WEIGHTED FROBENIUS-PERRON OPERATORS AND THEIR SPECTRA

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Abstract. First, some classic properties of a weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$ as a predual of weighted Koopman operator $W = uU_\varphi$ on $L^\infty(\Sigma)$ will be investigated using the language of the conditional expectation operator. Also, we determine the spectrum of $\mathcal{P}_\varphi^u$ under certain conditions.

Keywords: Frobenius-Perron operator; Fredholm operator; spectrum

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1. Introduction and preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the space $L^1(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^1(\mathcal{A})$, where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function f is defined by $\operatorname{supp}(f) = \{x \in X \colon f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ.

Recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \mathcal{A}, \mu)$ with no atoms is called non-atomic. It is a well known fact that every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is non-atomic over B and Z is a countable union of atoms of finite measure (see [16]). For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^1(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E_\mu^{\mathcal{A}}(f)$ such that

$$\int_A f \,\mathrm{d}\mu = \int_A E_\mu^{\mathcal{A}}(f) \,\mathrm{d}\mu,$$

where A is any $\mathcal{A}$-measurable set for which $\int_A f\,\mathrm{d}\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E_\mu^{\mathcal{A}}\colon L^1(\Sigma) \to L^1(\mathcal{A})$ uniquely defined by the assignment $f \mapsto E_\mu^{\mathcal{A}}(f)$ is called the conditional expectation operator with respect to $\mathcal{A}$.

From now on, we assume that φ is a nonsingular transformation on X, $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E = E_\mu^{\mathcal{A}}$. Ding in [4] proved that for each $f \in L^1(\Sigma)$ there exists a unique $g \in L^1(\Sigma)$ with $\mathrm{supp}(g) \subseteq \mathrm{supp}(h)$ such that $E(f) = g \circ \varphi$. As usual, we then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ. The mapping E acts on $L^1(\Sigma)$ as a projection onto $L^1(\varphi^{-1}(\Sigma))$. Note that $\mathcal{D}(E)$, the domain of E, contains $L^1(\Sigma) \cup \{f \in L^0(X, \Sigma, \mu)\colon f \geqslant 0\}$. Throughout this paper, we take u in $\mathcal{D}(E)$. The analysis of a (weighted) Frobenius-Perron operator is based on the concept of conditional expectation operator. Let $f, g \in \mathcal{D}(E)$. We list some useful properties of E.

- ▷ L(1) $E((f \circ \varphi)g) = (f \circ \varphi)E(g)$;
- ▷ L(2) If $f \geqslant 0$, then $E(f) \geqslant 0$; if $f > 0$, then $E(f) > 0$;
- ▷ L(3) $\mathrm{supp}(f) \subseteq \mathrm{supp}(E(f))$ for each $f \geqslant 0$;
- ▷ L(4) $((Ef) \circ \varphi^{-1}) \circ \varphi = E(f)$;
- ▷ L(5) $(E(\alpha f + g)) \circ \varphi^{-1} = \alpha(E(f)) \circ \varphi^{-1} + (E(g)) \circ \varphi^{-1}$;
- ▷ L(6) $|E(f) \circ \varphi^{-1}|^n = |E(f)|^n \circ \varphi^{-1} \leqslant E(|f|^n) \circ \varphi^{-1}$, $(n \in \mathbb{N})$.

For proofs and discussions on some of these elementary facts see [14].

The aim of this paper is to generalize some of the results obtained for the (classic) Frobenius-Perron operators in [7], [6], [3] to the weighted Frobenius-Perron operators.

2. Fredholm weighted Frobenius-Perron operators

Let a Σ-measurable transformation $\varphi\colon X \to X$ be nonsingular, i.e., $\mu \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$, and let $u \in \mathcal{D}(E)$. The linear operator $\mathcal{P}_\varphi^u\colon L^1(\Sigma) \to L^1(\Sigma)$ defined by

$$\int_A \mathcal{P}_\varphi^u f\,\mathrm{d}\mu = \int_{\varphi^{-1}(A)} uf\,\mathrm{d}\mu, \quad f \in L^1(\Sigma),\ A \in \Sigma$$

is called the weighted Frobenius-Perron operator associated with the pair (u, φ). By the Radon-Nikodym, $\mathcal{P}_\varphi^u$ is well defined [10]. When $u = 1$, $P_\varphi := \mathcal{P}_\varphi^1$ is called the (classical) Frobenius-Perron operator. As an application of the conditional expectation and using the change of variable formula we have

$$\int_A \mathcal{P}_\varphi^u f\,\mathrm{d}\mu = \int_{\varphi^{-1}(A)} uf\,\mathrm{d}\mu = \int_{\varphi^{-1}(A)} E(uf)\,\mathrm{d}\mu = \int_A hE(uf) \circ \varphi^{-1}\,\mathrm{d}\mu,$$

where $h = (\mathrm{d}\mu \circ \varphi^{-1})/\mathrm{d}\mu$. So, in the language of conditional expectation, $\mathcal{P}_\varphi^u$ can be presented as $\mathcal{P}_\varphi^u(f) = hE(uf) \circ \varphi^{-1}$. By L(5), $\mathcal{P}_\varphi^u$ is linear. Note that $\mathcal{P}_\varphi^u = P_\varphi M_u$, where $P_\varphi = hE(f) \circ \varphi^{-1}$ is the classic Frobenius-Perron operator and M_u is the multiplication operator.

The weighted Koopman operator on $L^\infty(\Sigma)$ with respect to the pair (u, φ) is defined by $W = M_u U_\varphi$, where U_φ is the (classical) Koopman operator defined by $U_\varphi(f) = f \circ \varphi$ for all $f \in L^\infty(\Sigma)$. Here, the nonsingularity of φ guarantees that W is well defined as a mapping of equivalence classes of functions on $\sigma(u)$. It is known that W is a bounded operator on $L^\infty(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$, and in this case $(\mathcal{P}_\varphi^u)^* = W$ and $\|\mathcal{P}_\varphi^u\| = \|u\|_\infty$. In particular, $(P_\varphi)^* = U_\varphi$ and $\|P_\varphi\| = 1$ (see [3], [10]).

Let X be a Banach space and X*, the Banach space of all bounded linear complex functionals on X, be the dual space of X. For $T \in B(\mathrm{X})$, the algebra of all bounded operators on X, the null-space, range and the dual operator of T are denoted by $\mathcal{N}(T)$, $\mathcal{R}(T)$ and T^*, respectively.

Lemma 2.1 (Banach's closed range theorem [15]). *Let $T \in B(\mathrm{X})$. The following statements are equivalent.*

(a) *T has closed range.*
(b) *T^* has closed range.*
(c) $\mathcal{R}(T) = {}^\perp\mathcal{N}(T^*)$.
(d) $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.

Theorem 2.2. *Let $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Then it is invertible if and only if the following conditions are all satisfied:*

(a) $\mu \ll \mu \circ \varphi^{-1}$.
(b) *For each set $F \in \Sigma$ there is a set $G \in \Sigma$ such that $\varphi^{-1}(G) = F$.*
(c) *There exists a constant $\delta > 0$ such that $|u| \geqslant \delta$ on X.*

P r o o f. Assume $\mathcal{P}_\varphi^u$ is invertible. We first show (a). Since $\mathcal{P}_\varphi^u$ is onto, then by Lemma 2.1 W is injective. Suppose $\mu \circ \varphi^{-1}(F) = \mu(\varphi^{-1}(F)) = 0$ for $F \in \Sigma$. Then $W(\chi_F) = u\chi_F \circ \varphi = u\chi_{\varphi^{-1}(F)} = 0$. The injectivity of W implies that $\mu(F) = 0$.

To prove (b), suppose $\varphi^{-1}(\Sigma) \subsetneqq \Sigma$. Then we can find $F \in \Sigma$ with $\mu(F) > 0$ such that F is disjoint with any $\varphi^{-1}(G)$. Since Σ is σ-finite, F can be written as $F = \bigcup_i F_i$, where $0 < \mu(F_i) < \infty$ and $F_i \cap F_j = \emptyset$. Put $f = \sum_i 2^{-i}\chi_{F_i}$. Then $f \in L^1(\Sigma)$ with $\mathrm{supp}(f) = F$. It follows that

$$\int_G \mathcal{P}_\varphi^u f \,\mathrm{d}\mu = \int_{\varphi^{-1}(G)} uf \,\mathrm{d}\mu = 0 \quad \text{for all } G \in \Sigma.$$

Hence $\mathcal{P}_\varphi^u f = 0$. But this contradicts $\mathcal{N}(\mathcal{P}_\varphi^u) = \{0\}$. Now we claim that u is bounded away from zero on X. Since $\mathcal{P}_\varphi^u$ is invertible, then so is W. Hence, W is bounded below. So there is a constant $c > 0$ such that

$$c\|f\|_\infty \leqslant \|W(f)\|_\infty \quad \text{for all } f \in \infty. \tag{2.1}$$

We claim $|u| \geqslant \frac{1}{2}c$ on X. Otherwise, there would be a set $G \in \Sigma$ with $\mu(G) > 0$ such that $|u| < \frac{1}{2}c$ on G. Using (b), $G = \varphi^{-1}(A)$ for some $A \in \Sigma$. By using (a), $\mu(A) > 0$ because $\mu(A) = 0$ implies that $\mu(\varphi^{-1}(A)) = 0$. Put $f = \chi_A$. Then by (2.1) we obtain

$$c = c\|\chi_A\|_\infty \leqslant \|u\chi_{\varphi^{-1}(A)}\|_\infty = \|u\chi_G\|_\infty \leqslant \frac{c}{2},$$

which is a contradiction and thus (c) holds.

Conversely, assume all three conditions hold. Firstly, we show that $\mathcal{P}_\varphi^u$ is injective. From (b) E is the identity operator. Then by the change of variable formula we have

$$\begin{aligned} 0 = \|\mathcal{P}_\varphi^u f\|_{L^1} &= \int_X |hE(uf)\circ\varphi^{-1}|\,\mathrm{d}\mu = \int_X |E(uf)|\circ\varphi^{-1}\,\mathrm{d}\mu\circ\varphi^{-1} \\ &= \int_X |E(uf)|\,\mathrm{d}\mu = \int_X |uf|\,\mathrm{d}\mu \Longrightarrow uf = 0 \Longrightarrow f = 0, \quad \text{by (c).} \end{aligned}$$

So $\mathcal{P}_\varphi^u$ is injective. Finally, we claim that $\mathcal{P}_\varphi^u$ is surjective, which is equivalent to the injectivity of $(\mathcal{P}_\varphi^u)^* = W$ on $L^\infty(\Sigma)$ (Lemma 2.1 (c)). Let $f \in \mathcal{N}(W)$. Then by (c), $f\circ\varphi = 0$. Using (a), φ is onto ([6], Lemma 2.3), and so $f = 0$. Now, by the bounded inverse theorem $\mathcal{P}_\varphi^u$ is invertible. □

Proposition 2.3. *Put $d\nu = |u|d\mu$ and let $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Then the following assertions hold.*

(a) $\operatorname{supp}(|uf|) \subseteq \operatorname{supp}(\mathcal{P}_\varphi^u(|f|))$ *for all* $f \in L^1(\Sigma)$.

(b) *If $\varphi^{-1}(\Sigma) = \Sigma$, then $\mathcal{P}_\varphi^u\colon L^1(X,\Sigma,\nu) \to L^1(X,\Sigma,\mu)$ is an isometry.*

(c) *If $|u| = 1$ and $\mu \ll \mu\circ\varphi^{-1}$, then W is an isometry on $L^\infty(\Sigma)$. Furthermore, if W is an isometry, then $\|u\|_\infty = 1$.*

P r o o f. (a) Let $f \in L^1(\Sigma)$. Since $\operatorname{supp}(h\circ\varphi) = X$, by L(3) we have

$$\begin{aligned} \varphi^{-1}(\operatorname{supp}(\mathcal{P}_\varphi^u(|f|)) &= \operatorname{supp}(\mathcal{P}_\varphi^u(|f|)\circ\varphi) = \operatorname{supp}(h\circ\varphi E(|uf|)) \\ &= \operatorname{supp}(E(|uf|)) \supseteq \operatorname{supp}(|uf|). \end{aligned}$$

(b) By hypothesis $E = I$. An easy computation shows that

$$\|\mathcal{P}_\varphi^u(f)\|_\mu = \int_X |E(uf)|\,\mathrm{d}\mu = \int_X |f|\,\mathrm{d}\nu = \|f\|_\nu.$$

(c) It was shown in [6], Lemma 2.3 that if $\mu \ll \mu \circ \varphi^{-1}$, then φ is onto. Hence,

$$\|W(f)\|_\infty = \|(uf)\circ\varphi\|_\infty = \|f\circ\varphi\|_\infty = \|f\|_\infty.$$

On the other hand, if W is an isometry, then $\|u\|_\infty = \|\mathcal{P}_\varphi^u\| = 1$. □

Definition 2.4. A sub-σ-finite algebra $\mathcal{A}$ is said to be rich subalgebra of Σ if for each $A \in \Sigma$ with positive measure there exists $K \in \mathcal{A}$ with positive measure such that $K \subseteq A$.

Note that if Σ contains a nontrivial rich subalgebra, then Σ is a non-atomic measure space.

Theorem 2.5. *Suppose* $\varphi(\Sigma) \subset \Sigma$ *and* $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. *Then the following assertions hold.*

(a) *If* $\varphi^{-1}(\Sigma)$ *is a non-atomic rich subalgebra of* Σ, *then* $\dim \mathcal{N}(\mathcal{P}_\varphi^u)$ *is either zero or infinite.*

(b) *If* (X, Σ, μ) *is a non-atomic measure space, then* $\operatorname{codim}(\overline{\operatorname{ran}(\mathcal{P}_\varphi^u)})$ *is either zero or infinite.*

Proof. (a) If $\mathcal{P}_\varphi^u$ is injective, then $\dim \mathcal{N}(\mathcal{P}_\varphi^u) = 0$. Otherwise, there is a nonzero element $f \in L^1(\Sigma)$ such that $\mathcal{P}_\varphi^u(f) = 0$. By hypothesis, there is $K \in \varphi^{-1}(\Sigma)$ with positive measure such that $K \subseteq \operatorname{supp}(f)$. So we may choose a sequence $\{K_n\}_{n=1}^\infty$ of pairwise disjoint $\varphi^{-1}(\Sigma)$-measurable sets in K with $0 < \mu(K_n) < \infty$. Set $f_n = f\chi_{K_n}$ for $n \in \mathbb{N}$. Evidently, f_n is in $L^1(\Sigma)$, and is nonzero. Moreover,

$$\begin{aligned}\|\mathcal{P}_\varphi^u f_n\|_{L^1} &= \int_X h|E(uf_n)\circ\varphi^{-1}|\,\mathrm{d}\mu = \int_X h|E(uf\chi_{K_n})\circ\varphi^{-1}|\,\mathrm{d}\mu \\ &= \int_X h|\chi_{K_n}\circ\varphi^{-1}E(uf)\circ\varphi^{-1}|\,\mathrm{d}\mu = \int_{\varphi(K_n)} h|E(uf)\circ\varphi^{-1}|\,\mathrm{d}\mu \\ &\leqslant \int_X h|E(uf)\circ\varphi^{-1}|\,\mathrm{d}\mu = \|\mathcal{P}_\varphi^u f\|_{L^1} = 0,\end{aligned}$$

so $f_n \in \mathcal{N}(\mathcal{P}_\varphi^u)$. Thus, the sequence $\{f_n\}$ forms a linearly independent subset of $\mathcal{N}(\mathcal{P}_\varphi^u)$, and hence $\dim \mathcal{N}(\mathcal{P}_\varphi^u) = \infty$.

(b) We suppose that $\operatorname{codim}(\overline{\operatorname{ran}(\mathcal{P}_\varphi^u)}) = \dim(\mathcal{N}(\mathcal{P}_\varphi^u)^*) = \dim(\mathcal{N}(W)) \neq 0$. Then there is a nonzero function $f \in L^\infty(\Sigma)$ such that $W(f) = 0$. By the same argument as in (a), we may choose a sequence $\{C_n\}_{n=1}^\infty \subseteq \operatorname{supp}(f)$ of pairwise disjoint Σ-measurable subsets in $\operatorname{supp}(f)$ with $0 < \mu(C_n) < \infty$. Put $f_n = f\chi_{C_n}$ for $n \in \mathbb{N}$. They are nonzero and linearly independent. Moreover,

$$\|W(f_n)\|_{L^\infty(X)} = \|W(f)\|_{L^\infty(\varphi^{-1}(C_n))} \leqslant \|W(f)\|_{L^\infty(X)} = 0.$$

So $f_n \in \mathcal{N}(W)$, and hence $\operatorname{codim}(\overline{\operatorname{ran}(\mathcal{P}_\varphi^u)}) = \infty$. □

Theorem 2.6. *Suppose (X, Σ, μ) is a non-atomic rich measure space and let $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Put $\mathrm{d}\nu = |u|\,\mathrm{d}\mu$. Then the following statements are equivalent.*

(a) *$\mathcal{P}_\varphi^u$ is invertible.*

(b) *$\mathcal{P}_\varphi^u$ is Fredholm operator.*

(c) (i) *There exists a constant $\delta > 0$ such that $\nu(F) \geqslant \delta\mu(F)$ for every set $F \in \Sigma$ with $\mu(F) < \infty$, and*

(ii) *$\varphi^{-1}(\Sigma) = \Sigma$.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious. We first show that (b) implies (c). Assume $\mathcal{P}_\varphi^u$ is Fredholm operator. Then $\mathcal{P}_\varphi^u$ has closed range and is injective by Theorem 2.5 (a), and so $\mathcal{P}_\varphi^u$ is bounded below with a lower bound $c > 0$. For $F \in \Sigma$ and $\mu(F) < \infty$ put $f = \chi_F$. Then by L(4) we have

$$\begin{aligned} c\mu(F) = c\|\chi_F\| \leqslant \|\mathcal{P}_\varphi^u \chi_F\| &= \int_X |E(u\chi_F)|\,\mathrm{d}\mu \\ &\leqslant \int_X E(|u\chi_F|)\,\mathrm{d}\mu = \int_X |u|\chi_F\,\mathrm{d}\mu = \int_F |u|\,\mathrm{d}\mu = \nu(F). \end{aligned}$$

Now let $\varphi^{-1}(\Sigma) \subsetneqq \Sigma$. Choose $F \in \Sigma \setminus \varphi^{-1}(\Sigma)$ with positive measure. Since (X, Σ, μ) is σ-finite, we can construct a nonnegative $f \in L^1(\Sigma)$ such that $\operatorname{supp}(f) = F$. It follows that

$$\int_G \mathcal{P}_\varphi^u f\,\mathrm{d}\mu = \int_{\varphi^{-1}(G)} uf\,\mathrm{d}\mu = 0$$

for all $G \in \Sigma$. Hence, $\mathcal{P}_\varphi^u(f) = 0$ and so $\mathcal{P}_\varphi^u$ is not injective. This contradiction implies that $\varphi^{-1}(\Sigma) = \Sigma$ and so $E = I$.

It remains to show that (c) implies (a). Let $f = \chi_{F\cup G}$, where F and G are disjoint measurable sets with finite measures. Since $\nu(F \cup G) \geqslant \delta\mu(F \cup G)$ and $E = I$, we obtain

$$\begin{aligned} \|\mathcal{P}_\varphi^u(f)\| = \|\mathcal{P}_\varphi^u(\chi_{F\cup G})\| &= \int_X |E(u\chi_{F\cup G})|\,\mathrm{d}\mu = \int_X |u\chi_{F\cup G}|\,\mathrm{d}\mu \\ &= \int_{F\cup G} |u|\,\mathrm{d}\mu \geqslant \delta \int_{F\cup G} \mathrm{d}\mu = \delta\|f\|. \end{aligned}$$

Since simple functions are dense in $L^1(\Sigma)$, then the above inequality holds for all $f \in L^1(\Sigma)$. Therefore $\mathcal{P}_\varphi^u$ is bounded below and thus $\mathcal{P}_\varphi^u$ is injective and has closed range. Finally, we claim that $\mathcal{P}_\varphi^u$ is surjective, which is equivalent to the injectivity of $(\mathcal{P}_\varphi^u)^* = W$. By hypothesis u is bounded away from zero on X and φ is onto. Thus, $(uf)\circ\varphi = 0$ implies that $f \circ \varphi = 0$ and so $f = 0$. This completes the proof. $\square$

3. Generalized weighted Frobenius-Perron operators

In [9], Ding and Hornor introduced the generalized Frobenius-Perron operators as a restriction of the adjoint of the Koopman operators into a nice closed subspace of complex charges. In this section, we extend this generalization for weighted Frobenius-Perron operator and we expect it to be a restriction of the adjoint of W into the mentioned subspace.

Suppose Σ is a σ-algebra of subsets of a set X. Then a complex charge on Σ is a map $\nu\colon \Sigma \to \mathbb{C}$ such that $\nu(\emptyset) = 0$, and if $A, B \in \Sigma$ with $A \cap B = \emptyset$, then $\nu(A \cup B) = \nu(A) + \nu(B)$. A charge ν on Σ is said to be bounded if $\sup\{|\nu(F)|\colon F \in \Sigma\} < \infty$. Let $M(X, \Sigma)$ denote the complex vector space of all complex measures on Σ. With the total variation norm $\|\mu\| = |\mu|(X)$, $M(X, \Sigma)$ is a Banach space. The collection of all bounded complex charges on Σ is denoted by $\mathrm{ba}(X, \Sigma)$. Define

$$\mathrm{ba}(X, \Sigma, \mu) = \{\nu \in \mathrm{ba}(X, \Sigma)\colon \nu \ll \mu\},$$
$$\mathrm{ca}(X, \Sigma, \mu) = \mathrm{ba}(X, \Sigma, \mu) \cap M(X, \Sigma).$$

It was shown that the complex vector space $\mathrm{ba}(X, \Sigma, \mu)$ with the total variation norm is also a Banach space and $\mathrm{ca}(X, \Sigma, \mu)$ is a closed subspace of $\mathrm{ba}(X, \Sigma, \mu)$. Let $\mathcal{P}^u_\varphi \in B(L^1(\Sigma))$. For $\nu \in \mathrm{ba}(X, \Sigma, \mu)$ we define the measure λ_ν by

$$\lambda_\nu(A) = \int_{\varphi^{-1}(A)} u \,\mathrm{d}\nu, \quad A \in \Sigma. \tag{3.1}$$

Then $\lambda_\nu \in M(X, \Sigma)$ and is absolutely continuous with respect to μ, because the assumption $\mu \ll \mu \circ \varphi^{-1}$ implies that for each $A \in \Sigma$ with $\mu(A) = 0$, $\mu(\varphi^{-1}(A)) = 0$, and so $\nu(\varphi^{-1}(A)) = 0$. Thus $\lambda_\nu(A) = 0$, and hence $\lambda_\nu \in \mathrm{ca}(X, \Sigma, \mu)$. Note that $\lambda_\nu(A) = \int_A E_\nu(u) \circ \varphi^{-1} \,\mathrm{d}\nu \circ \varphi^{-1}$. So $\mathrm{d}\lambda_\nu = E_\nu(u) \circ \varphi^{-1} \,\mathrm{d}\nu \circ \varphi^{-1}$. Take $f \in L^\infty(\Sigma)$ and $\nu \in \mathrm{ba}(X, \Sigma, \mu)$. As an application of properties of conditional expectation operators and using the change of variable formula, we have

$$\begin{aligned}\langle f, W^*(\nu)\rangle = \langle W(f), \nu\rangle &= \int_X (uf) \circ \varphi \,\mathrm{d}\nu = \int_X E_\nu(u) f \circ \varphi \,\mathrm{d}\nu \\ &= \int_X f E_\nu(u) \circ \varphi^{-1} \,\mathrm{d}\nu \circ \varphi^{-1} = \int_X f \,\mathrm{d}\lambda_\nu = \langle f, \lambda_\nu\rangle.\end{aligned}$$

Hence, $W^*(\nu) = \lambda_\nu$ is the adjoint of W. We refer to W^* as the generalized weighted Frobenius-Perron operator corresponding to the pair (u, φ). Now let $g \in L^1(\Sigma)$ and define $F_g(A) = \int_A g \,\mathrm{d}\mu$. Then $F_g \in b(X, \Sigma, \mu)$. So the mapping $g \to F_g$ is an isometry from $L^1(\Sigma)$ into a closed subspace of $\mathrm{ba}(X, \Sigma, \mu)$. Therefore $L^1(\Sigma)$ can be isometrically embedded into $b(X, \Sigma, \mu) \cong L^\infty(X, \Sigma, \mu)^* \cong L^1(X, \Sigma, \mu)^{**}$ (see [1]).

Define a mapping $\Psi\colon\ L^1(X,\Sigma,\mu) \longrightarrow \mathrm{ca}(X,\Sigma,\mu)$ by $\Psi(f) = \mu_f$, where $\mu_f(A) = \int_X f\,\mathrm{d}\mu$. Then μ_f is a complex measure on Σ and $\mu_f \ll \mu$. So $\Psi(L^1(X,\Sigma,\mu)) \subseteq \mathrm{ca}(X,\Sigma,\mu)$. On the other hand, let $\nu \in \mathrm{ca}(X,\Sigma,\mu)$. Then ν is a complex measure and $\nu \ll \mu$. Put $f_\nu = \mathrm{d}\nu/\mathrm{d}\mu$. Then $\Psi(f_\nu) = \mu_{f_\nu} = \nu$ because for each $A \in \Sigma$

$$\mu_{f_\nu}(A) = \int_A f_\nu\,\mathrm{d}\mu = \int_X \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_A \mathrm{d}\nu = \nu(A).$$

Moreover, if $\Psi(f) = 0$, then $\mu_f = 0$ and so $f = 0$. Thus, Ψ is an invertible operator with inverse $\Psi^{-1}(\nu) = \mathrm{d}\nu/\mathrm{d}\mu$. Therefore $L^1(\Sigma) \cong \mathrm{ca}(X,\Sigma,\mu)$. Let $f \in L^1(\Sigma)$. Then we have

$$\Psi^{-1}W^*\Psi(f) = \Psi^{-1}W^*(\mu_f) = \Psi^{-1}(\lambda_{\mu_f}) = \frac{\mathrm{d}\lambda_{\mu_f}}{\mathrm{d}\mu} = \mathcal{P}_\varphi^u(f)$$

because by (3.1),

$$\lambda_{\mu_f}(A) = \int_{\varphi^{-1}(A)} u\,\mathrm{d}\mu_f = \int_{\varphi^{-1}(A)} uf\,\mathrm{d}\mu = \int_A \mathcal{P}_\varphi^u(f)\,\mathrm{d}\mu.$$

So the compression of W^* on $\mathrm{ca}(X,\Sigma,\mu)$ is $\mathcal{P}_\varphi^u$. Now we define a mapping $Q_\varphi\colon L^1(X,\varphi^{-1}(\Sigma),\mu) \longrightarrow L^1(X,\Sigma,\mu)$ by $Q_\varphi f = h(f\circ\varphi^{-1})$, though we make no assumptions regarding the invertibility of φ (see [2]). Then

$$\|Q_\varphi f\| = \int_X h|f|\circ\varphi^{-1}\,\mathrm{d}\mu = \int_X |f|\,\mathrm{d}\mu = \|f\|.$$

So Q_φ is an isometry and $\mathcal{P}_\varphi^u f = Q_\varphi E M_u$. Consequently, we have the following diagram:

$$\begin{array}{ccccc}
L^1(X,\Sigma,\mu) & \xleftarrow{\;M_u\;} & L^1(X,\Sigma,\mu) & \xrightarrow{\;\Psi\;} & \mathrm{ca}(X,\Sigma,\mu) \\
\Big\downarrow E & & \Big\downarrow \mathcal{P}_\varphi^u & & \Big\downarrow W^* \\
L^1(X,\varphi^{-1}(\Sigma),\mu) & \xrightarrow{\;Q_\varphi\;} & L^1(X,\Sigma,\mu) & \xleftarrow{\;\Psi^{-1}\;} & \mathrm{ca}(X,\Sigma,\mu)
\end{array}$$

Furthermore, the operator $\mathcal{P}_\varphi^u$ is closely related to EM_u by the quantity

$$\|\mathcal{P}_\varphi^u f\| = \|Q_\varphi EM_u(f)\| = \|Q_\varphi E(uf)\| = \|E(uf)\|, \quad f \in L^1(\Sigma). \tag{3.2}$$

Therefore $\mathcal{N}(\mathcal{P}_\varphi^u) = \mathcal{N}(EM_u)$. Moreover, $\mathcal{P}_\varphi^u$ is compact if and only if the conditional type operator $EM_u\colon\ L^1(\Sigma) \to L^1(\varphi^{-1}(\Sigma))$ is compact. Thus, by Remark 2.3, Theorem 2.5 and Theorem 2.8 (ii) in [11] we have the following corollary.

Corollary 3.1. *Let $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Then the following assertions hold.*

(a) *$\mathcal{P}_\varphi^u$ is compact if and only if it is weakly compact if and only if $u(B) = 0$ and for any $\varepsilon > 0$ the set $\{x \in X\colon\ E(|u|)(x) \geqslant \varepsilon\}$ consists of finitely many atoms.*

(b) *Let $E(u)$ is bounded away from zero on its support. Then $\mathcal{P}_\varphi^u$ has closed range if and only if* $\mathrm{supp}(E(u)) = X$ *except for at most finitely many atoms.*

Theorem 3.2. *Let $\varphi(\Sigma) \subseteq \Sigma$, $u > 0$ and $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Then $\mathcal{P}_\varphi^u$ has closed range if and only if there exists a positive constant r such that $\varphi(U(r)) = \varphi(\operatorname{supp}(u))$, where $U(r) := \{x \in X\colon\ u(x) \geqslant r\}$.*

P r o o f. Suppose that $\mathcal{P}_\varphi^u$ has closed range. By the Banach closed range theorem, this implies that the range of $W = (\mathcal{P}_\varphi^u)^*$ is also closed. Thus, by [13], Theorem 2.8 there exists a positive constant r such that $\varphi(U(r)) = \varphi(\operatorname{supp}(u))$, where $U(r) := \{x \in X\colon\ u(x) \geqslant r\}$.

Conversely, suppose that there exists a positive constant r such that $\varphi(U(r)) = \varphi(\operatorname{supp}(u))$. Then by [13], Theorem 2.8 W and hence W^* have closed range. Let $\{f_n\} \subseteq L^1(\Sigma)$ and $\mathcal{P}_\varphi^u(f_n) = \Psi^{-1}W^*\Psi(f_n) \to g$ for some $g \in L^1(\Sigma)$. So $W^*(\Psi(f_n)) \to \Psi(g)$. Since $W^*(\operatorname{ca}(X,\Sigma,\mu)) \subseteq \operatorname{ca}(X,\Sigma,\mu)$, $\Psi(g) = W^*(\nu)$ for some $\nu \in \operatorname{ca}(X,\Sigma,\mu)$. It follows that $g = \Psi^{-1}W^*(\nu) = \Psi^{-1}W^*\Psi(\mathrm{d}\nu/\mathrm{d}\mu)$. Thus, $\Psi^{-1}W^*\Psi = \mathcal{P}_\varphi^u$ has closed range. This completes the proof. □

4. Spectrum of weighted Frobenius-Perron operators

The spectrum $\sigma(\mathcal{P}_\varphi^u)$ of $\mathcal{P}_\varphi^u$ is defined to be the set of all the complex numbers λ such that the linear operator $\lambda I - \mathcal{P}_\varphi^u$ does not have a bounded inverse defined on $L^1(\Sigma)$, where I is the identity operator. The complement of $\sigma(\mathcal{P}_\varphi^u)$ in the complex plane $\mathbb{C}$ is called the resolvent set of $\mathcal{P}_\varphi^u$ and is denoted by $\varrho(\mathcal{P}_\varphi^u)$. The spectrum $\sigma(\mathcal{P}_\varphi^u)$ is a disjoint union of the point spectrum $\sigma_p(\mathcal{P}_\varphi^u)$, the continuous spectrum $\sigma_c(\mathcal{P}_\varphi^u)$, and the residual spectrum $\sigma_r(\mathcal{P}_\varphi^u)$. The boundary of $\sigma(\mathcal{P}_\varphi^u)$ is denoted by $\partial\sigma(\mathcal{P}_\varphi^u)$. A number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum $\sigma_a(\mathcal{P}_\varphi^u)$ if there exists a sequence $\{f_n\}$ in $L^1(\Sigma)$ such that $\|f_n\| = 1$ for all n and $\|(\lambda I - \mathcal{P}_\varphi^u)f_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Obviously, $\sigma_a(\mathcal{P}_\varphi^u) \subset \sigma(\mathcal{P}_\varphi^u)$. A measurable set A is called wandering for φ if $\{\varphi^{-k}(A)\}_{k\geqslant 0}$ are disjoint (see [7]).

The spectrum problem of classic Frobenius-Perron operators is difficult. In fact, it is still an open problem, and so is the spectrum of weighted Frobenius-Perron operators. Some general properties and a partial spectral analysis of Frobenius-Perron operators and Koopman operators have been given in [7] and [8]. The spectrum of $\mathcal{P}_\varphi^u$ is determined in [12] for $\mathcal{P}_\varphi^u$ compact. In this section we obtain some results on the spectrum of $\mathcal{P}_\varphi^u$ under certain conditions, see [5].

Theorem 4.1. *Let (X,Σ,μ) be a σ-finite atomic measure space and $u \in L^\infty(X)$ with $\alpha = \operatorname{essinf}|u| > 0$. If φ is invertible and has a wandering set and μ is invariant under φ, then $\{\lambda \in \mathbb{C}\colon\ |\lambda| \leqslant \alpha\} \subseteq \sigma_p(\mathcal{P}_\varphi^u)$.*

P r o o f. Let $A_{n_0} \in \Sigma$ be an atomic and wandering set for φ. Put $\varphi^{-k}(A_{n_0}) = A_{n_k}$. Then $\{A_{n_k}\}_{k\geqslant 0}$ are disjoint. By the assumption we have $\mu(A_{n_k}) = \mu(\varphi^{-1}(A_{n_k}))$ for all $k \geqslant 0$. Set $G = \{\lambda \in \mathbb{C}\colon\ |\lambda| \leqslant \alpha\}$. Define $f\colon\ G \to L^1(X)$ by $f(\lambda) = f_\lambda$, where

$$f_\lambda|_{A_n} = \begin{cases} \dfrac{\lambda^k}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})}\Big|_{A_{n_0}} & n = n_k; \\ 0 & \text{otherwise.} \end{cases}$$

Then for each $\lambda \in G$, $f_\lambda \in L^1(X)$ because

$$\begin{aligned}\int_X |f_\lambda|\,\mathrm{d}\mu &= \int_X \Big|\sum_{k=0}^{\infty} (f_\lambda|_{A_{n_k}})\chi_{A_{n_k}}\Big|\,\mathrm{d}\mu < \sum_{k=0}^{\infty}\int_X |(f_\lambda|_{A_{n_k}})\chi_{A_{n_k}}|\,\mathrm{d}\mu \\ &= \sum_{k=0}^{\infty}\int_X \Big|\Big(\frac{\lambda^k}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})}\Big|_{A_{n_0}}\Big)\chi_{A_{n_k}}\Big|\,\mathrm{d}\mu \\ &= \sum_{k=0}^{\infty}\int_{A_{n_k}} \Big(\frac{|\lambda^k|}{|u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})|}\Big|_{A_{n_0}}\Big)\,\mathrm{d}\mu < \sum_{k=0}^{\infty}\frac{1}{\alpha}\frac{|\lambda^k|}{\alpha^k} < \infty.\end{aligned}$$

Moreover, for each $\lambda \in G$ we have

$$\begin{aligned}\mathcal{P}_\varphi^u f_\lambda &= \mathcal{P}_\varphi^u \sum_{k=0}^{\infty}(f_\lambda|_{A_{n_k}})\chi_{A_{n_k}} \\ &= \sum_{k=0}^{\infty}\Big(\frac{1}{\mu(A_{n_k})}u\Big|_{\varphi^{-1}(A_{n_k})} f_\lambda|_{\varphi^{-1}(A_{n_k})}\mu(\varphi^{-1}(A_{n_k}))\Big)\chi_{A_{n_k}} \\ &= \sum_{k=0}^{\infty}(u|_{A_{n_{k+1}}} f_\lambda|_{A_{n_{k+1}}})\chi_{A_{n_k}} \\ &= \sum_{k=0}^{\infty}\Big(u|_{\varphi^{-(k+1)}(A_{n_0})}\frac{\lambda^{k+1}}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-(k+1)})}\Big|_{A_{n_0}}\Big)\chi_{A_{n_k}} \\ &= \sum_{k=0}^{\infty}\Big(\frac{\lambda^{k+1}}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})}\Big|_{A_{n_0}}\Big)\chi_{A_{n_k}}.\end{aligned}$$

Then

$$\begin{aligned}(\lambda I - \mathcal{P}_\varphi^u)f_\lambda &= (\lambda I - \mathcal{P}_\varphi^u)\sum_{k=0}^{\infty}(f_\lambda|_{A_{n_k}})\chi_{A_{n_k}} \\ &= \sum_{k=0}^{\infty}\Big(\frac{\lambda^{k+1}}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})}\Big|_{A_{n_0}}\Big)\chi_{A_{n_k}} \\ &\quad - \sum_{k=0}^{\infty}\Big(\frac{\lambda^{k+1}}{u(u\circ\varphi^{-1})\ldots(u\circ\varphi^{-k})}\Big|_{A_{n_0}}\Big)\chi_{A_{n_k}} = 0.\end{aligned}$$

Thus, $\{\lambda \in \mathbb{C}\colon\ |\lambda| \leqslant \alpha\} \subseteq \sigma_p(\mathcal{P}_\varphi^u)$. □

Theorem 4.2. *If $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$, then $\sigma_p(\mathcal{P}_\varphi^u) \subset \partial\mathbb{D}_u \cup \{0\}$, where $\mathbb{D}_u = \{\lambda \in \mathbb{C}\colon |\lambda| \leqslant \|u\|_\infty\}$.*

Proof. Let $0 \neq \lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(\mathcal{P}_\varphi^u)$, then there exists a function $0 \neq f \in L^1(\Sigma)$ such that $(\lambda - \mathcal{P}_\varphi^u)f = 0$. Thus we have

$$\begin{aligned} 0 = \|\lambda f - \mathcal{P}_\varphi^u f\|_1 &\geqslant |\lambda|\|f\|_1 - \|\mathcal{P}_\varphi^u f\|_1 \geqslant |\lambda|\|f\|_1 - \|u\|_\infty\|f\|_1 \\ &= (|\lambda| - \|u\|_\infty)\|f\|_1. \end{aligned}$$

Thus, $|\lambda| = \|u\|_\infty$ and so $\lambda \in \partial\mathbb{D}_u$. □

Theorem 4.3. *If $W \in B(L^\infty(\Sigma))$ and $\mu \ll \mu \circ \varphi^{-1}$, then $\sigma_p(W) \subset \partial\mathbb{D}_u \cup \{0\}$.*

Proof. Since $\mu \ll \mu \circ \varphi^{-1}$, φ is onto. Hence,

$$\|W(f)\|_\infty = \|(uf)\circ\varphi\|_\infty = \|uf\|_\infty \leqslant \|u\|_\infty\|f\|_\infty.$$

Now let $0 \neq \lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(W)$, then there exists a function $0 \neq f \in L^\infty(\Sigma)$ such that $(\lambda I - W)f = 0$. Then

$$\begin{aligned} 0 = \|\lambda f - Wf\|_\infty &\geqslant |\lambda|\|f\|_\infty - \|Wf\|_\infty \geqslant |\lambda|\|f\|_\infty - \|u\|_\infty\|f\|_\infty \\ &= (|\lambda| - \|u\|_\infty)\|f\|_\infty \end{aligned}$$

and hence $|\lambda| = \|u\|_\infty$. □

Theorem 4.4. *Let $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$. Then the following assertions hold.*

(a) *If $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$ is not invertible, then $\sigma(\mathcal{P}_\varphi^u) = \mathbb{D}_u$.*

(b) *If $\mathcal{P}_\varphi^u \in B(L^1(\Sigma))$ is invertible, then $\sigma(\mathcal{P}_\varphi^u) \subset \partial\,\mathbb{D}_u$.*

Proof. Let $f \in L^1(\Sigma)$ and $\lambda \in \mathbb{C}$ with $|\lambda| < \|u\|_\infty$. Then

$$\|\lambda f - \mathcal{P}_\varphi^u f\|_1 \geqslant |\lambda|\|f\|_1 - \|\mathcal{P}_\varphi^u f\|_1 \geqslant |\lambda|\|f\|_1 - \|u\|_\infty\|f\|_1 = (|\lambda| - \|u\|_\infty)\|f\|_1.$$

Thus, $\lambda I - \mathcal{P}_\varphi^u$ is bounded from below and so $\lambda \notin \sigma_a(\mathcal{P}_\varphi^u)$. Since $\partial\sigma(\mathcal{P}_\varphi^u) \subset \sigma_a(\mathcal{P}_\varphi^u)$, $\lambda \notin \partial\sigma(\mathcal{P}_\varphi^u)$ for all $|\lambda| < \|u\|_\infty$. In particular, $0 \notin \partial\sigma(\mathcal{P}_\varphi^u)$. Now, let for $u \in L^\infty(\Sigma)$, $\mathcal{P}_\varphi^u$ is not invertible. Then $0 \in \sigma(\mathcal{P}_\varphi^u)$. If there exists $|\lambda| < \|u\|_\infty$ such that $\lambda \notin \sigma(\mathcal{P}_\varphi^u)$, then it is easy to see that there exists a $\lambda_1 \in \partial\sigma(\mathcal{P}_\varphi^u)$ such that $|\lambda_1| < \|u\|_\infty$. But this is a contradiction to the fact that $\lambda \notin \partial\sigma(\mathcal{P}_\varphi^u)$ for all $|\lambda| < \|u\|_\infty$. It follows that $\sigma(\mathcal{P}_\varphi^u) = \mathbb{D}_u$ because $\sigma(\mathcal{P}_\varphi^u)$ is a closed subset of $\mathbb{D}_u$.

Consider now the case when $\mathcal{P}_\varphi^u$ is invertible. Then $0 \in \varrho(\mathcal{P}_\varphi^u)$. If there exists $|\lambda| < \|u\|_\infty$ such that $\lambda \in \sigma(\mathcal{P}_\varphi^u)$, then there exists a $\lambda_2 \in \partial\sigma(\mathcal{P}_\varphi^u)$ with $|\lambda_2| < \|u\|_\infty$, which also contradicts the fact that $\lambda \notin \partial\sigma(\mathcal{P}_\varphi^u)$ for all $|\lambda| < \|u\|_\infty$. Therefore $|\lambda| < \|u\|_\infty$ implies that $\lambda \notin \sigma(\mathcal{P}_\varphi^u)$, and so $\sigma(\mathcal{P}_\varphi^u) \subset \partial\mathbb{D}_u$. □

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