doi: 10.14232/actasm-015-825-0

Acta Sci. Math. (*Szeged*)
83 (2017), 291–299

Reducibility of $M_u C_\varphi$ on $L^2(\Sigma)$

M. R. Jabbarzadeh and M. Jafari Bakhshkandi

Communicated by L. Kérchy

Abstract. In this note we study reducing subspaces for weighted composition operators defined on $L^2(\Sigma)$. Some necessary and sufficient conditions are given for such operators to have two types of reducing subspaces of the forms $L^2(\Sigma_A)$ and $L^2(\mathcal{A})$. This is basically discussed by using conditional expectation properties.

1. Introduction and preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. Also given $B \in \Sigma$, we shall abbreviate the subspace $L^2(B, \Sigma_B, \mu_{|\Sigma_B})$ to $L^2(\Sigma_B) = \{f \in L^2(\Sigma) : \chi_{B^c} f = 0\}$ where $\Sigma_B = \{A \cap B : A \in \Sigma\}$ and $B^c = X \setminus B$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The subspace $L^\infty(\Sigma)$ consists of those Σ-measurable functions on X which are essentially bounded. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each non-negative function $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon–Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu,$$

where A is any $\mathcal{A}$-measurable set for which $\int_F f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} \colon L^2(\Sigma) \to L^2(\mathcal{A})$, uniquely

Received October 16, 2015.
AMS Subject Classification (2000): 47B37, 47B38.
Key words and phrases: reducing subspace, weighted composition operators, conditional expectation.

defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. The mapping $E^{\mathcal{A}}$ is a linear orthogonal projection onto $L^2(\mathcal{A})$. If $\mathcal{B} \subseteq \mathcal{A} \subseteq \Sigma$, then $E^{\mathcal{B}}_{\mathcal{A}}$ denotes the appropriate conditional expectation from $L^2(\mathcal{A})$ onto $L^2(\mathcal{B})$. Put $E^{\mathcal{A}}_{\Sigma} = E^{\mathcal{A}}$. Then $E^{\mathcal{B}}_{\mathcal{A}} E^{\mathcal{A}} = E^{\mathcal{B}}$. Note that $\mathcal{D}(E^{\mathcal{A}})$, the domain of E, contains $L^2(\Sigma) \cup \{g \in L^0(\Sigma) : g \geq 0\}$. For more details on conditional expectation see [3, 10].

Let $\varphi \colon X \to X$ be a Σ-measurable transformation of X. Denote by $\mu \circ \varphi^{-1}$ the measure on Σ given by $\mu \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A))$ for $A \in \Sigma$. We say that φ is non-singular if $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. Put $h = d\mu \circ \varphi^{-1}/d\mu$. For a non-singular measurable transformation φ of X and a weight function $u : X \to [0, \infty)$, the weighted composition operator on $L^2(\Sigma)$ is defined by $W(f) = u.f \circ \varphi$. Note that $W = M_u \circ C_\varphi$, where M_u is the multiplication and C_φ is the composition operator. It is shown in [8] that W is bounded if and only if $J := hE^{\varphi^{-1}(\Sigma)}(u^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Throughout this paper we assume that φ is non-singular, $u \geq 0$ and $J \in L^\infty(\Sigma)$.

The role of the conditional expectation operator is important in this note. We shall use the following general properties of $E^{\mathcal{A}}$ and W acting on $L^2(\Sigma)$. For proofs and discussions of some of these facts see [1, 6–8, 10].

- L(1) If f is an $\mathcal{A}$-measurable function, then $E^{\mathcal{A}}(fg) = fE^{\mathcal{A}}(g)$;
- L(2) if $f \geq 0$ then $E^{\mathcal{A}}(f) \geq 0$; if $f > 0$ then $E^{\mathcal{A}}(f) > 0$;
- L(3) $\sigma(f) \subseteq \sigma(E^{\mathcal{A}}(f))$, for each $0 \leq f \in L^2(\Sigma)$;
- L(4) $E^{\mathcal{A}}(|f|^2) = |E^{\mathcal{A}}(f)|^2$ if and only if $f \in L^0(\mathcal{A})$;
- L(5) $\varphi^{-1}(\sigma(h)) = X$, i.e., $h \circ \varphi > 0$;
- L(6) (change of variable) $\int_{\varphi^{-1}(A)} gf \circ \varphi d\mu = \int_A hE^{\varphi^{-1}(\Sigma)}(g) \circ \varphi^{-1} f d\mu$, for all $g \in \mathcal{D}(E^{\varphi^{-1}(\Sigma)})$ and $A \in \Sigma$;
- L(7) $W^*f = hE^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1}$;
- L(8) $W^*Wf = hE^{\varphi^{-1}(\Sigma)}(u^2) \circ \varphi^{-1} f$;
- L(9) $WW^*f = u(h \circ \varphi)E^{\varphi^{-1}(\Sigma)}(uf)$;
- L(10) $E^{\varphi^{-1}(\mathcal{A})}(L^2(\mathcal{A})) = \overline{C_\varphi(L^2(\mathcal{A}))} = \{f \in L^2(\mathcal{A}) : f \in L^0(\varphi^{-1}(\Sigma))\}$.

Let $\mathcal{H}$ be a real or complex Hilbert space. The set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ is denoted by $B(\mathcal{H})$. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and range of an operator $T \in B(\mathcal{H})$. Recall that a closed subspace $M \subseteq \mathcal{H}$ is said to be invariant for an operator $T \in B(\mathcal{H})$ whenever $T(M) \subseteq M$. If M and its orthogonal complement $M^\perp$ are both invariant for T, then we say that M reduces T. So M is a reducing subspace of T if and only if $PT(I-P) = 0$ and $(I-P)TP = 0$, where P is an orthogonal projection onto M. The problem of classifying the reducing subspaces of T is equivalent to finding the orthogonal projections in $\{T\}'$, the commutant algebra of T. Reducibility of composition operators on $L^2(\Sigma)$ have been studied in [1]. In the next section some necessary and sufficient conditions are given for weighted composition operator $W \in B(L^2(\Sigma))$, that is a combination of a multiplication and a composition operator, to have two types of reducing subspaces of the forms $L^2(\Sigma_A)$ and $L^2(\mathcal{A})$.

2. Reducibility of *W*

In order to characterize the reducibility of weighted composition operators we first need to know the behavior of the orthogonal projections onto reducing subspaces. We shall need the following known facts.

Lemma 2.1. ([4]) *For a closed subspace M of $\mathcal{H}$ and $T \in B(\mathcal{H})$, let P be the orthogonal projection onto M. Then the following are equivalent.*

(a) *M is a reducing subspace of T;*
(b) *$TP = PT$;*
(c) *$T^*P = PT^*$.*

In this case, P commutes with TT^* and T^*T.

Lemma 2.2. ([1, Corollary 3]) *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-finite subalgebras in Σ. Then the following are equivalent.*

(a) *$E^{\mathcal{A}}E^{\mathcal{B}}$ is an orthogonal projection;*
(b) *$E^{\mathcal{A}}E^{\mathcal{B}}=E^{\mathcal{B}}E^{\mathcal{A}}$;*
(c) *$E^{\mathcal{A}}E^{\mathcal{B}}=E^{\mathcal{A}\cap\mathcal{B}}$.*

Let P be the orthogonal projection onto a reducing subspace of $L^2(\Sigma)$ for W. By Lemma 2.1 and L(7), L(8), L(9) we obtain the following proposition.

Proposition 2.3. *Let W be a weighted composition operator induced by the pair (u, φ), and let P be the orthogonal projection onto a reducing subspace of $L^2(\Sigma)$ for W. Then for each $f \in L^2(\Sigma)$,*

(a) *$P(uf \circ \varphi) = u(Pf) \circ \varphi$;*
(b) *$P(hE^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1}) = hE^{\varphi^{-1}(\Sigma)}(uPf) \circ \varphi^{-1}$;*
(c) *$P(Jf) = JPf$;*
(d) *$P(uh \circ \varphi E^{\varphi^{-1}(\Sigma)}(uf)) = uh \circ \varphi E^{\varphi^{-1}(\Sigma)}(uPf)$.*

For $B \in \Sigma$ with $\mu(B) > 0$, put $B^c = X \setminus B$. Then $L^2(\Sigma) = L^2(\Sigma_B) \oplus L^2(\Sigma_{B^c})$, where $L^2(\Sigma_B) = \{f \in L^2(\Sigma) : f = 0 \text{ on } B^c\}$. If $\varphi^{-1}(B) \subseteq B$, then $\varphi^{-1}(\Sigma_B) = \varphi^{-1}(\Sigma) \cap \varphi^{-1}(B) \subseteq \Sigma \cap B = \Sigma_B$. Since $(B, \Sigma_B, \mu_{|\Sigma_B})$ is a relatively complete σ-finite measure space, then by L(10), $C_\varphi(L^2(\Sigma_B)) \subseteq L^2(\varphi^{-1}(\Sigma_B))$, and so $C_\varphi(L^2(\Sigma_B)) \subseteq L^2(\Sigma_B)$. Hence $L^2(\Sigma_B)$ is an invariant subspace of C_φ. Now, assume $\varphi^{-1}(B) \supseteq B$. Then $\varphi^{-1}(\Sigma_{B^c}) \subseteq \varphi^{-1}(\Sigma) \cap B^c \subseteq \Sigma_{B^c}$, and hence we have $C_\varphi(L^2(\Sigma_{B^c})) \subseteq L^2(\varphi^{-1}(\Sigma_{B^c})) \subseteq L^2(\Sigma_{B^c})$. Consequently, if $\varphi^{-1}(B) = B$, then $L^2(\Sigma_B)$ reduces C_φ. On the other hand if $L^2(\Sigma_B)$ and $L^2(\Sigma_{B^c})$ are both invariant under C_φ, then by the same argument we get that $\varphi^{-1}(\Sigma_B) \subseteq \Sigma_B$ and $\varphi^{-1}(\Sigma_{B^c}) \subseteq \Sigma_{B^c}$. Thus, $\varphi^{-1}(B) = B$. These observations establish the following proposition.

Proposition 2.4. *Let $B \in \Sigma$ with $\mu(B) > 0$ and let $C_\varphi \in B(L^2(\Sigma))$. Then the following assertions hold.*

(a) $\varphi^{-1}(B) \subseteq B$ *if and only if* $L^2(\Sigma_B)$ *is an invariant subspace of* C_φ.

(b) $\varphi^{-1}(B) \supseteq B$ *if and only if* $L^2(\Sigma_B)$ *is an invariant subspace of* C_φ^*.

Consequently, $L^2(\Sigma_B)$ reduces C_φ if and only if $\varphi^{-1}(B) = B$. This fact was originally proved by Burnap and Lambert in [1, Theorem 5(a)]. In the following theorem we try to restate a similar fact for the combination of a multiplication and a composition operator.

Theorem 2.5. *Let $W \in B(L^2(\Sigma))$ and $B \in \Sigma$ with $\mu(B) > 0$. Then $L^2(\Sigma_B)$ reduces W if and only if $B \cap \sigma(u) = \varphi^{-1}(B) \cap \sigma(u)$. In particular, if $\sigma(u) \subseteq \varphi^{-1}(\sigma(u))$, then $L^2(\Sigma_{\sigma(u)})$ is reducing for W.*

Proof. Put $P = M_{\chi_B}$. Then P is an orthogonal projection onto $L^2(\Sigma_B)$. By Lemma 2.1, $L^2(\Sigma_B)$ reduces W if and only if $PW = WP$. Let $L^2(\Sigma_B)$ reduce W. Then for each $f \in L^2(\Sigma)$ we have $\chi_B u f \circ \varphi = \chi_{\varphi^{-1}(B)} u f \circ \varphi$. Let $X = \cup_{n=1}^\infty B_n$ where $B_n \in \Sigma$ with $\mu(B_n) < \infty$. It follows that $u\chi_{B \cap \varphi^{-1}(B_n)} = u\chi_{\varphi^{-1}(B) \cap \varphi^{-1}(B_n)}$. Hence $u\chi_B = u\chi_{\varphi^{-1}(B)}$, because $X = \cup_{n=1}^\infty \varphi^{-1}(B_n)$. Thus $B \cap \sigma(u) = \varphi^{-1}(B) \cap \sigma(u)$.

Conversely, let $B \cap \sigma(u) = \varphi^{-1}(B) \cap \sigma(u)$. Since $u = \chi_{\sigma(u)} u$, then for each $f \in L^2(\Sigma)$ we obtain

$$PW(f) = \chi_B W(f) = \chi_{\varphi^{-1}(B)} W(f) = u(\chi_B \circ \varphi)(f \circ \varphi) = u(f\chi_B) \circ \varphi = WP(f).$$

So, $L^2(\Sigma_B)$ reduces W. When $\sigma(u) \subseteq \varphi^{-1}(\sigma(u))$, φ maps $\sigma(u)$ into $\sigma(u)$ and so $L^2(\Sigma)$ can be decomposed as $L^2(\Sigma) = L^2(\Sigma_{\sigma(u)}) \oplus L^2(\Sigma_{\sigma(u)^c})$. Now, the desired conclusion follows from [3, Lemma 2.3]. ∎

Let $\mathcal{A} \subseteq \Sigma$ be a relatively complete σ-finite algebra. In what follows we give some necessary and sufficient conditions that the subspace $L^2(\mathcal{A})$ reduces W.

Theorem 2.6. *Let $W \in B(L^2(\Sigma))$. If $L^2(\mathcal{A})$ reduces W, then $(\varphi^{-1}(\mathcal{A}))_{\sigma(u)} \subseteq \mathcal{A}_{\sigma(u)}$ and u, $J \in L^0(\mathcal{A})$, where $J = hE^{\varphi^{-1}(\Sigma)}(u^2) \circ \varphi^{-1}$.*

Proof. The reducibility of W implies that $u\chi_{\varphi^{-1}(A)} = W(\chi_A) \in L^2(\mathcal{A})$, for all $A \in \mathcal{A}$ with finite measure. Therefore $\sigma(u\chi_{\varphi^{-1}(A)}) = \sigma(u) \cap \varphi^{-1}(A) \in \mathcal{A}$, and so $\varphi^{-1}(A) \cap \sigma(u) \in \mathcal{A}_{\sigma(u)}$ for each $A \in \mathcal{A}$. Thus, $(\varphi^{-1}(\mathcal{A}))_{\sigma(u)} \subseteq \mathcal{A}_{\sigma(u)}$. We now show that J is $\mathcal{A}$-measurable. Since $\mathcal{R}(E^{\mathcal{A}}) = L^2(\mathcal{A})$ reduces W, then by Lemma 2.1, $E^{\mathcal{A}}W^*W = W^*WE^{\mathcal{A}}$. By L(8), $W^*W = M_J$. It follows that $E^{\mathcal{A}}(Jf) = JE^{\mathcal{A}}(f)$, for each $f \in L^2(\mathcal{A})$. Let $\{B_n\}$ be a sequence in Σ increasing to X. Then $E^{\mathcal{A}}(\chi_{B_n}) \uparrow E^{\mathcal{A}}(1) = 1$ and hence $E^{\mathcal{A}}(J\chi_{B_n}) \uparrow J$. Since $E^{\mathcal{A}}(J\chi_{B_n})$ is $\mathcal{A}$-measurable for each $n \in \mathbb{N}$, we conclude that $J \in L^0(\mathcal{A})$. Finally, let $\{C_n\} \subseteq \mathcal{A}$, $\mu(C_n) < \infty$ and $X = \cup C_n$. Thus, $X = \cup\varphi^{-1}(C_n)$. Hence we get that $u\chi_{\varphi^{-1}(C_n)\cap\sigma(u)} = u\chi_{\varphi^{-1}(C_n)} = W(\chi_{c_n}) \in L^2(\mathcal{A})$, for each $n \in \mathbb{N}$. This implies that $u \in L^0(\mathcal{A})$. ■

Corollary 2.7. *Let C_φ be a bounded composition operator on $L^2(\Sigma)$. If $L^2(\mathcal{A})$ reduces C_φ, then*

(a) $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ *and* $h \in L^\infty(\mathcal{A})$;
(b) $E^{\mathcal{A}}E^{\varphi^{-1}(\mathcal{A})} = E^{\varphi^{-1}(\mathcal{A})}$;
(c) $E^{\mathcal{A}}E^{\varphi^{-1}(\Sigma)} = E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}$;
(d) $E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)} = E^{\varphi^{-1}(\mathcal{A})}$;
(e) $C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}}C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}}C_\varphi$.

Proof. (a) In Theorem 2.6, it suffices to put $u = 1$.

(b) The inclusion $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ in (a) implies in turn that $L^2(\varphi^{-1}(\mathcal{A}))$ reduces $E^{\mathcal{A}}$. Now, by Lemma 2.1 and Lemma 2.2, $E^{\mathcal{A}}E^{\varphi^{-1}(\mathcal{A})} = E^{\varphi^{-1}(\mathcal{A})}E^{\mathcal{A}} = E^{\varphi^{-1}(\mathcal{A})}$.

(c) Put $u = 1$ and $P = E^{\mathcal{A}}$ in Proposition 2.3(d). Then by L(5) and Lemma 2.2 we obtain $E^{\mathcal{A}}E^{\varphi^{-1}(\Sigma)} = E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}} = E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}$.

(d) By the inclusion $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ in (a), $E^{\varphi^{-1}(\mathcal{A})} = E^{\varphi^{-1}(\mathcal{A})\cap\varphi^{-1}(\Sigma)} \leq E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}$. To establish the reverse inequality, it is sufficient to show that $L^2(\mathcal{A}\cap\varphi^{-1}(\Sigma)) \subseteq L^2(\varphi^{-1}(\mathcal{A}))$. Let $f \in L^2(\mathcal{A}\cap\varphi^{-1}(\Sigma))$. Then $E^{\mathcal{A}}(f) = f = g\circ\varphi$, for some $g \in L^2(\Sigma)$. Note that g is uniquely determined in $\sigma(h)$ ([2]). Since h is $\mathcal{A}$-measurable and $L^2(\Sigma) \cap L^\infty(\Sigma)$ is dense in $L^2(\Sigma)$, then by [9, Proposition 3] we have

$$E^{\varphi^{-1}(\mathcal{A})}_{\varphi^{-1}(\Sigma)}(g\circ\varphi) = E^{\mathcal{A}}(g)\circ\varphi.$$

It follows that

$$E^{\varphi^{-1}(\mathcal{A})}(f) = E^{\varphi^{-1}(\mathcal{A})}_{\varphi^{-1}(\Sigma)}E^{\varphi^{-1}(\Sigma)}(g\circ\varphi) = E^{\mathcal{A}}(g)\circ\varphi,$$

because $g\circ\varphi$ and $E^{\mathcal{A}}(g)\circ\varphi$ are $\varphi^{-1}(\Sigma)$-measurable. Now, by Proposition 2.3(a),

$$E^{\varphi^{-1}(\mathcal{A})}(f) = E^{\mathcal{A}}(g\circ\varphi) = E^{\mathcal{A}}(f) = f.$$

Thus, $f \in L^2(\varphi^{-1}(\mathcal{A}))$.

(e) Observe that $L^2(\Sigma)$ may be decomposed as the orthogonal direct sum $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E^{\mathcal{A}})$, where $L^2(\mathcal{A}) = \mathcal{R}(E^{\mathcal{A}})$ and $\mathcal{N}(E^{\mathcal{A}}) = \{f - E^{\mathcal{A}}(f) : f \in L^2(\Sigma)\}$. Put $P = E^{\mathcal{A}}$ and $T = C_\varphi$. Now by Lemma 2.1, we have $TP = PTP = PT$ and this completes the proof. ■

Theorem 2.8. *If u, $J \in L^0(\mathcal{A})$ and $E^{\varphi^{-1}(\Sigma)} E^{\mathcal{A}} M_u = E^{\varphi^{-1}(\mathcal{A})} M_u$ on $L^0(\Sigma)$, then $L^2(\mathcal{A})$ reduces W.*

Proof. Since $W(L^2(\Sigma)) \subseteq L^2(\Sigma)$ and $\mathcal{A}$-measurable simple functions are dense in $L^2(\mathcal{A})$, it is sufficient to show that $W(\chi_A)$ and $W^*(\chi_A)$ are $\mathcal{A}$-measurable for each $A \in \mathcal{A}$ with finite measure. By hypostases, after taking adjoint, $M_u E^{\mathcal{A}} E^{\varphi^{-1}(\Sigma)}(f) = M_u E^{\varphi^{-1}(\mathcal{A})}(f)$ for each $f \in L^0(\Sigma)$. Take $f = \chi_{\varphi^{-1}(A)}$. Since $E^{\varphi^{-1}(\Sigma)}(\chi_A \circ \varphi) = \chi_A \circ \varphi = \chi_{\varphi^{-1}(A)} = f$ and u is $\mathcal{A}$-measurable, then $M_u E^{\mathcal{A}} E^{\varphi^{-1}(\Sigma)}(f) = M_u E^{\mathcal{A}}(f) = E^{\mathcal{A}}(uf)$ and $M_u E^{\varphi^{-1}(\mathcal{A})}(f) = uf$. It follows that $E^{\mathcal{A}}(uf) = uf$, and so $W(\chi_A) = u\chi_{\varphi^{-1}(A)} = uf \in L^0(\mathcal{A})$.

Now, let $E^{\varphi^{-1}(\mathcal{A})}(u\chi_A) = g \circ \varphi$ for some $g \in L^0(\mathcal{A})$. Since $u\chi_A = E^{\mathcal{A}}(u\chi_A)$, we obtain

$$\begin{aligned} W^*(\chi_A) &= hE^{\varphi^{-1}(\Sigma)}(u\chi_A) \circ \varphi^{-1} = hE^{\varphi^{-1}(\Sigma)} M_u(\chi_A) \circ \varphi^{-1} \\ &= hE^{\varphi^{-1}(\mathcal{A})}(u\chi_A) \circ \varphi^{-1} = h(g \circ \varphi) \circ \varphi^{-1} = hg \in L^0(\mathcal{A}). \end{aligned}$$

This completes the proof. ■

Corollary 2.9. *The following assertions hold.*

(a) *Let $\varphi^{-2}(\Sigma) \subseteq \Sigma$ be a complete σ-finite subalgebra and u, $h \in L^0(\varphi^{-1}(\Sigma))$. If $M_u E^{\varphi^{-1}(\Sigma)} = E^{\varphi^{-2}(\Sigma)} M_u$, then $L^2(\mathcal{A})$ reduces W.*

(b) *If $u \in L^0(\mathcal{A})$ and $L^2(\mathcal{A})$ reduces C_φ, then $L^2(\mathcal{A})$ reduces W.*

(c) *If $\sigma(u) = X$ and $L^2(\mathcal{A})$ reduces W, then $L^2(\mathcal{A})$ reduces C_φ.*

(d) *$L^2(\mathcal{A})$ reduces C_φ if and only if $h \in L^0(\mathcal{A})$ and $E^{\varphi^{-1}(\Sigma)} E^{\mathcal{A}} = E^{\varphi^{-1}(\mathcal{A})}$.*

(e) *$L^2(\mathcal{A})$ reduces M_u if and only if $u \in L^0(\mathcal{A})$.*

Proof. (a) Put $\mathcal{A} = \varphi^{-1}(\Sigma)$. Because u is $\varphi^{-1}(\Sigma)$-measurable, $E^{\varphi^{-1}(\Sigma)} M_u = M_u E^{\varphi^{-1}(\Sigma)}$. Now, the desired conclusion follows by Theorem 2.8.

(b) Let $f \in L^2(\Sigma)$. By Proposition 2.3(a), $E^{\mathcal{A}}(f) \circ \varphi = E^{\mathcal{A}}(f \circ \varphi)$. Hence $WE^{\mathcal{A}}(f) = uE^{\mathcal{A}}(f) \circ \varphi = uE^{\mathcal{A}}(f \circ \varphi) = E^{\mathcal{A}}(u.f \circ \varphi) = E^{\mathcal{A}} W(f)$.

(c) $L^2(\mathcal{A})$ reduces W, then $WE^{\mathcal{A}} = E^{\mathcal{A}} W$ and $u \in L^0(\mathcal{A})$. Thus, $uE^{\mathcal{A}}(f) \circ \varphi = E^{\mathcal{A}}(u.f \circ \varphi) = uE^{\mathcal{A}}(f \circ \varphi)$ for each $f \in L^2(\Sigma)$. Because $u > 0$, we have $E^{\mathcal{A}}(f) \circ \varphi = E^{\mathcal{A}}(f \circ \varphi)$, and so $C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}} C_\varphi$.

(d) In Theorem 2.8, put $u = 1$. Then $h \in L^0(\mathcal{A})$ and $E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}} = E^{\varphi^{-1}(\mathcal{A})}$. The converse follows from Theorem 2.5 and Corollary 2.7(d). This result is originally due to Burnap and Lambert [1, Theorem 5(b)].

(e) This follows from Theorem 2.6 and Theorem 2.8. ■

Example 2.10. Let $X = [-\frac{1}{2}, \frac{1}{2}]$, $d\mu = dx$, Σ be the Lebesgue sets, and let $\mathcal{A} \subseteq \Sigma$ be the σ-algebra generated by the symmetric sets about the origin. Let $0 < a \le \frac{1}{2}$ and $f \in L^2(\Sigma)$. Then

$$\int_{-a}^{a} E^{\mathcal{A}}(f)(x)dx = \int_{-a}^{a} f(x)dx = \int_{-a}^{a} \Big\{\frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}\Big\}dx$$
$$= \int_{-a}^{a} \frac{f(x)+f(-x)}{2}dx.$$

Thus, $E^{\mathcal{A}}(f)(x) = \frac{f(x)+f(-x)}{2}$. Therefore, by Corollary 2.9(e), $L^2(\mathcal{A})$ reduces M_u if and only if u is an even function.

Burnap and Lambert in [1] proved that if $\mathcal{M}(\neq L^2(\Sigma_B)$ for each $B \in \Sigma)$ reduces C_φ, then there is a nontrivial σ-algebra $\mathcal{A} \subseteq \Sigma$ such that $L^2(\mathcal{A})$ reduces C_φ. In the following theorem we extend this to the case of weighted composition operators.

Theorem 2.11. *Let (X, Σ, μ) be a probability space and let $M_u, C_\varphi \in B(L^2(\Sigma))$. If $\mathcal{M}$ is a reducing subspace for M_u and C_φ, then there is a σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ such that $L^2(\mathcal{A})$ reduces W.*

Proof. To prove the theorem, we adopt the method by Burnap and Lambert [1]. Let P be the orthogonal projection onto $\mathcal{M}$. Put

$$\mathcal{V} = \{M_\nu : \nu \in L^\infty(\Sigma_{\sigma(u)}) \text{ and } M_\nu P = PM_\nu\}.$$

Note that $I, M_u \in \mathcal{V}$ and so $\mathcal{V} \neq \emptyset$. Also $\mathcal{V}$ is a weakly closed C^*-algebra of all bounded multiplication operators on $L^2(\Sigma_{\sigma(u)})$ (see [1]). Put

$$\mathcal{A}_0 = \{A \in \Sigma_{\sigma(u)} : M_{\chi_A}P = PM_{\chi_A}\}.$$

Then by [5], $\mathcal{V} = \{M_\nu : \nu \in L^\infty(\mathcal{A}_0)\}$. Hence for $\nu \in L^\infty(\mathcal{A}_0)$ we have $M_\nu P = PM_\nu$ and so $M_{u\nu}P = PM_{u\nu}$. Since $C_\varphi P = PC_\varphi$, then by [1, Theorem 6], $C_\varphi M_{C_\varphi^*(u\nu)}P = C_\varphi PM_{C_\varphi^*(u\nu)}$. Then for $f \in L^2(\Sigma)$, we obtain

$$WM_{W^*(\nu)}P(f) = M_uC_\varphi M_{C_\varphi^*(u\nu)}P(f) = M_uC_\varphi PM_{C_\varphi^*(u\nu)}(f) = WPM_{W^*(\nu)}(f).$$

Then $C_\varphi M_{W^*(\nu)} P(f) = C_\varphi P M_{W^*(\nu)}(f)$ on $\sigma(u)$, because $\mathcal{N}(M_u) = \{f \in L^2(\Sigma) : f\chi_{\sigma(u)} = 0\}$. It follows that

$$M_{W^*(\nu)} P(f) = P M_{W^*(\nu)}(f) \quad \text{on } \sigma(u) \cap \sigma(h). \tag{2.1}$$

On the other hand, since $hE(u\nu)\circ\varphi^{-1}P(f) = P(hE(u\nu)\circ\varphi^{-1}f)$ and ν is supported on $\sigma(u)$, so (2.1) holds on X. Consequently, $M_{W^*(\nu)} \in \mathcal{V}$ and thus, $W^*(\nu) \in L^\infty(\mathcal{A}_0)$. Now, define the smallest sub-σ-algebra of Σ containing $\bigcup_{n=0}^\infty \varphi^{-n}(\mathcal{A}_0)$ by $\mathcal{A} = \bigvee_{n=0}^\infty \varphi^{-n}(\mathcal{A}_0)$. Note that $L^2(\mathcal{A})$ is generated by functions of the form $\chi_{\cap_{k=0}^n \varphi^{-k}(B_k)}$, where $B_k \in \mathcal{A}_0$ and $n \in \mathbb{N}$. Now, we claim that $L^2(\mathcal{A})$ is invariant for W and W^*. Let $f = \chi_{\cap_{k=0}^n \varphi^{-k}(B_k)}$. Then $W(f) = u\chi_{\cap_{k=0}^n \varphi^{-k-1}(B_k)} \in L^2(\mathcal{A})$ because $u \in L^\infty(\mathcal{A}_0) \subseteq L^\infty(\mathcal{A})$ and $f\circ\varphi \in L^2(\mathcal{A})$. Moreover,

$$\begin{aligned} W^*(f) &= hE(u\chi_{\cap_{k=0}^n \varphi^{-k}(B_k)})\circ\varphi^{-1} \\ &= (hE(u\chi_{B_0})\circ\varphi^{-1})\chi_{\cap_{k=1}^n \varphi^{-k-1}(B_k)} = (W^*(\chi_{B_0}))\chi_{\cap_{k=1}^n \varphi^{-k-1}(B_k)}. \end{aligned}$$

Since $W^*(\chi_{B_0}) \in L^\infty(\mathcal{A}_0) \subseteq L^\infty(\mathcal{A})$, so $W^*(f) \in L^2(\mathcal{A})$. Consequently, $L^2(\mathcal{A})$ reduces W.

■

References

[1] C. Burnap and A. Lambert, Reducibility of composition operators on L^2, *J. Math. Anal. Appl.*, **178** (1993), 87–101.

[2] J. T. Campbell and J. Jamison, On some classes of weighted composition operators, *Glasgow Math. J.*, **32** (1990), 87–94.

[3] J. T. Campbell and W. E. Hornor, Localising and seminormal composition operators on L^2, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), 301–316.

[4] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1990.

[5] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1969.

[6] P. Dodds, C. Huijsmans and B. de Pagter, Characterizations of conditional expectation-type operators, *Pacific J. Math.*, **141** (1990), 55–77.

[7] J. Herron, Weighted conditional expectation operators, *Oper. Matrices*, **5** (2011), 107–118.

[8] T. Hoover, A. Lambert and J. Quinn, The Markov process determined by a weighted composition operator, *Studia Math.*, **72** (1982), 225–235.

[9] A. Lambert and B. M. Weinstock, Descriptions of conditional expectations induced by non-measure preserving transformations, *Proc. Amer. Math. Soc.*, **123** (1995), 897–903.

[10] M. M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

M. R. Jabbarzadeh, Faculty of mathematical sciences, University of Tabriz, P.O. Box: 5166615648, Tabriz, Iran; *e-mail*: mjabbar@tabrizu.ac.ir

M. Jafari Bakhshkandi, Faculty of mathematical sciences, University of Tabriz, P.O. Box: 5166615648, Tabriz, Iran; *e-mail*: m_jafari@tabrizu.ac.ir