

Therefore $M \leq S_u^\varphi \leq \|h_0\|_\infty B$, and so S_u^φ is invertible. □

Definition 2.6. Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that the pair $(u, \mathcal{H})$ has absolute property, if for each $f \in L^2(X)$, there exists $h_f \in \mathcal{H}_1$ such that $\sup_{h\in\mathcal{H}_1}\int_X |\langle u,h\rangle||f\circ\varphi|d\mu = \int_X |\langle u,h_f\rangle||f\circ\varphi|d\mu$, and $\langle u,h_f\rangle = e^{i(-\arg f\circ\varphi+\theta_f)}|\langle u,h_f\rangle|$, for some constant $\theta_f \in \mathbb{C}$.

Corollary 2.7. *Assume that the pair $(u, \mathcal{H})$ has the absolute property. Then*

$$\sup_{h\in\mathcal{H}_1}|\int_X \langle u,h\rangle f\circ\varphi d\mu| = \sup_{h\in\mathcal{H}_1}\int_X |\langle u,h\rangle||f\circ\varphi|d\mu.$$

Proof. Let f be an arbitrary and fixed element of $L^2(X)$. Then

$$\sup_{h\in\mathcal{H}_1}\int_X |\langle u,h\rangle||f\circ\varphi|d\mu = \int_X |\langle u,h_f\rangle||f\circ\varphi|d\mu$$

$$= |\int_X e^{i(\arg f\circ\varphi-\theta_f)}\langle u,h_f\rangle|f\circ\varphi|d\mu|$$

$$= |\int_X e^{-i\theta_f}\langle u,h_f\rangle f\circ\varphi d\mu|$$

$$\leq \sup_{h\in\mathcal{H}_1}|\int_X \langle u,h\rangle f\circ\varphi d\mu|.$$

The inverse of the inequality is clear. □

From now on we assume that the pair $(u, \mathcal{H})$ has the absolute property. In the following theorem we give some necessary and sufficient conditions for the boundedness of T_u^φ.

Theorem 2.8. T_u^φ *is bounded if and only if* $\sup_{h\in\mathcal{H}_1}\|h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}\|_2 < \infty$.

Proof. Let $M := \sup_{h\in\mathcal{H}_1}\|h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}\|_2 < \infty$ and $f \in L^2(X)$. Then By Hölder's inequality and change of variable formula we have

$$\|T_u^\varphi f\| = \sup_{h\in\mathcal{H}_1}\int_X h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}|f|d\mu$$

$$\le \sup_{h\in\mathcal{H}_1}\left(\int_X (h_0E(|\langle u,h\rangle|)\circ\varphi^{-1})^2 d\mu\right)^{\frac{1}{2}}\left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}}$$
$$= \sup_{h\in\mathcal{H}_1}\|h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}\|_2\|f\|_2.$$

Consequently, for each $f\in L^2(X)$, $\|T_u^\varphi f\|\le M\|f\|_2$. Conversely, assume that T_u^φ is bounded. Take an arbitrary and fixed $h\in\mathcal{H}_1$ and define a linear functional Λ_h on $L^2(X)$ by

$$\Lambda_h(f)=\int_X h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} f d\mu, \qquad f\in L^2(X).$$

By a similar argument as in the proof of Theorem 2.4(ii), it can be proved that $h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}\in L^2(X)$, for each $h\in\mathcal{H}_1$. □

Theorem 2.9. *Let (X,Σ,μ) be partitioned as (1.1) and let T_u^φ be a compact operator on $L^2(X)$. Then:*

(i) $M:=\sup_{h\in\mathcal{H}_1}(h_0E(|\langle u,h\rangle|)\circ\varphi^{-1})^2=0$ *on B;*
(ii) $N:=\sup_{h\in\mathcal{H}_1}\sum_{i\in\mathbb{N}}(h_0E(|\langle u,h\rangle|)\circ\varphi^{-1})^2(A_i)\mu(A_i)<\infty$.

Proof. Let T_u^φ be a compact operator. First, we prove (i) by contradiction. Assume $\mu(\{x\in B: M(x)>0\})>0$. Then there exist $\delta>0$ and $h_1\in\mathcal{H}_1$ such that the set $C:=\{x\in B: h_0(x)E(|\langle u,h_1\rangle|)\circ\varphi^{-1})^2(x)\ge\delta\}$ has positive measure. We may also assume $\mu(C)<\infty$. Since $C\subseteq B$, we can find $E_n\in\Sigma$ of positive measure satisfying $E_{n+1}\subseteq E_n\subseteq C$ with $\mu(E_{n+1})=\frac{\mu(E_n)}{2}$, for all $n\in\mathbb{N}$. Put $f_n=\frac{h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}}{\|h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}\|_2}\chi_{E_n}$. Note that $\{f_n\}_n\subseteq L^2(X)$ is a bounded sequence. Then for each $n,m\in\mathbb{N}$ with $n>m$, we have

$$\begin{aligned}\|T_u^\varphi f_m - T_u^\varphi f_n\| &= \sup_{h\in\mathcal{H}_1}\int_X |\langle u,h\rangle||f_m-f_n|\circ\varphi d\mu\\ &\ge \int_X |\langle u,h_1\rangle||f_m-f_n|\circ\varphi d\mu\\ &= \int_X h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}|f_m-f_n|d\mu\\ &= \int_X \frac{(h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1})^2}{\|h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}\|_2}(\chi_{E_m}-\chi_{E_n})d\mu\\ &\ge \int_{E_m\setminus E_n}\frac{(h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1})^2}{\|h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}\|_2}d\mu\\ &> \frac{\delta}{\|h_0E(|\langle u,h_1\rangle|)\circ\varphi^{-1}\|_2}(\mu(E_m)-\mu(E_n)).\end{aligned}$$

Since $\mu(E_n)<\frac{\mu(E_m)}{2}$, then $\|T_u^\varphi f_m-T_u^\varphi f_n\|\ge k$ for some $k>0$. This implies that the sequence $\{T_u^\varphi f_n\}_n$ does not contain a convergent subsequence, but this shows that T_u^φ is not compact. Next, we prove (ii). Since T_u^φ is compact, so T_u^φ is bounded and by (i), $M=0$ on B. Now, by Theorem 2.8, $N^2<\infty$ and this complete the proof. □

Theorem 2.10. *Let* $\{i \in \mathbb{N} : \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(A_i) > 0\}$ *be a finite subset of* $\mathbb{N}$ *and let* $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} = 0$ *on* B*. Then* T_u^φ *is compact.*

Proof. We show that T_u^φ is a finite rank operator. To show that $T_u^\varphi(L^2(X))$ is finite-dimensional, we only need to prove that the set $U := \{h \in T_u^\varphi(L^2(X)) : \|h\| \le 1\}$ is compact in $\mathcal{H}$. If the set $\{i \in \mathbb{N} : \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(A_i) > 0\}$ is empty, then T_u^φ is the zero operator. Otherwise we may assume that there exists $k \in \mathbb{N}$ such that $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} > 0$ for $1 \le i \le k$ and $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} = 0$ for any $i > k$. Let $\{T_u^\varphi f_n\}_n$ be an arbitrary sequence in U. Then we have

$$\begin{aligned}
\|T_u^\varphi f_n\| &= \sup_{h\in\mathcal{H}_1} \int\limits_X h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}|f_n|d\mu \\
&= \sup_{h\in\mathcal{H}_1} \left(\int\limits_{\cup_{i\in\mathbb{N}}A_i} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}|f_n|d\mu + \int\limits_B h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}|f_n|d\mu \right) \\
&= \sum_{i=1}^k \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(A_i)\mu(A_i)|f_n(A_i)|.
\end{aligned}$$

Put $\alpha_i = \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(A_i)\mu(A_i)$. Then:

$$\|T_u^\varphi f_n\| = \sum_{i=1}^k \alpha_i|f_n(A_i)|. \tag{2.2}$$

Since $\|T_u^\varphi f_n\| \le 1$ for all $n \in \mathbb{N}$, then $|f_n(A_i)| \le \frac{1}{\alpha_i}$ for each $1 \le i \le k$ and each $n \in \mathbb{N}$. Hence, by Bolzano–Weierstrass theorem, there exists a subsequence of natural numbers $\{n_j\}_{j\in\mathbb{N}}$ such that for each fixed $1 \le i \le k$, the sequence $\{f_{n_j}(A_i)\}_{j\in\mathbb{N}}$ converges. Assume that $\lim_{j\to\infty} f_{n_j}(A_i) = \xi_i$ and let $f := \sum_{i=1}^k \xi_i\chi_{A_i}$. Now, from (2.2) we have $\|T_u^\varphi f\| = \sum_{i=1}^k \alpha_i|\xi_i| \le 1$. Hence, $T_u^\varphi f \in U$ and so we get that

$$\|T_u^\varphi f_{n_j} - T_u^\varphi f\| = \sum_{i=1}^k \alpha_i|f_{n_j}(A_i) - f(A_i)| = \sum_{i=1}^k \alpha_i|f_{n_j}(A_i) - \xi_i| \to 0,$$

as $j \to \infty$. It follows that U is compact in $\mathcal{H}$. This completes the proof of the theorem. □

Corollary 2.11. *Let* (X, Σ, μ) *be a non-atomic* σ*-finite measure space. Then no bounded substitution vector-valued integral operator* T_u^φ *on* $L^2(X)$ *is compact unless it is a zero operator.*

Proof. Let T_u^φ be a compact operator on $L^2(X)$. Then by Theorem 2.9, for each $h \in \mathcal{H}_1$, $h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} = 0$ on X. For each $f \in L^2(X)$, we have

$$\|T_u^\varphi f\| = \sup_{h\in\mathcal{H}_1} \int\limits_X h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}|f|d\mu.$$

Therefore $T_u^\varphi f = 0$, for each $f \in L^2(X)$, and thus T_u^φ is the zero operator. □

Theorem 2.12. *Let* T_u^φ *be a bounded operator from* $L^2(X)$ *into* $\mathcal{H}$*. Then the following statements are equivalent.*

(i) T_u^φ *is injective.*
(ii) *For each* $h \in \mathcal{H}_1$, $h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} > 0$ *on* X*.*

(iii) *For $A \in \Sigma$, if $\mu\left(\varphi^{-1}(A) \cap \sigma(\langle u, h\rangle)\right) = 0$ for each $h \in \mathcal{H}_1$, then $\mu(A) = 0$.*
(iv) *There exists $h_1 \in \mathcal{H}_1$, such that $h_0 E(|\langle u, h_1\rangle|) \circ \varphi^{-1} > 0$ on X.*

Proof. Direction (ii) ⇒ (iv) is trivial. For (iv) ⇒ (i), let f be a non-zero element in $\ker T_u^\varphi$. Then we have

$$\begin{aligned} 0 = \|T_u^\varphi f\| &= \sup_{h \in \mathcal{H}_1} \int_X |\langle u, h\rangle||f \circ \varphi| d\mu \\ &\geq \int_X |\langle u, h_1\rangle||f \circ \varphi| d\mu \\ &= \int_X h_0 E(|\langle u, h_1\rangle|) \circ \varphi^{-1} |f| d\mu \\ &= \int_{\sigma(f)} h_0 E(|\langle u, h_1\rangle|) \circ \varphi^{-1} |f| d\mu. \end{aligned}$$

This means that $f = 0$ on X. For (i) ⇒ (iii), let $\mu\left(\varphi^{-1}(A) \cap \sigma(\langle u, h\rangle)\right) = 0$ for each $h \in \mathcal{H}_1$ and $A \in \Sigma$ with $\mu(A) < \infty$. Since $\chi_A \in L^2(X)$, we have

$$\|T_u^\varphi \chi_A\| = \sup_{h \in \mathcal{H}_1} |\int_{\varphi^{-1}(A)} \langle u, h\rangle d\mu| = \sup_{h \in \mathcal{H}_1} |\int_{\varphi^{-1}(A)\cap\sigma(\langle u,h\rangle)} \langle u, h\rangle d\mu| = 0.$$

Thus $T_u^\varphi \chi_A = 0$. Now, the injectivity of T_u^φ implies that $\chi_A = 0$ and so $\mu(A) = 0$.

For (iii) ⇒ (ii), put $J = \sigma(h_0 E(|\langle u, h\rangle|) \circ \varphi^{-1})$ for an arbitrary and fixed $h \in \mathcal{H}_1$. Then

$$\begin{aligned} \int_{\varphi^{-1}(X\setminus J)\cap\sigma(\langle u,h\rangle)} |\langle u, h\rangle| d\mu &\leq \int_{\varphi^{-1}(X\setminus J)} |\langle u, h\rangle| d\mu \\ &= \int_{X\setminus J} h_0 E(|\langle u, h\rangle|) \circ \varphi^{-1} d\mu \\ &= 0. \end{aligned}$$

Therefore we deduce that for each $h \in \mathcal{H}_1$, $\mu\left(\varphi^{-1}(X \setminus J) \cap \sigma(\langle u, h\rangle)\right) = 0$, and so by (iii), $\mu(X \setminus J) = 0$. This implies that for each $h \in \mathcal{H}_1$, $h_0 E(|\langle u, h_1\rangle|) \circ \varphi^{-1} > 0$ on X. □

Lemma 2.13. *Let (X, Σ, μ) be a non-atomic σ-finite measure space. Then the nullity of T_u^φ is either zero or infinite.*

Proof. If T_u^φ is injective, then $\dim \ker T_u^\varphi = 0$. Since otherwise, there must exist a non-zero function $f \in L^2(X)$ such that $T_u^\varphi f = 0$. Since $\sigma(|f|)$ has positive measure, by hypothesis we can find $\{S_n\}_{n=1}^\infty$ of pairwise disjoint Σ-measurable subsets in $\sigma(|f|)$ with $0 < \mu(S_n) < \infty$ and $\sigma(|f|) = \cup_n S_n$. Set $f_n = f\chi_{S_n}$. Thus we have

$$\begin{aligned} \|T_u^\varphi f_n\| &= \sup_{h \in \mathcal{H}_1} \int_X |\langle u, h\rangle||f \circ \varphi|(\chi_{S_n} \circ \varphi) d\mu \\ &= \sup_{h \in \mathcal{H}_1} \int_{\varphi^{-1}(Sn)} |\langle u, h\rangle||f \circ \varphi| d\mu \end{aligned}$$

$$\leq \sup_{h\in\mathcal{H}_1} \int_X |\langle u,h\rangle||f\circ\varphi|d\mu = \|T_u^\varphi f\| = 0.$$

Thus for each n, $f_n \in \ker T_u^\varphi$ and hence $\dim\ker T_u^\varphi = \infty$. □

Corollary 2.14. *Let $T_u^\varphi : L^2(X) \to \mathcal{H}$ be a bounded operator. Then the following hold.*

(i) *Let (X,Σ,μ) be partitioned as (1.1) and let the set*

$$\{\mathrm{i}\in\mathbb{N} : \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(A_i) > 0\}$$

be finite and $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} = 0$ on B. If $h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} > 0$ on X for some $h\in\mathcal{H}_1$, then T_u^φ is a semi-Fredholm operator (i.e., $\mathcal{R}(T_u^\varphi)$, is closed and $\dim\mathcal{N}(T_u^\varphi) < \infty$).

(ii) *Let (X,Σ,μ) be a non-atomic σ-finite measure space and let T_u^φ be a semi-Fredholm operator. Then there exists a constant $\lambda > 0$ such that*

$$\sup_{h\in\mathcal{H}_1} \int_{\varphi^{-1}(C)} |\langle u,h\rangle|d\mu \geq \lambda\mu(C)^{\frac{1}{2}},$$

for each $C\in\Sigma$ with $\mu(C) < \infty$.

Proof. (i) This follows immediately from Theorem 2.10 and Theorem 2.12.

(ii) Since T_u^φ is a semi-Fredholm operator, then $\mathcal{R}(T_u^\varphi)$ is closed and by Lemma 2.13, T_u^φ is injective. It follows that there exists a constant $\lambda > 0$ such that for each $f\in L^2(\Sigma)$, $\|T_u^\varphi f\| \geq \lambda\|f\|_2$. For $C\in\Sigma$ with $\mu(C) < \infty$, take $f = \chi_C$. Then

$$\sup_{h\in\mathcal{H}_1} \int_{\varphi^{-1}(C)} |\langle u,h\rangle|d\mu = \|T_u^\varphi \chi_C\| \geq \lambda\|\chi_C\|_2 = \lambda\mu(C)^{\frac{1}{2}}. \qquad \square$$

Example 2.15. Let $X = [0,1]$, Σ be the Lebesgue subsets of X and let μ be the Lebesgue measure on X. For $a\in[\frac{1}{2},1]$, define $\varphi : X\to X$ by $\varphi(x) = ax$. Also, let $u : X\to\ell^2(\mathbb{N})$ be defined by $u(x) = (\frac{1}{x+1},\ldots)$. It is easy to check that φ is a non-singular measurable transformation and $h_0 = \frac{1}{a}$. In addition, for each $h = (h_1,h_2,\ldots)\in\ell^2(\mathbb{N})$, we have

$$\int_X |\langle h,u(x)\rangle|^2 d\mu = \int_0^1 \left(\frac{h_1}{x+1}\right)^2 dx \leq \|h\|^2 \int_0^1 \left(\frac{1}{x+1}\right)^2 dx = \frac{1}{2}\|h\|^2.$$

Hence u is a semi-weakly bounded function and so by Theorem 2.4 (i), $T_u^\varphi : L^2(X)\to\ell^2(\mathbb{N})$ is a bounded operator.

Example 2.16. Let (X,Σ,μ) be the Lebesgue space, where $X = [0,1]$ and $d\mu(x) = dx$. Let $\varphi : X\to X$ be a non-singular measurable transformation with $h_0\in L^\infty(\Sigma)$ and also let $u : X\to L^2(X)$ be a semi-weakly bounded function. For $A\subseteq X$, put $h = \chi_A$. Then for every $f\in L^2(X)$ we obtain

$$\langle T_u^\varphi f,\chi_A\rangle = \int_0^1 \langle u,\chi_A\rangle f\circ\varphi d\mu = \int_0^1\int_A u(x)dyf\circ\varphi dx.$$

Since $u(x)(f \circ \varphi) \in L^1(X \times X)$, then by Fubini's theorem we get that

$$\int_A T_u^\varphi f dx = \int_0^1 \int_A u(x) f \circ \varphi dy dx = \int_A \int_0^1 u(x) f \circ \varphi dx dy.$$

Consequently, we obtain the expressive formula for the substitution vector-valued integral operator, $T_u^\varphi f = \int_0^1 u(x) f \circ \varphi dx$, for each $f \in L^2(X)$.

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