Stability constants for weighted composition operators on $L^p(\Sigma)$

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Abstract

In this note we give an explicit formula for the Moore-Penrose inverse $W^\dagger$ of a weighted composition operator W on $L^2(\Sigma)$ and then we obtain the stability constant K_W of W on $L^p(\Sigma)$, where $1 \leq p \leq \infty$. Moreover, we determine, under certain conditions, the essential norm of W acting on $L^\infty(\Sigma)$.

1 Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For a sub-σ-algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation mapping, associated with $\mathcal{A}$ is a mapping $E^{\mathcal{A}} : f \to E^{\mathcal{A}} f$, defined for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ $(1 \leq p)$, where $E^{\mathcal{A}} f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

Let f be a real valued Σ-measurable function on X, if $\mu(\{x : E^{\mathcal{A}}(f^+(x)) = E^{\mathcal{A}}(f^-(x)) = \infty\}) = 0$, then we define $E^{\mathcal{A}}(f) := E^{\mathcal{A}}(f^+) - E^{\mathcal{A}}(f^-)$. In the case of complex-valued function f, if $\mu(\{x : E^{\mathcal{A}}((\operatorname{Im} f)^+(x)) = E^{\mathcal{A}}((\operatorname{Im} f)^-(x)) = \infty\}) = 0$ and $\mu(\{x : E^{\mathcal{A}}((\operatorname{Re} f)^+(x)) = E^{\mathcal{A}}((\operatorname{Re} f)^-(x)) = \infty\}) = 0$, then $E^{\mathcal{A}} := E^{\mathcal{A}}(\operatorname{Re} f) + iE^{\mathcal{A}}(\operatorname{Im} f)$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is an orthogonal projection and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. For an introduction to as well as for a deep

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study of conditional expectation operator, we refer the reader to the Lambert papers, for example [11] and the monograph [15].

Let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, that is, φ is non-singular. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued, which is, equivalent to the fact that $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite. In the setting of L^p-spaces the so called conditional expectation operator $E^{\varphi^{-1}(\Sigma)}$ with respect to $\varphi^{-1}(\Sigma)$ plays an important role. If there is no possibility of confusion, we write Ef in place of $E^{\varphi^{-1}(\Sigma)}f$. Denote the complement of B by B^c. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^p(\Sigma)$ with $1 \leq p \leq \infty$, induced by u and the non-singular measurable function φ is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on $L^p(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively. It is a classical fact that $W \in \mathrm{B}(L^2(\Sigma))$, the C^*-algebra of all bounded linear operators on $L^2(\Sigma)$, if and only if $J := hE(|u|^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$ and $W \in \mathrm{B}(L^\infty(\Sigma))$ if and only if $u \in L^\infty(\Sigma)$ (see [6]). Throughout this paper we assume that $\varphi : X \to X$ is a non-singular transformation and $u \geq 0$.

Now, let $\mathcal{H}$ be a complex Hilbert space. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and range of an operator $T \in \mathrm{B}(\mathcal{H})$. Let $T \in \mathrm{B}(\mathcal{H})$ have closed range. Then the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in \mathrm{B}(\mathcal{H})$ which satisfies $TT^\dagger T = T$, $T^\dagger T T^\dagger = T^\dagger$, $(TT^\dagger)^* = TT^\dagger$ and $(T^\dagger T)^* = T^\dagger T$. For other important properties of $T^\dagger$, see [4, 13].

The study of Hyers-Ulam stability of mappings has a quite long and rich history (see [7, 20]). The Hyers-Ulam stability of linear operators was considered for the first time in the paper by Takagi, Miura and Takahasi in [17]. Let $\mathcal{X}$ be a Banach space. We recall that $T \in B(\mathcal{X})$ has the Hyers-Ulam stability, if there exists $K > 0$ such that, for any $f \in \mathcal{X}$, there exists $f_0 \in \mathcal{N}(T)$ with $\|f - f_0\| \leq K\|Tf\|$. We call K a Hyers-Ulam stability (HUS) constant for T, and denote the infimum of all HUS constants for T by K_T. By [17, Theorem 2.1], $T \in B(\mathcal{X})$ has the Hyers-Ulam stability if and only if T has closed range if and only if $\widetilde{T}^{-1}$ is bounded, where $\widetilde{T}$ is the one-to-one operator from the quotient Banach space $\mathcal{X}/\mathcal{N}(T)$ onto $\mathcal{R}(T)$ defined by $\widetilde{T}(f + \mathcal{N}(T)) = Tf$. Moreover, in this case they proved that $K_T = \|\widetilde{T}^{-1}\|$. After then, Hirasawa and Miura [5] gave some necessary and sufficient conditions under which a closed operator in a Hilbert space has the Hyers-Ulam stability. They showed that $K_T = \gamma(T)^{-1}$, where $\gamma(T)$ is the reduced minimum modulus of T. Also Rakocevic in [14] shows that $\gamma(T)^{-1} = \|T^\dagger\|$. Thus $K_T = \|T^\dagger\|$. In [8], Hyers-Ulam stability of weighted composition operators acting on L^p-spaces with $1 \leq p < \infty$ have been studied under certain conditions. Some good sources about the Hyers-Ulam stability of weighted composition operators acting between various function spaces are [18] and [19].

2 $W^\dagger$ and HUS Constants and Essential Norm of W

Let $0 \leq u \in L^0(\Sigma)$. Then the multiplication operator M_u has closed range on $L^2(\Sigma)$ if and only if u is bounded away from zero on $\sigma(u)$ (see [2]). Since $\|Wf\| = \|\sqrt{J}f\|$ (see [6]), so W has closed range on $L^2(\Sigma)$ if and only if J is bounded away from zero on $\sigma(J)$. Let $W \in \mathrm{B}(L^2(\Sigma))$ have closed range. A result of Hoover, Lambert and Quinn [6] shows that the adjoint W^* of $W \in B(L^2(\Sigma))$ is given by $W^*f = hE(uf) \circ \varphi^{-1}$. Put $S = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$. Thus $S \in \mathrm{B}(L^2(\Sigma))$, since $\frac{\chi_{\sigma(J)}}{J} \in L^\infty(\Sigma)$. Then we have

$$\begin{aligned} WSWf &= u(SWf) \circ \varphi \\ &= u(\frac{\chi_{\sigma(J)}}{J} hE(u^2 f \circ \varphi) \circ \varphi^{-1}) \circ \varphi \\ &= u(\frac{\chi_{\sigma(J)}}{J} hE(u^2)\varphi^{-1} f) \circ \varphi \\ &= u\chi_{\sigma(J\circ\varphi)} f \circ \varphi. \end{aligned}$$

Since $u \geq 0$ and $\sigma(h \circ \varphi) = X$, hence $\sigma(J \circ \varphi)) = \sigma(h \circ \varphi E(u^2)) = \sigma(E(u^2)) \supseteq \sigma(u)$. It follows that

$$\begin{aligned} WSWf &= (u\chi_{\sigma(u)})\chi_{\sigma(E(u^2))} f \circ \varphi \\ &= (u\chi_{\sigma(u)}) f \circ \varphi = Wf, \end{aligned}$$

$$\begin{aligned} SWSf &= \frac{\chi_{\sigma(J)}}{J} hE(uWSf) \circ \varphi^{-1} \\ &= \frac{\chi_{\sigma(J)}}{J} hE(u^2(Sf) \circ \varphi) \circ \varphi^{-1} \\ &= \frac{\chi_{\sigma(J)}}{J} h(E(u^2)(Sf) \circ \varphi) \circ \varphi^{-1} \\ &= \frac{\chi_{\sigma(J)}}{J} (hE(u^2) \circ \varphi^{-1}) Sf \\ &= \chi_{\sigma(J)} Sf = Sf, \end{aligned}$$

$$\begin{aligned} (WS)^* &= (M_u E M_{\frac{u}{E(u^2)}})^* \\ &= M_{\frac{u}{E(u^2)}} E M_u \\ &= M_u E M_{\frac{u}{E(u^2)}} = WS, \end{aligned}$$

and $SW = M_{\chi_{\sigma(J)}} = (SW)^*$. These observations establish the following theorem.

Theorem 2.1. *Let $W \in B(L^2(\Sigma))$ have closed range. Then $W^\dagger = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$. In particular, if φ is a measure-preserving map, then $C_\varphi^\dagger = C_\varphi^* = E(\cdot) \circ \varphi^{-1}$.*

Lemma 2.2. *Let* $1 \leq p < \infty$ *and* $W \in B(L^p(\Sigma))$*. Then*

$$\|f + \mathcal{N}(W)\|^p = \int_{\sigma(J)} |f|^p d\mu.$$

Proof. Since for each $f \in L^p(\Sigma)$, $\|Wf\| = \|\sqrt[p]{J}f\|$, where $J = hE(u^p) \circ \varphi^{-1}$, it follows that

$$\mathcal{N}(W) = \mathcal{N}(M_{\sqrt[p]{J}}) = \{f \in L^p(\Sigma) : f_{|\sigma(J)} = 0\} = L^p(\sigma(J)^c).$$

Let $g \in \mathcal{N}(W)$. Then we have

$$\int_{\sigma(J)} |f|^p d\mu \leq \inf_{g \in \mathcal{N}(W)} \int_X |f + g|^p d\mu = \|f + \mathcal{N}(W)\|^p.$$

On the other hand, since for each $f \in L^p(\Sigma)$, $\chi_{\sigma(J)^c} f \in \mathcal{N}(W)$, then we get that

$$\|f + \mathcal{N}(W)\|^p \leq \|f - \chi_{\sigma(J)^c} f\|^p = \|f\chi_{\sigma(J)}\|^p = \int_{\sigma(J)} |f|^p d\mu. \qquad \blacksquare$$

Theorem 2.3. *Let* $1 \leq p < \infty$ *and* $W \in B(L^p(\Sigma))$ *have closed range. Then* $K_W = \frac{1}{R} = \|\frac{\chi_{\sigma(J)}}{\sqrt[p]{J}}\|_\infty$*, where* $R = \sup\{r > 0 : J_{|\sigma(J)} \geq r^p\}$*.*

Proof. First we show that $K_W = \frac{1}{R}$. Since W has closed range, hence J is bounded away from zero on $\sigma(J)$. Let $J_{|\sigma(J)} \geq r^p$ for some $r > 0$ and $\tilde{f} = f + \mathcal{N}(W) \in X/\mathcal{N}(W)$. Then by Lemma 2.2, we have

$$\begin{aligned}\|f + \mathcal{N}(W)\|^p &= \int_{\sigma(J)} |f|^p d\mu \leq \frac{1}{r^p} \int_{\sigma(J)} |\sqrt{J} f|^p d\mu \\ &\leq \frac{1}{r^p} \int_X |\sqrt{J} f|^p d\mu = \frac{1}{r^p} \|Wf\|^p.\end{aligned}$$

It follows that $\|\tilde{f}\| \leq \frac{1}{r}\|\widetilde{W}\tilde{f}\|$, and so $\|\widetilde{W}^{-1}\| \leq \frac{1}{r}$. Now, by Takagi-Miura-Takahasi equality $K_W = \|\widetilde{W}^{-1}\|$ (see [17]), if r is taken over all numbers satisfying $J_{|\sigma(J)} \geq r^p$, we obtain $K_W \leq \frac{1}{R}$. If $\|\widetilde{W}^{-1}\| < \frac{1}{R}$, then, by definition of R, there exists $A \subseteq \sigma(J)$ with $0 < \mu(A) < \infty$ such that $J_{|A} < \frac{1}{\|\widetilde{W}^{-1}\|^p}$. Put $f_0 = \chi_A/\mu(A)^{1/p}$. Then $\|Wf_0\| < \frac{1}{\|\widetilde{W}^{-1}\|}$, and so

$$1 = \|f_0\| = (\int_{\sigma(J)} |f_0|^p d\mu)^{\frac{1}{p}} = \|f_0 + \mathcal{N}(W)\| \leq \|\widetilde{W}^{-1}\| \, \|Wf_0\| < 1.$$

But this is a contradiction, and hence $K_W = \frac{1}{R}$. Finally

$$\begin{aligned}R = \sup\{r > 0 : J_{|\sigma(J)} \geq r^p\} &= \sup\{r > 0 : \frac{\chi_{\sigma(J)}}{J} \leq \frac{1}{r^p}\} \\ &= \frac{1}{\inf\{r > 0 : \frac{\chi_{\sigma(J)}}{J} \leq r^p\}} \\ &= \frac{1}{\|\frac{\chi_{\sigma(J)}}{\sqrt[p]{J}}\|_\infty}.\end{aligned}$$

So, $K_W = \frac{1}{R} = \|\frac{\chi_{\sigma(J)}}{\sqrt[p]{J}}\|_\infty$. $\blacksquare$

Lemma 2.4. *[9, Proposition 2.1(b)] For $w \in L^0(\Sigma)$, $T = EM_w$ defines a bounded linear operator on $L^2(\Sigma)$ if and only if $E(w^2) \in L^\infty(\varphi^{-1}(\Sigma))$. In this case $\|T\| = \sqrt{\|E(w^2)\|_\infty}$.*

Corollary 2.5. *Let $W \in B(L^2(\Sigma))$ have closed range. Then $K_W = \|\frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}}\|_\infty$. Moreover, if $u \in L^0(\varphi^{-1}(\Sigma))$, then $K_W = \|\frac{\chi_{\sigma(u)}}{\sqrt{J\circ\varphi}}\|_\infty$.*

Proof. For each $f \in L^2(\Sigma)$, we have

$$\begin{aligned}
\|W^\dagger f\|^2 = \int_X |W^\dagger f|^2 d\mu &= \int_X |h\frac{\chi_{\sigma(J)}}{J} E(uf)\circ\varphi^{-1}|^2 d\mu \\
&= \int_X h|\sqrt{h}\frac{\chi_{\sigma(J)}}{J} E(uf)\circ\varphi^{-1}|^2 d\mu \\
&= \int_X |\sqrt{h\circ\varphi}\frac{\chi_{\sigma(J)\circ\varphi}}{J\circ\varphi} E(uf)|^2 d\mu \\
&= \int_X |E(u\frac{\chi_{\sigma(E(u^2)}}{J\circ\varphi}\sqrt{h\circ\varphi} f)|^2 d\mu \\
&= \|EM_w f\|^2,
\end{aligned}$$

where $w = u\frac{\chi_{\sigma(E(u^2)}}{J\circ\varphi}\sqrt{h\circ\varphi}$. Hence by Lemma 2.3 we get that

$$\|W^\dagger\| = \sqrt{\|E(w^2)\|_\infty} = \|\frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}}\|_\infty.$$

Now, the desired conclusion follows from the equality $K_W = \|W^\dagger\|$. Moreover, If u is $\varphi^{-1}(\Sigma)$-measurable, then $E(u) = u$, hence $\sigma(J\circ\varphi) = \sigma(E(u^2)) = \sigma(E(u)) = \sigma(u)$. So $K_W = \|\frac{\chi_{\sigma(u)}}{\sqrt{J\circ\varphi}}\|_\infty$. ∎

Corollary 2.6. *(i) Let $C_\varphi \in B(L^2(\Sigma))$ have closed range. Then*

$$K_{C_\varphi} = \sup_{0<\mu(A)<\infty} \frac{\mu(A)}{\int_A \sqrt{h\circ\varphi}\,d\mu}.$$

(ii) If φ is a measure-preserving map, then $K_{C_\varphi} = \|C_\varphi\| = 1$.

At this stage, we determine the stability constant K_W of W on $L^\infty(\Sigma)$.

Lemma 2.7. *Assume $\varphi(\Sigma) \subseteq \Sigma$ and $\mu(\varphi(A)) = 0$ for every null set $A \in \Sigma$, and let $W \in B(L^\infty(\Sigma))$. Then for each $f \in L^\infty(\Sigma)$,*

$$\|f + \mathcal{N}(W)\| = \operatorname{ess\,sup}\{|f(x)| : x \in \varphi(\sigma(u))\}.$$

Proof. Pick $g \in \mathcal{N}(W)$ and take $\alpha = \operatorname{ess\,sup}\{|f(x)| : x \in \varphi(\sigma(u))\}$. Since $\mathcal{N}(W) = \{f \in L^\infty(\Sigma) : (f \circ \varphi)_{|_{\sigma(u)}} = 0\} = \{f \in L^\infty(\Sigma) : f_{|_{\varphi(\sigma(u))}} = 0\} = L^\infty(\varphi(\sigma(u))^c)$, hence $g_{|\varphi(\sigma(u))} = 0$. It follows that

$$\alpha = \operatorname{ess\,sup}\{|(f+g)(x)| : x \in \varphi(\sigma(u))\} \leq \|f+g\|_\infty,$$

and so $\alpha \leq \inf\{\|f+g\|_\infty : g \in \mathcal{N}(W)\} = \|f + \mathcal{N}(W)\|$. For the opposite inequality, put $g = -f\chi_{\varphi(\sigma(u))^c}$. Then $g \in \mathcal{N}(W)$ and

$$\|f+g\|_\infty = \operatorname{ess\,sup}\{|f(1-\chi_{\varphi(\sigma(u))^c})(x)| : x \in \varphi(\sigma(u))\} = \alpha.$$

Thus, $\|f + \mathcal{N}(W)\| \leq \alpha$. ∎

Theorem 2.8. *Let $W \in B(L^\infty(\Sigma))$. If $\varphi(\Sigma) \subseteq \Sigma$ and $\mu(\varphi(A)) = 0$ for every null set $A \in \Sigma$, then W has Hyers-Ulam stability if and only if there exists a positive constant r such that $\varphi(U(r)) = \varphi(\sigma(u))$, where $U(r) := \{x \in X : |u(x)| \geq r\}$. Moreover, in this case $K_W = \frac{1}{R}$, where $R = \sup\{r > 0 : \varphi(U(r)) = \varphi(\sigma(u))\}$.*

Proof. Suppose that there exists an $r > 0$ such that $\varphi(U(r)) = \varphi(\sigma(u))$. Then by Lemma 2.7 we have

$$\begin{aligned}
\|f + \mathcal{N}(W)\| &= \operatorname{ess\,sup}\{|f(x)| : x \in \varphi(U(r))\} \\
&= \operatorname{ess\,sup}\{|f \circ \varphi(x)| : x \in U(r)\} \\
&= \operatorname{ess\,sup}\{\frac{1}{|u(x)|}|Wf(x)| : x \in U(r)\} \\
&\leq \frac{1}{r}\ \operatorname{ess\,sup}\{|Wf(x)| : x \in U(r)\} \\
&\leq \frac{1}{r}\ \|Wf\|_\infty.
\end{aligned}$$

It follows that $\widetilde{W}^{-1}$ from $\mathcal{R}(W)$ into $L^\infty(\Sigma)/\mathcal{N}(W)$ is bounded and $\|\widetilde{W}^{-1}\| \leq \frac{1}{r}$. Thus $\|\widetilde{W}^{-1}\| \leq \frac{1}{R}$. Conversely, suppose that W has closed range. Then $\widetilde{W}^{-1}$ is bounded [17, Theorem 2.1]. Assume $\|\widetilde{W}^{-1}\| < \frac{1}{r}$ for some $r > 0$. We show that $\varphi(\sigma(u)) = \varphi(U(r))$. For this, we assume that $\varphi(\sigma(u)) \neq \varphi(U(r))$. Take $A = \varphi(\sigma(u)) \setminus \varphi(U(r))$. Put $f_0 = \chi_A$. Then $|Wf_0| \leq |u|\chi_{U(r)^c} \leq r$. Thus, we get that

$$\begin{aligned}
1 &= \operatorname{ess\,sup}\{|f_0(y)| : y \in \varphi(\sigma(u)))\} \\
&= \|f_0 + \mathcal{N}(W)\| \leq \|\widetilde{W}^{-1}\|\ \|Wf_0\| < 1.
\end{aligned}$$

But this is a contradiction. Finally, by a similar argument we show that $\frac{1}{R} \leq \|\widetilde{W}^{-1}\|$. Suppose, to the contrary, $\|\widetilde{W}^{-1}\| < \beta < \frac{1}{R}$ for some $\beta > 0$. Then $\varphi(U(\frac{1}{\beta})) \neq \varphi(\sigma(u))$. Now, take $B = \varphi(\sigma(u)) \setminus \varphi(U(\frac{1}{\beta}))$ and put $f_1 = \chi_B$. Then $\|Wf_1\| \leq \frac{1}{\beta}$, and so $1 = \|f_1 + \mathcal{N}(W)\| \leq \|\widetilde{W}^{-1}\|\ \|Wf_1\| < 1$. Thus, $K_W = \|\widetilde{W}^{-1}\| = \frac{1}{R}$. ∎

Let $\mathcal{K}$ be the set of all compact operators on $L^\infty(\Sigma)$. For $W \in B(L^\infty(\Sigma))$ the essential norm of W means the distance from W to $\mathcal{K}$ in the operator norm, namely $\|W\|_e = \inf\{\|W - S\| : S \in \mathcal{K}\}$. Many people have computed the essential norm of (weighted) composition operators on various function spaces. In [10], the essential norm of W on $L^p(\Sigma)$ with $1 < p < \infty$ has been computed. At this stage, we determine the essential norm of W on $L^\infty(\Sigma)$. Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as $X = Z \cup Y$, where $Z = \cup\{A_j : j \in \mathbb{N}\}$ is a union of pairwise disjoint atoms and $Y \in \Sigma$, being disjoint from each A_j, is non-atomic (see [21]). Since Σ is σ-finite, so $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$. Note that $\varphi(Y)$ is not necessarily subset of Y, but $\varphi(Z)$ is essentially subset of Z. In other words, if $A \notin \{A_i : i \in \mathbb{N}\}$, then $\varphi^{-1}(A)$ is not an atom (see [3]). Also, every $L^\infty(\Sigma)$-function is constant on any atom in Z.

Theorem 2.9. *Assume $\varphi(\Sigma) \subseteq \Sigma$ and $\mu(\varphi(A)) = 0$ for every null set $A \in \Sigma$, and let $W \in B(L^\infty(\Sigma))$. The essential norm of W is given by*

$$\|W\|_e = \inf\{r > 0 : \varphi(G_r) \text{ consists of only finitely many atoms}\}, \tag{2.1}$$

where $G_r = \{x \in X : |u(x)| \geq r\}$.

Proof. Denote the right side of (2.1) by α. We first show that $\|W\|_e \geq \alpha$. If $\alpha = 0$, there is nothing to prove, so we assume that $\alpha > 0$. Take $\varepsilon > 0$ arbitrarily. The definition of α implies that $\varphi(G_{\alpha-\varepsilon/2})$ either contains a non-atomic subset or has infinitely many atoms. So we can find mutually disjoint measurable subsets $\{F_n\}_n \subseteq Y \cap \varphi(G_{\alpha-\varepsilon/2})$ or $\{B_n\}_n \subseteq \{A_i \cap \varphi(G_{\alpha-\varepsilon/2}) : i \in \mathbb{N}\}$. For $\{C_n\}_n \subseteq \{F_n, B_n\}_n$, put $f = \chi_{C_n}$. Then $\|f_n\|_\infty = 1$ and $f_n \to 0$ weakly (see [12, p. 54-55]). Now, take a compact operator T on $L^\infty(\Sigma)$ such that $\|W - T\| < \|W\|_e + \frac{\varepsilon}{2}$. Then we have

$$\begin{aligned}
\|W\|_e &> \|W - T\| - \frac{\varepsilon}{2} \geq \|Wf_n - Tf_n\|_\infty - \frac{\varepsilon}{2} \\
&\geq \|(Wf_n)\chi_{G_{\alpha-\varepsilon/2}}\|_\infty - \|Tf_n\|_\infty - \frac{\varepsilon}{2} \\
&= \|u\chi_{\varphi^{-1}(C_n)\cap G_{\alpha-\varepsilon/2}}\|_\infty - \|Tf_n\|_\infty - \frac{\varepsilon}{2} \\
&\geq (\alpha - \frac{\varepsilon}{2}) - \|Tf_n\|_\infty - \frac{\varepsilon}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|Tf_n\|_\infty \to 0$. Hence $\|W\|_e \geq \alpha - \varepsilon$. Since ε was arbitrary, we obtain $\|W\|_e \geq \alpha$.

For the opposite inequality, take ε arbitrarily. By definition of α, there is $m \in \mathbb{N}$ such that $\varphi(G_{\alpha+\varepsilon}) = \bigcup_{i=1}^m A_{j_i}$. Put $v = u\chi_{G_{\alpha+2\varepsilon}}$. Note that if $\varphi(G_{\alpha+\varepsilon}) = \emptyset$, then $G_{\alpha+2\varepsilon} \subseteq G_{\alpha+\varepsilon} = \emptyset$, and so $v = 0$. Take $F_i = \varphi^{-1}(A_{j_i}) \cap G_{\alpha+\varepsilon}$ and $v_i = u\chi_{G_{\alpha+2\varepsilon}\cap F_i}$. Since $G_{\alpha+2\varepsilon} \subseteq \varphi^{-1}(\varphi(G_{\alpha+\varepsilon})) = \bigcup_{i=1}^m \varphi^{-1}(A_{j_i})$, hence $\cup_{i=1}^m(G_{\alpha+2\varepsilon} \cap$

$F_i) = G_{\alpha+2\varepsilon} \cap \varphi^{-1}(\cup_{i=1}^m A_i) \cap G_{\alpha+\varepsilon} = G_{\alpha+2\varepsilon}$. It follows that $v = \sum_{i=1}^m v_i$. Moreover, since $\varphi(F_i) \subseteq A_i$, then for each $f \in L^\infty(\Sigma)$, $f(\varphi(F_i)) = f(A_i)$ is constant. This implies that $vC_\varphi f = \sum_{i=1}^m f(A_i)v_i$. Hence vC_φ has finite rank and so is compact. Then we have

$$\|W - vC_\varphi\| = \|M_{u-v}C_\varphi\| \leq \|u - v\|_\infty = \|(1 - \chi_{G_{\alpha+2\varepsilon}})u\|_\infty \leq \alpha + \varepsilon.$$

It follows that $\|W\|_e \leq \alpha$. ∎

Corollary 2.10. *Assume $\varphi(\Sigma) \subseteq \Sigma$ and $\mu(\varphi(A)) = 0$ for every null set $A \in \Sigma$, and let $W \in B(L^\infty(\Sigma))$. Then W is compact if and only if for each $\varepsilon > 0$, $\varphi(\{x \in X : |u(x)| \geq \varepsilon\})$ consists of only finitely many atoms.*

Note that, for $1 \leq p < \infty$, Chan in [1] obtains a characterization of the weighted composition operators on $L^p(\Sigma)$ that are compact. He proved that $W \in B(L^p(\Sigma))$ is compact if and only if for any $\varepsilon > 0$ the set $\{x \in X : h(x) (E(|u|^p) \circ \varphi^{-1})(x) \geq \varepsilon\}$ consists essentially of finitely many atoms. The same characterization is contained in a paper by Takagi [16].

Example 2.11. (a) Let $w := \{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the space $\ell^2(w) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u(j)\}_{j=1}^\infty$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation. Direct computations show that (see [10])

$$h(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j;$$

$$E(f)(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j};$$

$$J(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} (u(j))^2 m_j.$$

Thus either $\sigma(J)^c = \{k \in \mathbb{N} : \varphi^{-1}(k) = \emptyset \text{ or } u(\{\varphi^{-1}(k)\}) = \{0\}\}$. Hence $\sigma(J) = \{n \in \mathbb{N} : \varphi^{-1}(\{n\}) \cap \sigma(u) \neq \emptyset\} = \varphi(\sigma(u))$. It follows that $W \in B(\ell^2(w))$ has closed range if and only if

$$\inf\{\frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} (u(j))^2 m_j;\ k \in \sigma(J)\} > 0.$$

So by Theorem 2.4, $\|W^\dagger\| = \frac{1}{\sqrt{\alpha}}$, where

$$\alpha := \inf\{\frac{1}{m_{\varphi(k)}} \sum_{j \in \varphi^{-1}(\varphi(k))} (u(j))^2 m_j;\ k \in \sigma(J \circ \varphi)\}.$$

Note that $\sigma(J \circ \varphi) = \sigma(E(u^2)) = \{k \in \mathbb{N} : u(\{\varphi^{-1}(\varphi(k))\}) \neq \{0\}\}$. In particular, if for each $k \in \mathbb{N}$, $\varphi^{-1}(k) \neq \emptyset$, equivalently $h > 0$, then $\|C_\varphi^\dagger\| = \frac{1}{\sqrt{\beta}}$, where

$$\beta = \inf\{\frac{1}{m_{\varphi(k)}} \sum_{j \in \varphi^{-1}(\varphi(k))} m_j;\ k \in \mathbb{N}\}.$$

(b) Let $X = (0,1)$ equipped with the Lebesgue measure μ on the Lebesgue measurable subsets. Set $u(x) = \sqrt{x}$ and let $\varphi : X \to X$ be defined by

$$\varphi(x) = \begin{cases} 2x & 0 < x < \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x < 1. \end{cases}$$

Direct computations show that

$$J(x) = \frac{1}{2}\left(u^2(\frac{x}{2}) + u^2(1 - \frac{x}{2})\right) = \frac{1}{2}$$

and for each $f \in L^2(\Sigma)$,

$$(W^\dagger f)(x) = \sqrt{\frac{x}{2}} f(\frac{x}{2}) + \sqrt{\frac{2-x}{2}} f(1 - \frac{x}{2}).$$

Thus W has closed range with $\|W\| = \frac{\sqrt{2}}{2}$ and $\|W^\dagger\| = \sqrt{2}$.

Here, there are a few examples to show that some of our results may be not true without some assumptions.

Example 2.12. (a) Take $X = [0,1]$, $\Sigma = \{\emptyset, X\}$, $\mu(X) = 1$, $\varphi(x) = \frac{x}{2}$, and $u = 1$. Here $\varphi(\Sigma)$ is not a subset of Σ , $G_1 = \{x : |u(x)| \geq 1\} = X$, and $\varphi(X) = [0, \frac{1}{2}]$ does not consist of finitely many atoms. But since $L^\infty(\Sigma)$ is finite dimensional, so W is compact operator.

(b) Consider $X = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$. Let E denote the set of even numbers and define $\mu(A) = \text{card}(A \cap E)$, where A is a subset of $\mathbb{N}$. Define $\varphi(n) = 2n$ for every $n \in \mathbb{N}$. It is clear that $\varphi(\Sigma) \subseteq \Sigma$. Consider $u = \chi_{E^c}$. Then $G_1 = E^c$ and $\varphi(E^c)$ is an infinite subset of even numbers, but $W = 0$ is compact.

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