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Some Weak p-Hyponormal Classes of Weighted Composition Operators

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Abstract. In this note, we discuss measure theoretic weighted composition operators on $L^2(\Sigma)$ in some operator classes that are weaker than p-hyponormal.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a sigma finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is sigma finite. We use the notation $L^2(\varphi^{-1}(\Sigma))$ for $L^2(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$ and henceforth we write μ in place of $\mu_{|\varphi^{-1}(\Sigma)}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote that the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^2(\Sigma)$ induced by φ and u is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on $L^2(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively.

Let $\mathcal{A} = \varphi^{-1}(\Sigma)$. If $\varphi^{-1}(\Sigma) \subseteq \Sigma$, there exists an operator $E := E^{\varphi^{-1}(\Sigma)} : L^p(\Sigma) \to L^p(\mathcal{A})$ which is called conditional expectation operator. $\mathcal{D}(E)$, the domain of E, contains the set of all non-negative measurable functions and each $f \in L^p(\Sigma)$ with $1 \leq p \leq \infty$, which satisfies

$$\int_A f d\mu = \int_A E(f) d\mu, \quad A \in \varphi^{-1}(\Sigma).$$

Recall that $E : L^2(\Sigma) \to L^2(\mathcal{A})$ is a surjective, positive and contractive orthogonal projection. For more details on the properties of E see [10, 14, 16]. Since by the change of variable formula,

$$\int_X f \circ \varphi d\mu = \int_X h f d\mu, \quad f \in L^1(\Sigma),$$

then $\|Wf\|_2 = \|\sqrt{hE(|u|^2) \circ \varphi^{-1}} f\|_2$. Put $J = hE(|u|^2) \circ \varphi^{-1}$. It follows that W is bounded on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [11] and also [4] for a discussion of $E(\cdot) \circ \varphi^{-1}$ when φ is not invertible). Thus, W^n

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is a bounded operator precisely when $J_n := h_n E_n(|u_n|^2) \circ \varphi^{-n} \in L^\infty(\Sigma)$, where $n \in \mathbb{N}$, $h_n = d\mu \circ \varphi^{-n}/d\mu$, $u_n = u(u \circ \varphi)(u \circ \varphi^2) \cdots (u \circ \varphi^{n-1})$ and $E_n = E^{\varphi^{-n}(\Sigma)}$. From now on, we assume that $J \in L^\infty(\Sigma)$ and $u \geq 0$. Put $h_0 = 1$, $J_1 = J$, $h_1 = h$ and $E_1 = E$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition for $T \in \mathcal{B}(\mathcal{H})$, where U is a partial isometry and $|T| = (T^*T)^{1/2}$.

Definition 1.1. *Let $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and let $p > 0$. We denote by $\mathbb{H}(p, n, k)$ the set of all operators T on $\mathcal{H}$ that $T^{*k}(T^{*n}T^n)^p T^k \geq T^{*k}(T^n T^{*n})^p T^k$.*

Note that $\mathbb{H}(p, n, k_1) \subseteq \mathbb{H}(p, n, k_2)$, for all $k_1 \leq k_2$. Let r and q be two positive real numbers. An operator T is absolute (p, r)-paranormal if $\||T|^p U|T|^r x\| \geq \||T|^r x\|^{p+r}$, and T is absolute (p, r)-$*$-paranormal if $\||T|^p U|T|^r x\|^r \geq \||T|^r U^* x\|^{p+r}$, for all unit vectors x in $\mathcal{H}$. An operator T is (p, r, q)-paranormal if $\||T|^p U|T|^r x\|^{\frac{1}{q}} \geq \||T|^{\frac{p+r}{q}} x\|$, and T is (p, r, q)-$*$-paranormal if $\||T|^p U|T|^r x\|^{\frac{1}{q}} \geq \||T|^{\frac{p+r}{q}} U^* x\|$, for all unit vectors $x \in \mathcal{H}$.

Composition operators as an extension of shift operators are a good tool for separating weak hyponormal classes. Classic seminormal (weighted) composition operators have been extensively studied by Harrington and Whitley [9], Lambert [11, 14], Singh [18], Campbell [4–6] and Stochel [8]. In [2] and [3] some weak hyponormal classes of composition operators are studied. In those works, examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through w-hyponormal. But in some cases composition operators can not be separated some of these classes. Hence, it is better that we consider the weighted case of composition operators. In [7] and [12], the authors generalized the work done in [2] and have obtained some characterizations of related p-hyponormal weighted composition operators as separately. In [12] some examples were presented to illustrate that weighted composition operators lie between those classes.

This note is a continuation of the work done in [12]. The plan of this note is to present some characterizations of weak p-hyponormal and weak p-paranormal classes of weighted composition operators on $L^2(\Sigma)$. We then give specific examples illustrating these classes.

2. Characterizations

In the following two theorems, to avoid tedious calculations, we investigate only $\mathbb{H}(p, 1, k)$ and $\mathbb{H}(p, n, 0)$ classes of weighted composition operators. Note that $T \in \mathbb{H}(p, 1, 0)$ if and only if T is p-hyponormal and $T \in \mathbb{H}(p, 1, 1)$ if and only if T is p-quasihyponormal. Recall that $T \in \mathbb{H}(p, 1, k)$ if and only if $T^{*k}(T^*T)^p T^k \geq T^{*k}(TT^*)^p T^k$, and $T \in \mathbb{H}(p, n, 0)$ if and only if $(T^{*n}T^n)^p \geq (T^n T^{*n})^p$. We assume throughout this note that W is a bounded weighted composition operator on $L^2(\Sigma)$.

Theorem 2.1. *Let $k \in \mathbb{N}$. Then $W \in \mathbb{H}(p, 1, k)$ if and only if*

$$E_k({u_k}^2 J^p) \geq E_k(u_k u(h^p \circ \varphi)(E(u^2))^{p-1} E(u u_k)), \quad \text{on } \sigma(h_k).$$

Proof. Let $f \in L^2(\Sigma)$. Then we have

$$\begin{aligned} W^{*k}(W^*W)^p W^k f &= h_k E_k(u_k M_{J^p}(u_k f \circ \varphi^k)) \circ \varphi^{-k} f \\ &= h_k E_k(u_k^2 J^p) \circ \varphi^{-k} f, \end{aligned}$$

and

$$\begin{aligned} W^{*k}(WW^*)^p W^k f &= h_k E_k(u_k u(h^p \circ \varphi)(E(u^2))^{p-1} E(u u_k f \circ \varphi^k)) \circ \varphi^{-k} f \\ &= h_k E_k(u_k u(h^p \circ \varphi)(E(u^2))^{p-1} E(u u_k)) \circ \varphi^{-k} f. \end{aligned}$$

Thus, $W \in \mathbb{H}(p, 1, k)$ if and only if

$$h_k E_k(u_k^2 J^p) \geq h_k E_k(u_k u(h^p \circ \varphi)(E(u^2))^{p-1} E(uu_k)).$$

□

Corollary 2.2. *Let $C_\varphi \in \mathcal{B}(L^2(\Sigma))$. Then $C_\varphi \in \mathbb{H}(p, 1, k)$ if and only if $E_k(h^p) \geq E_k(h^p \circ \varphi)$ on $\sigma(h_k)$. In particular (see [2]), if $k = 1$, then C_φ is p-quasihyponormal if and only if $E(h^p) \geq h^p \circ \varphi$ on $\sigma(h)$.*

Lemma 2.3. *[15] Let α and β be nonnegative and measurable functions. Then for every $f \in L^2(\Sigma)$,*

$$\int_X \alpha |f|^2 d\mu \geq \int_X |E_n(\beta f)|^2 d\mu$$

if and only if $\sigma(\beta) \subseteq \sigma(\alpha)$ and $E_n(\frac{\beta^2}{\alpha} \chi_{\sigma(\alpha)}) \leq 1$.

Theorem 2.4. *$W \in \mathbb{H}(p, n, 0)$ if and only if $\sigma(u_n) \subseteq \sigma(J_n)$ and*

$$(h_n^p \circ \varphi^n)(E_n({u_n}^2))^{p-1} E_n(\frac{u_n^2}{J_n^p} \chi_{\sigma(J_n)}) \leq 1.$$

Proof. Let $f \in L^2(\Sigma)$. Then we have

$$\langle (W^{*^n} W^n)^p f, f \rangle = \int_X J_n^p |f|^2 d\mu,$$

and

$$\begin{aligned} \langle (W^n W^{*^n})^p f, f \rangle &= \int_X u_n (h_n^p \circ \varphi^n)(E_n(u_n^2))^{p-1} (E_n(u_n f)) \bar{f} d\mu \\ &= \int_X |E_n((h_n^{\frac{p}{2}} \circ \varphi^n)(E_n(u_n^2))^{\frac{p-1}{2}} u_n f)|^2 d\mu. \end{aligned}$$

Put $\alpha = J_n^p$ and $\beta = (h_n^{\frac{p}{2}} \circ \varphi^n)(E_n(u_n^2))^{\frac{p-1}{2}} u_n$. Then $\sigma(\alpha) = \sigma(J_n)$ and $\sigma(\beta) = \sigma(u_n)$. Now, the desired conclusion follows from Lemmamama 2.3. □

Corollary 2.5. *[7] W is p-hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and*

$$E\left(\left(\frac{J \circ \varphi}{J}\right)^p \frac{u^2}{E(u^2)}\right) \leq 1.$$

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $U|T|$ be its polar decomposition. Suppose p, r and q are positive real numbers. Then the following hold:*

(i) [17] T is absolute (p, r)-paranormal if and only if for each $\lambda > 0$,

$$r|T|^r U^* |T|^{2p} U |T|^r - (p + r)\lambda^p |T|^{2r} + p\lambda^{p+r} \geq 0.$$

(ii) [1] T is absolute (p, r)-$*$-paranormal if and only if for each $\lambda > 0$,

$$r|T|^r U^* |T|^{2p} U |T|^r - (p + r)\lambda^p |T^*|^{2r} + p\lambda^{p+r} \geq 0.$$

(iii) [13] T is (p, r, q)-paranormal if and only if for each $\lambda > 0$,

$$|T|^r U^* |T|^{2p} U |T|^r - q\lambda^{q-1} |T|^{\frac{2(p+r)}{q}} + (q - 1)\lambda^q \geq 0.$$

(iv) [1] T is (p, r, q)-$*$-paranormal if and only if for each $\lambda > 0$,

$$|T|^r U^* |T|^{2p} U |T|^r - q\lambda^{r(q-1)} |T^*|^{\frac{2(p+r)}{q}} + (q - 1)\lambda^{rq} \geq 0.$$

Theorem 2.7. *The following statements are equivalent.*

(i) W is absolute (p, r)-paranormal.

(ii) $E(u^2J^p) \geq (h^p \circ \varphi)(E(u^2))^{p+1}$, on $\sigma(h) \cap \sigma(E(u))$.

(iii) W is (p, r, q)-paranormal.

Proof. (i)⇔ (ii) Let $f \in L^2(\Sigma)$. It is easy to check that

$$Uf = \frac{uf \circ \varphi}{\sqrt{h \circ \varphi E(u^2)}};$$

$$\begin{aligned} U^*f &= h^{\frac{1}{2}}((E(u^2))^{\frac{-1}{2}}E(uf)) \circ \varphi^{-1}; \\ |W|^r f &= J^{\frac{r}{2}} f; \\ |W^*|^{2r} &= u(h^r \circ \varphi)(E(u^2))^{r-1}E(uf). \end{aligned}$$

It follows that

$$|W|^rU^*|W|^{2p}U|W|^r f = rh^r((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-1} f := af.$$

Now, by Lemmamama 2.6(i), W is absolute (p, r)-paranormal if and only if

$$A(\lambda) := a - (p + r)b\lambda^p + p\lambda^{p+r} \geq 0, \quad \lambda \in [0, \infty),$$

where $b = J^r$. Since this function takes its minimum value at $\lambda = \sqrt[r]{b}$, then

$$\begin{aligned} A(J) \geq 0 &\Longleftrightarrow a \geq rJ^{p+r} \\ &\Longleftrightarrow rh^r((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-1} \geq r(h^{p+r}(E(u^2))^{p+r} \circ \varphi^{-1}) \\ &\Longleftrightarrow (h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (h^{p+r} \circ \varphi)(E(u^2))^{p+r}, \qquad \text{on } \sigma(h) \\ &\Longleftrightarrow E(u^2J^p) \geq (h^p \circ \varphi)(E(u^2))^{p+1}, \qquad \text{on } \sigma(h) \cap \sigma(E(u)). \end{aligned}$$

(iii)⇔ (ii) It is easy to check that

$$|W|^rU^*|W|^{2p}U|W|^r f = h^r((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-1} f$$

$$|W|^{\frac{2(p+r)}{q}} f = J^{\frac{p+r}{q}} f.$$

By Lemmamama 2.6(iii), W is (p, r, q)-paranormal if and only

$$B(\lambda) := a - q\lambda^{q-1}b + (q - 1)\lambda^q \geq 0, \qquad \lambda \in [0, \infty),$$

where $a := h^r((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-1}$ and $b = J^{\frac{p+r}{q}}$. This function has a minimum at $\lambda = b$. Then we have

$$\begin{aligned} B(J^{\frac{p+r}{q}}) \geq 0 &\Longleftrightarrow a \geq b^q \\ &\Longleftrightarrow h^r((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-1} \geq J^{p+r} \\ &\Longleftrightarrow (h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (h^{p+r} \circ \varphi)(E(u^2))^{p+r}, \quad \text{on } \sigma(h) \\ &\Longleftrightarrow E(u^2J^p) \geq (h^p \circ \varphi)(E(u^2))^{p+1}, \qquad \text{on } \sigma(h) \cap \sigma(E(u)). \end{aligned}$$

Hence the proof is complete. □

Corollary 2.8. *The following are equivalent.*

(i) C_φ is absolute (p,r)-paranormal.

(ii) $E(h^p) \geq h^p \circ \varphi$ on $\sigma(h)$.

(iii) C_φ is (p,r,q)-paranormal.

Note that by [2, Theorem 2.3], $E(h^p) \geq h^p \circ \varphi$ on $\sigma(h)$ if and only if C_φ is absolute p-paranormal if and only if C_φ is p-paranormal. So for each r and q, these classes can not be separated by composition operators.

Theorem 2.9. *The following assertions hold.*

(i) W is absolute (p,r)-$*$-paranormal if and only if

$$(h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (h^{p+r} \circ \varphi^2)(\{E(u)^{\frac{2(p+r)}{r}}(E(u^2))^{\frac{(p+r)(r-1)}{r}}\} \circ \varphi)$$

on $\sigma(h)$.

(ii) W is (p,r,q)-$*$-paranormal if and only if

$$(h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (h^{p+r} \circ \varphi^2)(\{E(u)^{2q}(E(u^2))^{(p+r-q)}\} \circ \varphi)$$

on $\sigma(h)$.

Proof. (i) Let $f \in L^2(\Sigma)$. Direct computations show that

$$|W|^rU^*|W|^{2p}U|W|^rf = h^r\{(E(u^2))^{r-1}E(u^2J^p)\} \circ \varphi^{-1}f := Cf,$$
$$|W^*|^{2r}f = \lambda^p u(h^r \circ \varphi)(E(u^2))^{r-1}E(uf).$$

Then by Lemmama ma 2.6 (ii), W is absolute (p,r)-$*$-paranormal if and only if

$$\langle rCf - (p+r)\lambda^p u(h^r \circ \varphi)(E(u^2))^{r-1}E(uf) + p\lambda^{p+r}, f\rangle \geq 0,$$

for each $\lambda \in (0,\infty)$. Put $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (2.1) holds if and only if

$$\int_{\varphi^{-1}B}\{rC - (p+r)\lambda^p u(h^r \circ \varphi)(E(u^2))^{r-1}E(u) + p\lambda^{p+r}\}d\mu \geq 0.$$

Equivalently,

$$\int_B\{rC \circ \varphi^{-1} - (p+r)\lambda^p(E(u) \circ \varphi^{-1})h^r((E(u^2))^{r-1} \circ \varphi^{-1})(E(u) \circ \varphi^{-1}) + p\lambda^{p+r}\}hd\mu \geq 0.$$

But, this is equivalent to

$$rC \circ \varphi^{-1} - (p+r)\lambda^p(E(u) \circ \varphi^{-1})^2h^r(E(u^2))^{r-1} \circ \varphi^{-1} + p\lambda^{p+r} \geq 0$$

on $\sigma(h)$. Set $rC \circ \varphi^{-1} = a$, and

$$b = (E(u) \circ \varphi^{-1})^2h^r(E(u^2))^{r-1} \circ \varphi^{-1}.$$

Then W is absolute (p,r)-$*$-paranormal if and only if

$$D(\lambda) := a - (p+r)b\lambda^p + p\lambda^{p+r} \geq 0, \quad \lambda \in [0,\infty).$$

Since

$$\min_{\lambda\in[0,\infty)} A(\lambda) = A(\sqrt[r]{b}),$$

it follows that

$$\begin{aligned} D(\sqrt[r]{b}) \geq 0 &\Longleftrightarrow a \geq rb^{\frac{p+r}{r}} \\ &\Longleftrightarrow (h^r \circ \varphi^{-1})((E(u^2))^{r-1}E(u^2J^p)) \circ \varphi^{-2} \geq (E(u)^{\frac{2(p+r)}{r}} \circ \varphi^{-1})h^{p+r}((E(u^2))^{\frac{(p+r)(r-1)}{r}} \circ \varphi^{-1}) \\ &\Longleftrightarrow (h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (\{E(u)^{\frac{2(p+r)}{r}}(E(u^2))^{\frac{(p+r)(r-1)}{r}}\} \circ \varphi)(h^{p+r} \circ \varphi^2), \end{aligned}$$

on $\sigma(h)$.

(ii) The proof is similar to the proof of part (i). Let $f \in L^2(\Sigma)$. By 2.6 (iv), W is (p, r, q)-$*$-paranormal if and only if for each $\lambda \in [0, \infty)$,

$$G(\lambda) := a - qb\lambda^{r(q-1)} + (q-1)\lambda^{rq} \geq 0,$$

where

$$\begin{aligned} a &= (h^r \circ \varphi^{-1})\{(E(u^2))^{r-1}E(u^2J^p)\} \circ \varphi^{-2}, \\ b &= (E(u) \circ \varphi^{-1})^2 h^{\frac{p+r}{q}}((E(u^2))^{\frac{p+r}{q}-1} \circ \varphi^{-1}). \end{aligned}$$

Since this function takes its minimum value at $\lambda = \sqrt[r]{b}$, then we have

$$\begin{aligned} G(\sqrt[r]{b}) \geq 0 &\Longleftrightarrow a \geq b^q \\ &\Longleftrightarrow (h^r \circ \varphi^{-1})(\{(E(u^2))^{r-1}E(u^2J^p)\} \circ \varphi^{-2}) \geq ((E(u))^{2q} \circ \varphi^{-1})h^{p+r}((E(u^2))^{(p+r-q)} \circ \varphi^{-1}) \\ &\Longleftrightarrow (h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^p) \geq (\{(E(u))^{2q}(E(u^2))^{(p+r-q)}\} \circ \varphi)(h^{p+r} \circ \varphi^2), \end{aligned}$$

on $\sigma(h)$. This completes the proof.

Corollary 2.10. *The following are equivalent.*

(i) C_φ *is absolute* (p, r)-$*$-*paranormal.*

(ii) $(h^r \circ \varphi)E(h^p) \geq h^{p+r} \circ \varphi^2$ *on* $\sigma(h)$.

(iii) C_φ *is* (p, r, q)-$*$-*paranormal.*

3. Examples

Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider the space $\ell^2(m) = L^2(\mathbb{N}, 2^{\mathbb{N}}, m)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and m is a measure on $2^{\mathbb{N}}$ defined by $m(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Direct computation shows that (see [14])

$$\begin{aligned} h_n(k) &= \frac{1}{m_k}\sum_{j\in\varphi^{-n}(k)} m_j; \\ E_n(f)(k) &= \frac{\sum_{j\in\varphi^{-n}(\varphi^n(k))} f_j m_j}{\sum_{j\in\varphi^{-n}(\varphi^n(k))} m_j}; \\ J_n(k) &= \frac{1}{m_k}\sum_{j\in\varphi^{-n}(k)} (u_n(j))^2 m_j, \end{aligned}$$

for all non-negative sequence $f = \{f_n\}_{n=1}^{\infty} \in \ell^2(m)$ and $k \in \mathbb{N}$.

Example 3.1. *Let X be the set of nonnegative integers, let Σ be the σ-algebra of all subsets of X, take m to be the point mass measure determined by the*

$$m = 1, 1, 1, c, d, c^2, d^2, c^3, d^3, \cdots,$$

where c and d are fixed positive real numbers. The powers of c occur for odd integers and those of d for even integers. Our point transformation φ is defined by

$$\varphi(n) = \begin{cases} 0 & n = 0, 1, \\ n-1 & n \geq 2. \end{cases}$$

Note that this example was used in [2] and [3] to show that composition operators can separate almost all weak hyponormality classes. Define u by

$$u(n) = \begin{cases} 1 & n = 0; \\ c & n \geq 1 \ (odd); \\ d & n \geq 2 \ (even). \end{cases}$$

Then we get that

$$J^p(n) = \begin{cases} (1+c^2)^p & n = 0; \\ (\frac{d^{\frac{n+3}{2}}}{c^{\frac{n-1}{2}}})^p & n \geq 1 \ (odd); \\ (\frac{c^{\frac{n+4}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2 \ (even), \end{cases}$$

$$E(u^2 J^p)(n) = \begin{cases} \frac{(1+c^2)^p + c^2 d^{2p}}{2} & n = 0, 1; \\ c^2(\frac{d^{\frac{n+3}{2}}}{c^{\frac{n-1}{2}}})^p & n \geq 3 \ (odd); \\ d^2(\frac{c^{\frac{n+4}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2 \ (even), \end{cases}$$

$$(E(u^2))^{p+1}(n) = \begin{cases} (\frac{1+c^2}{2})^{p+1} & n = 0, 1; \\ c^{2p+2} & n \geq 3 \ (odd); \\ d^{2p+2} & n \geq 2 \ (even). \end{cases}$$

Since

$$(h \circ \varphi)(n) = \frac{\sum_{j \in \varphi^{-1}(\varphi(n))} m_j}{m_{\varphi(n)}},$$

then we have

$$(h^p \circ \varphi)(n) = \begin{cases} 2^p & n = 0, 1; \\ (\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n \geq 3 \ (odd); \\ ((\frac{d}{c})^{\frac{n-2}{2}})^p & n \geq 2 \ (even) \end{cases},$$

and so

$$(h^p \circ \varphi)(E(u^2))^{p+1}(n) = \begin{cases} \frac{(1+c^2)^{p+1}}{2} & n = 0, 1; \\ c^{2p+2}(\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n \geq 3 \ (odd); \\ d^{2p+2}((\frac{d}{c})^{\frac{n-2}{2}})^p & n \geq 2 \ (even). \end{cases}$$

Then by Theorem 2.7, W is (p, r, q)-paranormal if and only if $1 + c^2 \leq d^2$. Also, direct computations show that

$$(h^r \circ \varphi)(E(u^2))^{r-1} E(u^2 J^p)(n) = \begin{cases} (1+c^2)^{r-1}\{(1+c^2)^p + c^2 d^{2p}\} & n = 0, 1; \\ c^{2r}(\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^r (\frac{c^{\frac{n+3}{2}}}{d^{\frac{n-1}{2}}})^p & n \geq 3 \ (odd); \\ d^{2r}((\frac{d}{c})^{\frac{n-2}{2}})^r (\frac{c^{\frac{n+4}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2 \ (even), \end{cases}$$

$$(h^{p+r}\circ\varphi^2)(n)=\begin{cases}2^{p+r} & n=0,1,2;\\ (\frac{c^{\frac{n-2}{2}}}{d^{\frac{n-4}{2}}})^{p+r} & n\geq 3\ (odd);\\ ((\frac{d}{c})^{\frac{n-3}{2}})^{p+r} & n\geq 4\ (even),\end{cases}$$

Set $P(u)=((E(u^2)^{p+r-q}(E(u))^{2q})\circ\varphi$. *Then*

$$(P(u))(n)=\begin{cases}\frac{(1+c)^{2q}(1+c^2)^{p+r-q}}{2^{p+r+q}} & n=0,1,2;\\ d^{2(p+r)} & n\geq 3\ (odd);\\ c^{2(p+r)} & n\geq 4\ (even),\end{cases}$$

$$\{(h^{p+r}\circ\varphi^2)P(u)\}(n)=\begin{cases}\frac{(1+c)^{2q}(1+c^2)^{p+r-q}}{2^{q}} & n=0,1,2;\\ d^{2(p+r)}(\frac{c^{\frac{n-2}{2}}}{d^{\frac{n-4}{2}}})^{p+r} & n\geq 3\ (odd);\\ c^{2(p+r)}((\frac{d}{c})^{\frac{n-3}{2}})^{p+r} & n\geq 4\ (even).\end{cases}$$

Then by Theorem 2.9, W is (p,r,q)*-$*$-paranoramal if and only if*

$$1+c^2(\frac{d^2}{1+c^2})^p\geq(1+c^2)\{\frac{(1+c)^2}{2(1+c^2)}\}^q. \tag{1}$$

Now,

$$u_2(n)=u(n)u(\varphi(n))=\begin{cases}1 & n=0;\\ c & n=1;\\ (cd) & n\geq 2.\end{cases}$$

$$u(n)u_2(n)=\begin{cases}1 & n=0;\\ c^2 & n=1;\\ (c^2d) & n\geq 3\ (odd);\\ (cd^2) & n\geq 2\ (even),\end{cases}$$

$$E(uu_2)(n)=\begin{cases}\frac{1+c^2}{2} & n=0,1;\\ (c^2d) & n\geq 3\ (odd);\\ (cd^2) & n\geq 2\ (even).\end{cases}$$

Take $Q(u)=(uu_2)E(uu_2)(h^p\circ\varphi)(E(u^2))^{p-1}$. *Then*

$$(Q(u))(n)=\begin{cases}(1+c^2)^p & n=0;\\ c^2(1+c^2)^p & n=1;\\ c^{2p+2}d^2(\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n\geq 3\ (odd);\\ d^{2p+2}c^2((\frac{d}{c})^{\frac{n-2}{2}})^p & n\geq 2\ (even),\end{cases}$$

and so

$$E_2(Q(u))(n)=\begin{cases}\frac{(1+c^2)^p+c^2(1+c^2)^p+c^2d^{2p+2}}{3} & n=0,1,2;\\ c^{2p+2}d^2(\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n\geq 3\ (odd);\\ d^{2p+2}c^2((\frac{d}{c})^{\frac{n-2}{2}})^p & n\geq 2\ (even).\end{cases}$$

Also, it is easy to check that

$$u_2^2 J^p(n) = \begin{cases} (1+c^2)^p & n=0; \\ c^2 d^{2p} & n=1; \\ c^2 d^2 (\frac{d^{\frac{n+3}{2}}}{c^{\frac{n-1}{2}}})^p & n \geq 3\ (odd); \\ c^2 d^2 (\frac{c^{\frac{n+4}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2\ (even). \end{cases}$$

$$E_2(u_2^2 J^p)(n) = \begin{cases} \frac{(1+c^2)^p + c^2 d^{2p} + (c^2 d^2 c^{3p})}{3} & n=0,1,2; \\ c^2 d^2 (\frac{d^{\frac{n+3}{2}}}{c^{\frac{n-1}{2}}})^p & n \geq 3\ (odd); \\ c^2 d^2 (\frac{c^{\frac{n+4}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 4\ (even). \end{cases}$$

Then by Theorem 2.1, $W \in \mathbb{H}(p,1,2)$ *if and only if*

$$d^2 + (\frac{1+c^2}{d^2})^p - d^2(\frac{c^3}{d^2})^p \leq 1. \tag{2}$$

By similar computations we have

$$\varphi^2(n) = \begin{cases} 0 & n=0,1,2; \\ n-2 & n \geq 3, \end{cases}$$

$$h_2(\varphi^2(n)) = \begin{cases} 3 & n=0,1,2; \\ c & n \geq 3\ (odd); \\ d & n \geq 4\ (even), \end{cases}$$

$$E_2(u_2^2)(n) = \begin{cases} \frac{1+c^2+(cd)^2}{3} & n=0,1,2; \\ (cd)^2 & n \geq 3\ (odd); \\ (cd)^2 & n \geq 4\ (even), \end{cases}$$

$$J_2(n) = \begin{cases} 1+c^2+(cd)^2 & n=0; \\ (cd)^2 c & n \geq 1\ (odd); \\ (cd)^2 d & n \geq 2\ (even), \end{cases}$$

$$\frac{u_2^2}{J_2^p}(n) = \begin{cases} \frac{1}{(1+c^2+(cd)^2)^p} & n=0; \\ \frac{c^2}{(cd)^{2p} c^p} & n=1; \\ \frac{(cd)^2}{(cd)^{2p} c^p} & n \geq 3\ (odd); \\ \frac{(cd)^2}{(cd)^{2p} d^p} & n \geq 2\ (even), \end{cases}$$

$$E_2(\frac{u_2^2}{J_2^p})(n) = \begin{cases} \frac{\frac{1}{(1+c^2+(cd)^2)^p} + \frac{c^2}{(cd)^{2p} c^p} + \frac{(cd)^2}{(cd)^{2p} d^p}}{3} & n=0,1,2; \\ \frac{(cd)^2}{(cd)^{2p} c^p} & n \geq 3\ (odd); \\ \frac{(cd)^2}{(cd)^{2p} d^p} & n \geq 4\ (even). \end{cases}$$

Put $R(u) = (h_2^p \circ \varphi^2)(E_2(u_2^2))^{p-1} E_2(\frac{u_2^2}{J_2^p})$. *Then*

$$(R(u))(n) = \begin{cases} (1+c^2+(cd)^2)^{p-1}\{\frac{1}{(1+c^2+(cd)^2)^p} + \frac{c^2}{c^p (cd)^{2p}} + \frac{(cd)^2}{d^p (cd)^{2p}}\} & n=0,1,2; \\ 1 & n \geq 3. \end{cases}$$

Then by Theorem 2.4, $W \in \mathbb{H}(p,2,0)$ if and only if

$$(1+c^2+(cd)^2)^{p-1}\{\frac{1}{(1+c^2+(cd)^2)^p}+\frac{c^2}{c^p(cd)^{2p}}+\frac{(cd)^2}{d^p(cd)^{2p}}\}\leq 1. \tag{3}$$

In particular, if $p = q = 2$ and $c = d \in (0, 0.5)$, then by (1), (2), (3), W is (p,r,q)-$*$-paranormal but W neither in $\mathbb{H}(p,1,2)$ nor in $\mathbb{H}(p,2,0)$. Also if $c = d \in (0.5, 2.3)$, then W is not (p,r,q)-$*$-paranormal but W is in $\mathbb{H}(p,1,2)\cap\mathbb{H}(p,2,0)$. Now let

$$u(n)=\begin{cases}1 & n=0,1;\\ n & n\geq 2.\end{cases}$$

Then we have

$$u_2(n)=\begin{cases}1 & n=0,1;\\ 2 & n=2;\\ n(n-1) & n\geq 3,\end{cases}$$

$$E_2(u_2^2)(n)=\begin{cases}2 & n=0,1,2;\\ (n(n-1))^2 & n\geq 3,\end{cases}$$

$$J_2(n)=\begin{cases}6 & n=0;\\ c((n+2)(n+1))^2 & n\geq 1\ (odd);\\ d((n+2)(n+1))^2 & n\geq 2\ (even),\end{cases}$$

$$(\frac{u_2^2(E_2(u_2^2))^{p-1}}{J_2^p})(n)=\begin{cases}\frac{1}{2\times 3^p} & n=0;\\ \frac{1}{3^{2p}\times 2^{p+1}}\times\frac{1}{c^p} & n=1;\\ \frac{1}{3^{2p}\times 2^{3p-1}}\times\frac{1}{d^p} & n=2;\\ (\frac{n(n-1)}{(n+1)(n+2)})^{2p}\times\frac{1}{c^p} & n\geq 3\ (odd);\\ (\frac{n(n-1)}{(n+1)(n+2)})^{2p}\times\frac{1}{d^p} & n\geq 4\ (even),\end{cases}$$

$$E_2(\frac{u_2^2(E_2(u_2^2))^{p-1}}{J_2^p})(n)=\begin{cases}\frac{(\frac{1}{2\times 3^p})+(\frac{1}{3^{2p}\times 2^{p+1}}\times\frac{1}{c^p})+(\frac{1}{3^{2p}\times 2^{3p-1}}\times\frac{1}{d^p})}{3} & n=0,1,2;\\ (\frac{n(n-1)}{(n+1)(n+2)})^{2p}\times\frac{1}{c^p} & n\geq 3\ (odd);\\ (\frac{n(n-1)}{(n+1)(n+2)})^{2p}\times\frac{1}{d^p} & n\geq 4\ (even).\end{cases}$$

Put $S(u)=(h_2\circ\varphi^2)^p(E_2(u_2^2))^{p-1}E_2(\frac{u_2^2\chi_{\sigma(J_2)}}{J_2^p})$. Then

$$(S(u))(n)=\begin{cases}\frac{1}{6}+\frac{1}{2^{p+1}3^{p+1}c^p}+\frac{1}{2^{3p-1}3^{p+1}d^p} & n=0,1,2;\\ (\frac{n(n-1)}{(n+1)(n+2)})^{2p} & n\geq 3.\end{cases}$$

Then by Theorem 2.4, $W \in \mathbb{H}(p,2,0)$ if and only if

$$(\frac{4}{6c})^p+4(\frac{1}{6d})^p\leq 5\times 4^p. \tag{4}$$

Put $T(u)=(u_2u)(h^p\circ\varphi)(E(u^2))^{p-1}E(u_2u)$. Since

$$(h^p\circ\varphi)(n)=\begin{cases}2^p & n=0,1;\\ \{(\frac{d}{c})^{\frac{n-2}{2}}\}^p & n\geq 2\ (even);\\ \{\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}}\}^p & n\geq 3\ (odd),\end{cases}$$

$$J(n) = \begin{cases} 2 & n = 0; \\ (n+1)^2 \frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}} & n \geq 2\ (even); \\ (n+1)^2 (\frac{d}{c})^{\frac{n-1}{2}} & n \geq 1\ (odd), \end{cases}$$

$$u_2^2 J^p(n) = \begin{cases} 2^p & n = 0; \\ 2^{2p} & n = 1; \\ 3^{2p} \times c^p \times 2^2 & n = 2; \\ (n(n-1))^2 (n+1)^{2p} (\frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 4\ (even); \\ (n(n-1))^2 (n+1)^{2p} ((\frac{d}{c})^{\frac{n-1}{2}})^p & n \geq 3\ (odd), \end{cases}$$

Hence

$$E_2(u_2^2 J^p)(n) = \begin{cases} \frac{2^p + 2^{2p} + (3^{2p} \times c^p \times 2^2)}{3} & n = 0, 1, 2; \\ n^2 (n-1)^2 (n+1)^{2p} (\frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 4\ (even); \\ n^2 (n-1)^2 (n+1)^{2p} ((\frac{d}{c})^{\frac{n-1}{2}})^p & n \geq 3\ (odd), \end{cases}$$

$$(T(u))(n) = \begin{cases} 2^p & n = 0, 1; \\ n^{2p+2} (n-1)^2 ((\frac{d}{c})^{\frac{n-2}{2}})^p & n \geq 2\ (even); \\ n^{2p+2} (n-1)^2 (\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n \geq 3\ (odd), \end{cases}$$

and so

$$E_2(T(u))(n) = \begin{cases} \frac{2^p + 2^p + 2^{2p+2}}{3} & n = 0, 1, 2; \\ n^{2p+2} (n-1)^2 ((\frac{d}{c})^{\frac{n-2}{2}})^p & n \geq 4\ (even); \\ n^{2p+2} (n-1)^2 (\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^p & n \geq 3\ (odd). \end{cases}$$

Then by Theorem 2.1, $W \in \mathbb{H}(p, 1, 2)$ if and only if

$$4(\frac{9}{4}c)^p - (\frac{1}{2})^p \geq 3, \tag{5}$$

and by Theorem 2.7, for each p and r, W is absolute (p, r)-paranormal. Again, by similar computations we have

$$E(u^2)(n) = \begin{cases} 1 & n = 0, 1; \\ n^2 & n \geq 2, \end{cases}$$

$$(h^{p+r} \circ \varphi)(n) = \begin{cases} 2^{p+r} & n = 0, 1; \\ ((\frac{d}{c})^{\frac{n-2}{2}})^{p+r} & n \geq 2\ (even); \\ (\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^{p+r} & n \geq 3\ (odd), \end{cases}$$

$$(u^2 J^p)(n) = \begin{cases} 2^p & n = 0; \\ 2^{2p} & n = 1; \\ n^2 (n+1)^{2p} (\frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2\ (even); \\ n^2 (n+1)^{2p} ((\frac{d}{c})^{\frac{n-1}{2}})^p & n \geq 1\ (odd), \end{cases}$$

$$E(u^2J^P)(n) = \begin{cases} \frac{2^p+2^{2p}}{2} & n = 0,1; \\ n^2(n+1)^{2p}(\frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}})^p & n \geq 2\ (even); \\ n^2(n+1)^{2p}((\frac{d}{c})^{\frac{n-1}{2}})^p & n \geq 3\ (odd). \end{cases}$$

$$(h^r \circ \varphi)(E(u^2))^{r-1}E(u^2J^P)(n) = \begin{cases} 2^r(\frac{2^p+2^{2p}}{2}) & n = 0,1; \\ n^{2r}(n+1)^{2p}(\frac{c^{\frac{n}{2}}}{d^{\frac{n-2}{2}}})^p((\frac{d}{c})^{\frac{n-2}{2}})^r & n \geq 2\ (even); \\ n^{2r}(n+1)^{2p}((\frac{d}{c})^{\frac{n-1}{2}})^p(\frac{c^{\frac{n-1}{2}}}{d^{\frac{n-3}{2}}})^r & n \geq 3\ (odd). \end{cases}$$

Then

$$(U(u))(n) = \begin{cases} 2^{p+r} & n = 0,1,2; \\ (n-1)^{2(p+r)}((\frac{d}{c})^{\frac{n-3}{2}})^{p+r} & n \geq 4\ (even); \\ (n-1)^{2(p+r)}(\frac{c^{\frac{n-2}{2}}}{d^{\frac{n-4}{2}}})^{p+r} & n \geq 3\ (odd), \end{cases}$$

where $U(u) = (h^{p+r} \circ \varphi^2)((E(u))^{\frac{2(p+r)}{r}}(E(u^2))^{\frac{(p+r)(r-1)}{r}} \circ \varphi)$. *Thus by Theorem 2.9, W is absolute* (p,r)-$(*)$-*paranormal if and only if*

$$2^r(\frac{9c}{2})^p \geq 1. \tag{6}$$

Accorollaryding to (4), (5) and (6), if $p = q = r = 2$ *and* $c = d \in (0, 0.11)$, *then* $W \in \mathbb{H}(p,2,0) \setminus \mathbb{H}(p,1,2)$, *W is absolute* (p,r)-*paranormal but it is not absolute* (p,r)-$*$-*paranormal operator. However if* $c = d \in (15,\infty)$, *then W is in* $\mathbb{H}(p,1,2) \setminus \mathbb{H}(p,2,0)$ *and W not only is absolute* (p,r)-*paranormal but it is also absolute* (p,r)-$*$-*paranormal.*

Example 3.2. *Let* $X = [0,1]$ *equipped with the Lebesgue measure* μ *on the Lebesgue measurable subsets and let* $\varphi : X \to X$ *is defined by*

$$\varphi(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2}, \\ 2-2x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then

$$E(f)(x) = \frac{f(x)+f(1-x)}{2},$$

$$\varphi^2(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{4}; \\ 2-4x & \frac{1}{4} \leq x \leq \frac{1}{2}; \\ -2+4x & \frac{1}{2} \leq x \leq \frac{3}{4}; \\ 4-4x & \frac{3}{4} \leq x \leq 1, \end{cases}$$

and so $h(x) = h_2(x) = 1$ *and for each* $f \in L^2(\Sigma)$

$$(E(f) \circ \varphi^{-1})(x) = \frac{f(\frac{x}{2}) + f(1-\frac{x}{2})}{2},$$

$$E_2(f)(x) = \begin{cases} \frac{f(x)+f(1-x)+f(\frac{1}{2}-x)+f(\frac{1}{2}+x)}{4} & 0 \leq x \leq \frac{1}{2}, \\ \frac{f(x)+f(1-x)+f(\frac{-1}{2}+x)+f(\frac{3}{2}-x)}{4} & \frac{1}{2} \leq x < 1. \end{cases}$$

Put

$$u(x) = \begin{cases} 0 & 0 \leq x \leq \frac{3}{4}; \\ -x & \frac{3}{4} \leq x \leq 1, \end{cases}$$

and take $V(u) = E_2(u_2u(h^p \circ T)(E(u^2))^{p-1}E(u_2u))$. *Then*

$$u_2(x) = \begin{cases} 0 & 0 \leq x \leq \frac{3}{4}; \\ x(2-2x) & \frac{3}{4} \leq x \leq 1, \end{cases}$$

$$J_2(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{4}; \\ 4x^2(1-x)^2 & \frac{3}{4} \le x \le 1. \end{cases}$$

$$E(u^2)(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{4}; \\ \frac{2x^2-2x+1}{2} & \frac{3}{4} \le x \le 1, \end{cases}$$

$$E(u_2 u)(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{4}; \\ -x(1-x) & \frac{3}{4} \le x \le 1. \end{cases}$$

Hence we obtain

$$E_2(u_2^2 J^p)(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{4}; \\ \frac{x^2(1-x)^2\{(x^2-2x+2)^p+(1+x^2)^p\}+(\frac{-1}{2}+x)^2(\frac{3}{2}-x)^2\{(x^2-3x+\frac{13}{4})^p+(x^2-x+\frac{5}{4})^p\}}{2^{2p}} & \frac{3}{4} \le x \le 1, \end{cases}$$

$$V(u)(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{4}; \\ \frac{x^2(1-x)^2(2x^2-2x+1)^{p-1}+(\frac{-1}{2}+x)^2(\frac{3}{2}-x)^2(2x^2-4x+\frac{5}{2})^{p-1}}{2^p} & \frac{3}{4} \le x \le 1. \end{cases}$$

Thus by Theorem 2.1, $W \in \mathbb{H}(p,1,2)$ if and only if $E_2(u_2^2 J^p) \ge V(u)$, i.e.,

$$x^2(1-x)^2\{(x^2-2x+2)^p + (1+x^2)^p\} + (\frac{-1}{2}+x)^2(\frac{3}{2}-x)^2\{(x^2-3x+\frac{13}{4})^p + (x^2-x+\frac{5}{4})^p\}$$

$$\ge 2^p\{x^2(1-x)^2(2x^2-2x+1)^{p-1} + (\frac{-1}{2}+x)^2(\frac{3}{2}-x)^2(2x^2-4x+\frac{5}{2})^{p-1}\}.$$

But the above inequality is not holds on $X = [0,1]$. So W is not in $\mathbb{H}(p,1,2)$. Also, by Theorem 2.7, W is (p,r,q)-paranormal if and only if

$$x^2(x^2-2x+2)^p + (1-x)^2(1+x^2)^p \ge 2^p(2x^2-2x+1)^{p+1}. \tag{7}$$

By Theorem 2.9, W is (p,r,q)-$*$-paranormal if and only if

$$2^p(8x^2-12x+5)^{p+r-q} \le 2^q(2x^2-2x+1)^{r-1}\{x^2(x^2-2x+2)^p + (1-x)^2(1+x^2)^p\}. \tag{8}$$

In particular, if $p = q = 100$ and $r = 2$, then by (7) and (8), W is (p,r,q)-$*$-paranormal but W is not (p,r,q)-paranormal. However if $p = 1023.15$, $q = 2$, $r = 2$, then W is (p,r,q)-paranormal but W is not (p,r,q)-$*$-paranormal.

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