



## Centered Operators Via Moore-Penrose Inverse and Aluthge Transformations

M. R. Jabbarzadeh<sup>a</sup>, M. Jafari Bakhshkandi<sup>a</sup>

<sup>a</sup>Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 5166615648, Tabriz, Iran

**Abstract.** In this paper, we obtain some characterizations of centered and binormal operators via Moore-Penrose inverse and Aluthge transform.

### 1. Introduction and Preliminaries

Let  $B(\mathcal{H})$  denote the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $\mathcal{H}$ . We write  $\mathcal{N}(T)$  and  $\mathcal{R}(T)$  for the null-space and the range of an operator  $T \in B(\mathcal{H})$ , respectively. Recall that for  $T \in B(\mathcal{H})$ , there is a unique factorization  $T = U|T|$ , where  $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$ ,  $U$  is a partial isometry, i.e.  $UU^*U = U$  and  $|T| = (T^*T)^{1/2}$  is a positive operator. This factorization is called the polar decomposition of  $T$ . As a consequence,  $U^*U|T| = |T|$ . Also, it is a classical fact that the polar decomposition of  $T^*$  is  $U^*|T|$ , and so  $UU^*|T^*| = |T^*|$ .

In [8] Morrel and Muhly introduced the concept of a centered operator. An operator  $T$  on a Hilbert space  $\mathcal{H}$  is said to be centered if the doubly infinite sequence  $\{T^n T^{*n}, T^{*m} T^m : n, m \geq 0\}$  consists of mutually commuting operators. It is shown in [4] that if  $T = U|T|$  is an operator on  $\mathcal{H}$  such that for each  $n \in \mathbb{N}$ ,  $T^n$  has polar decomposition  $U_n|T^n|$ , then  $T$  is centered if and only if  $U_n = U^n$  for each  $n \in \mathbb{N}$ .

Associated with  $T \in B(\mathcal{H})$  there is a useful related operator  $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ , called the Aluthge transform of  $T$  as it has been studied by Aluthge in [1]. Binormality of operators was defined by Campbell in [3]. An operator  $T$  is said to be binormal or weakly centered [9], if  $[|T|, |T^*|] = 0$ , where  $[A, B] = AB - BA$  for operators  $A$  and  $B$ . Let  $T \in B(\mathcal{H})$  have closed range. Then the Moore-Penrose inverse of  $T$ , denoted by  $T^\dagger$ , is the unique operator  $T^\dagger \in B(\mathcal{H})$  which satisfies  $TT^\dagger T = T$ ,  $T^\dagger T T^\dagger = T^\dagger$ ,  $(TT^\dagger)^* = TT^\dagger = P_{\mathcal{R}(T)}$  and  $(T^\dagger T)^* = T^\dagger T = P_{\mathcal{R}(T^\dagger)}$ , where the  $P_{\mathcal{M}}$  means the orthogonal projection onto a closed subspace  $\mathcal{M}$ .

In this paper, we study the centered and binormal bounded linear operators on a Hilbert space  $\mathcal{H}$  via Moore-Penrose inverse and Aluthge transformation. The work is organized as follows. In section 2, firstly, we give the polar decomposition of  $T^\dagger$ , and then we show that  $T^\dagger$  is centered if and only if  $T$  is centered. Secondly, we introduce the notion  $\dagger$ -Aluthge transformation  $\widetilde{T}^{(\dagger)}$  of  $T$  by setting  $\widetilde{T}^{(\dagger)} = (\widetilde{T}^\dagger)^\dagger$ . We show

2010 *Mathematics Subject Classification.* Primary 47B33; Secondary 47B38.

*Keywords.* Aluthge transform, Moore-Penrose inverse, binormal operator, centered operators.

Received: 05 May 2016; Revised: 04 July 2016; Accepted: 30 December 2016

Communicated by Dragana Cvetković Ilić

*Email addresses:* mjabbar@tabrizu.ac.ir (M. R. Jabbarzadeh), m\_jafari@tabrizu.ac.ir (M. Jafari Bakhshkandi)

that if  $T$  is a closed range binormal operator, then the  $*$ -Aluthge and  $\dagger$ -Aluthge transformations (see [11]) coincide. Also, we show that the reverse order law holds for  $|T|$  and  $|T^*|$ ; i.e.  $(|T||T^*|)^\dagger = |T^*|^\dagger |T|^\dagger$ , whenever  $T$  is a closed range binormal operator. Finally, we give the polar decomposition of powers of  $\widetilde{T}$  and then we find some conditions under which  $\widetilde{T}$  be centered. Also we show that if  $T$  is quasinormal operator, then  $\widetilde{T}$  is centered.

## 2. On Some Characterizations of $T^\dagger$

Let  $CR(\mathcal{H})$  be the subset of all bounded linear operators on  $\mathcal{H}$  with closed range. In the following proposition we obtain the polar decomposition of  $T^\dagger$ . The following lemma is significant for amount of consideration for the next results and computations.

**Lemma 2.1.** *Let  $T \in CR(\mathcal{H})$ . Then the following assertions hold.*

- (a) If  $T \geq 0$ , then  $T^\dagger \geq 0$ ,  $\mathcal{N}(T) = \mathcal{N}(T^{\frac{1}{2}})$ ,  $T^{\frac{1}{2}} \in CR(\mathcal{H})$  and  $(T^\dagger)^{\frac{1}{2}} = (T^{\frac{1}{2}})^\dagger$ .
- (b)  $|T^\dagger| = |T^*|^\dagger$ .
- (c)  $\mathcal{R}(|T|) = \mathcal{R}(|T^{\frac{1}{2}}|)$  and  $\mathcal{R}(|T^\dagger|) = \mathcal{R}(|T^*|) = \mathcal{R}(|T^{\frac{1}{2}}|)$ .
- (d) If  $T = T^*$ , then  $TP_{\mathcal{R}(T)} = T$ .
- (e)  $|T^\dagger|^{\frac{1}{2}} = (|T^*|^{\frac{1}{2}})^\dagger$ ,  $|T^\dagger|^{\frac{1}{2}} P_{\mathcal{R}(|T^\dagger|)} = |T^{\frac{1}{2}}|$  and  $|T^\dagger|^{\frac{1}{2}} |T^*|^{\frac{1}{2}} = P_{\mathcal{R}(|T^\dagger|)}$ .

*Proof.* (a) Let  $f \in \mathcal{H}$ . Then  $\langle T^\dagger f, f \rangle = \langle T^\dagger T T^\dagger f, f \rangle = \langle T T^\dagger f, T^\dagger f \rangle \geq 0$ , and so  $\langle T^\dagger f, f \rangle \geq 0$ . Also from  $\langle T f, f \rangle = \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} f \rangle = \|T^{\frac{1}{2}} f\|^2$  we deduce that  $T^{\frac{1}{2}} f = 0$  if and only if  $T f = 0$ . Now, from this and the inequality  $\|T f\| = \|T^{\frac{1}{2}}(T^{\frac{1}{2}} f)\| \leq \|T^{\frac{1}{2}}\| \|T^{\frac{1}{2}} f\|$ , we conclude that the range of  $T^{\frac{1}{2}}$  is also closed. Finally, Since for each  $n \in \mathbb{N}$ ,  $(T^n)^\dagger = (T^\dagger)^n$ , we have  $(T^{\frac{1}{2}})^\dagger (T^{\frac{1}{2}})^\dagger = ((T^{\frac{1}{2}})^\dagger)^2 = ((T^{\frac{1}{2}})^2)^\dagger = T^\dagger = ((T^\dagger)^{\frac{1}{2}})^2$ .

(b) It is sufficient to show that  $(T T^*)^\dagger = (T^*)^\dagger T^\dagger$ . Since  $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$ , so  $T T^* (T^*)^\dagger T^\dagger T T^* = TP_{\mathcal{R}(T^*)} P_{\mathcal{R}(T^*)} T^* = T T^*$  and

$$(T^*)^\dagger T^\dagger T T^* (T^*)^\dagger T^\dagger = (T^*)^\dagger P_{\mathcal{R}(T^*)} P_{\mathcal{R}(T^*)} T^\dagger = (T^*)^\dagger T^\dagger.$$

Hence,  $|T^\dagger| = ((T^*)^\dagger T^\dagger)^{\frac{1}{2}} = ((T T^*)^\dagger)^{\frac{1}{2}} = ((T T^*)^{\frac{1}{2}})^\dagger = |T^*|^\dagger$ .

(c) By part (a),  $\mathcal{N}(|T^*|) = \mathcal{N}(|T^{\frac{1}{2}}|)$ . Hence it follows that  $\overline{\mathcal{R}(|T^*|)} = \overline{\mathcal{R}(|T^{\frac{1}{2}}|)}$ . By hypotheses  $\mathcal{R}(T T^*)$ ,  $\mathcal{R}(|T^*|)$  and so  $\mathcal{R}(|T^{\frac{1}{2}}|)$  are closed. Thus  $\mathcal{R}(|T^*|) = \mathcal{R}(|T^{\frac{1}{2}}|)$ . The equality  $\mathcal{R}(|T^\dagger|) = \mathcal{R}(|T^*|)$  follows from (b).

(d) Since  $\mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{R}(T)^\perp$ , hence  $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ . So for each  $f \in \mathcal{H}$  there exists a unique  $g \in \mathcal{R}(T)$  and a unique  $h \in \mathcal{R}(T)^\perp$  such that  $f = g + h$ . It follows that  $TP_{\mathcal{R}(T)}(f) = T(g) = T(g + h) = T(f)$ .

(e) It follows from the previous parts.  $\square$

**Proposition 2.2.** *Let  $U|T|$  be the polar decomposition of an operator  $T \in CR(\mathcal{H})$ . Then  $T^\dagger = U^*|T^*|^\dagger = U^*|T^\dagger|$  is the polar decomposition for  $T^\dagger$  and hence the Aluthge transformation of  $T^\dagger$  is  $|T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}}$ .*

*Proof.* Put  $S = |T^*|^\dagger U$ . Since  $\mathcal{R}(U) = \mathcal{R}(T) = \mathcal{R}(|T^*|)$  and  $\mathcal{R}(|T^*|)^\perp = \mathcal{N}(|T^*|)$ , so  $\mathcal{R}(S) = \mathcal{R}(U) = \mathcal{R}(|T^*|)$ . Moreover, we have

$$\begin{aligned} T^* S T^* &= U^* |T^*| (|T^*|^\dagger U) U^* |T^*| \\ &= U^* |T^*| |T^*|^\dagger |T^*| \\ &= U^* |T^*| = T^*, \end{aligned}$$

$$\begin{aligned} ST^*S &= |T^*|^\dagger(UU^*|T^*|)|T^*|^\dagger U \\ &= |T^*|^\dagger |T^*| |T^*|^\dagger U \\ &= |T^*|^\dagger U = S, \end{aligned}$$

$$\begin{aligned} T^*S &= U^*|T^*| |T^*|^\dagger U \\ &= U^*P_{\mathcal{R}(|T^*|)}U \\ &= U^*U \\ &= P_{\mathcal{R}(T^*)}, \end{aligned}$$

and

$$\begin{aligned} ST^* &= |T^*|^\dagger UU^*|T^*| \\ &= |T^*|^\dagger P_{\mathcal{R}(|T^*|)}|T^*| \\ &= |T^*|^\dagger |T^*| \\ &= P_{\mathcal{R}(|T^*|^\dagger)} = P_{\mathcal{R}(|T^*|)} = P_{\mathcal{R}(S)}. \end{aligned}$$

These equalities show that  $(T^\dagger)^* = (T^*)^\dagger = S$ , and hence  $T^\dagger = S^* = U^*|T^*|^\dagger = U^*|T^\dagger|$  is the polar decomposition for  $T^\dagger$ . Thus, the Aluthge transformation of  $T^\dagger$  is  $|T^\dagger|^\frac{1}{2}U^*|T^\dagger|^\frac{1}{2}$ .  $\square$

**Corollary 2.3.** *Let  $T \in CR(\mathcal{H})$ . Then  $T^\dagger$  is centered if and only if  $T$  is centered.*

*Proof.* By Proposition 2.2,  $U_n|T^n|$  is the polar decomposition of  $T^n$  if and only if  $U_n^*(T^*)^n|^\dagger$  is the polar decomposition of  $(T^\dagger)^n$ , for each  $n \in \mathbb{N}$ . Now, the desired conclusion follows from the Morrel-Muhly Theorem [8].  $\square$

**Proposition 2.4.** *Let  $T = U|T| \in B(\mathcal{H})$  and  $|T|^\frac{1}{2}|T^*|^\frac{1}{2} = V||T|^\frac{1}{2}|T^*|^\frac{1}{2}|$  be the polar decompositions. Suppose that  $\widetilde{T} \in CR(\mathcal{H})$ , then  $\widetilde{T}^\dagger = U^*V^*|\widetilde{T}^\dagger| = U^*(|T^*|^\frac{1}{2}|T|^\frac{1}{2})^\dagger$  is the polar decomposition.*

*Proof.* It is sufficient to show that  $\widetilde{T} = VU|\widetilde{T}|$  is the polar decomposition. This has been proved in [7, Theorem 2.1]. Here we give a new proof. Since  $U|T|^\frac{1}{2} = |T^*|^\frac{1}{2}U$ , we obtain  $(\widetilde{T})^* = |T|^\frac{1}{2}U^*|T|^\frac{1}{2} = U^*|T^*|^\frac{1}{2}|T|^\frac{1}{2} = U^*V^*||T^*|^\frac{1}{2}|T|^\frac{1}{2}|$ . But  $||T^*|^\frac{1}{2}|T|^\frac{1}{2}|^2 = |T|^\frac{1}{2}|T^*||T|^\frac{1}{2} = (|T|^\frac{1}{2}U|T|^\frac{1}{2})(|T|^\frac{1}{2}U^*|T|^\frac{1}{2}) = |(\widetilde{T})^*|^2$ . Hence  $(\widetilde{T})^* = U^*V^*|(\widetilde{T})^*|$ . Also it is easy to check that  $(U^*V^*)(VU)(U^*V^*) = U^*V^*$  and  $\mathcal{N}((\widetilde{T})^*) = \mathcal{N}(U^*V^*)$ . Hence  $\widetilde{T} = VU|\widetilde{T}|$  is the polar decomposition. By using Proposition 2.2, we have that  $\widetilde{T}^\dagger = U^*V^*|\widetilde{T}^\dagger| = U^*(|T^*|^\frac{1}{2}|T|^\frac{1}{2})^\dagger$  is the polar decomposition.  $\square$

**Theorem 2.5.** *If  $T \in CR(\mathcal{H})$  is binormal, then  $\widetilde{T}^\dagger = (|T^\dagger|^\frac{1}{2}U^*(|T^\dagger|^\frac{1}{2}))^\dagger$ .*

*Proof.* First, we note that by a modification of [5, Theorem 2] we obtain

$$P_{\mathcal{N}(|T|)^\perp}P_{\mathcal{N}(|T^*|)^\perp} = P_{\mathcal{N}(|T^*|)^\perp}P_{\mathcal{N}(|T|)^\perp}.$$

Since for each  $0 \leq A \in B(\mathcal{H})$ ,  $\overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp$ , this implies that

$$P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)} = P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}.$$

Put  $S = (|T^\dagger|^\frac{1}{2}U^*(|T^\dagger|^\frac{1}{2}))^\dagger$ . Then we have

$$\begin{aligned} S\widetilde{T}S &= (|T^\dagger|^\frac{1}{2}U^*(|T^\dagger|^\frac{1}{2})^\frac{1}{2}|T|^\frac{1}{2}U|T|^\frac{1}{2}(|T^\dagger|^\frac{1}{2}U^*(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} \\ &= (|T^\dagger|^\frac{1}{2}U^*P_{\mathcal{R}(|T|)}UP_{\mathcal{R}(|T|)}U^*(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} \\ &= (|T^\dagger|^\frac{1}{2}U^*P_{\mathcal{R}(|T|)}UU^*(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} \\ &= (|T^\dagger|^\frac{1}{2}U^*P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} \\ &= (|T^\dagger|^\frac{1}{2}U^*P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} \\ &= (|T^\dagger|^\frac{1}{2}U^*(|T^\dagger|^\frac{1}{2}))^\frac{1}{2} = S, \end{aligned}$$

$$\begin{aligned}
 \widetilde{TS}\widetilde{T} &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}(|T|^{\dagger})^{\frac{1}{2}}U^*(|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
 &= |T|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}U^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}} \\
 &= |T|^{\frac{1}{2}}UU^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}} \\
 &= |T|^{\frac{1}{2}}P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}} \\
 &= |T|^{\frac{1}{2}}P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}U|T|^{\frac{1}{2}} \\
 &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \widetilde{T},
 \end{aligned}$$

and

$$\begin{aligned}
 S\widetilde{T} &= (|T|^{\dagger})^{\frac{1}{2}}U^*(|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
 &= (|T|^{\dagger})^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}U|T|^{\frac{1}{2}} \\
 &= (|T|^{\dagger})^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}}U \\
 &= (|T|^{\dagger})^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}P_{\mathcal{R}(|T|)}U \\
 &= (|T|^{\dagger})^{\frac{1}{2}}|T|^{\frac{1}{2}}U^*P_{\mathcal{R}(|T|)}U \\
 &= P_{\mathcal{R}(|T|)}U^*P_{\mathcal{R}(|T|)}U = U^*P_{\mathcal{R}(|T|)}U.
 \end{aligned}$$

By similar computation we have  $\widetilde{TS} = P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}$ . Hence  $\widetilde{TS}$  and  $S\widetilde{T}$  are self-adjoint operators. From the uniqueness of Moore-Penrose inverse we have  $\widetilde{T}^{\dagger} = S$ .  $\square$

In [11], Yamazaki introduce the notion of the  $*$ -Aluthge transform  $\widetilde{T}^{(*)}$  of  $T$  by setting  $\widetilde{T}^{(*)} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ . With the motivation of this definition we define  $\dagger$ -Aluthge transformation of  $T$  by setting  $\widetilde{T}^{(\dagger)} := (\widetilde{T}^{\dagger})^{\dagger}$ . In the following theorem we show that if  $T \in CR(\mathcal{H})$  is binormal, then the  $*$ -Aluthge and  $\dagger$ -Aluthge transformations coincide.

**Theorem 2.6.** *If  $T \in CR(\mathcal{H})$  is binormal, then  $\widetilde{T}^{(\dagger)} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}} = \widetilde{T}^{(*)}$ .*

*Proof.* Since  $T$  is binormal, then we have

$$\begin{aligned}
 \widetilde{T}^{(*)}\widetilde{T}^{\dagger}\widetilde{T}^{(*)} &= (|T^{\dagger}|^{\frac{1}{2}})^{\dagger}U(|T^{\dagger}|^{\frac{1}{2}})^{\dagger}|T^{\dagger}|^{\frac{1}{2}}U^*|T^{\dagger}|^{\frac{1}{2}}(|T^{\dagger}|^{\frac{1}{2}})^{\dagger}U(|T^{\dagger}|^{\frac{1}{2}})^{\dagger} \\
 &= |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^{\dagger}|^{\frac{1}{2}}U^*|T^{\dagger}|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}U^*P_{\mathcal{R}(|T^*|)}U|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}U^*U|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}UP_{\mathcal{R}(|T|)}|T^*|^{\frac{1}{2}} \\
 &= |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}} = \widetilde{T}^{(*)},
 \end{aligned}$$

and

$$\begin{aligned} \widetilde{T}^\dagger \widetilde{T}^{(*)} \widetilde{T}^\dagger &= |T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}} (|T^\dagger|^{\frac{1}{2}})^\dagger U (|T^\dagger|^{\frac{1}{2}})^\dagger |T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} U^* P_{\mathcal{R}(|T^*|)} U P_{\mathcal{R}(|T^*|)} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} U^* U P_{\mathcal{R}(|T^*|)} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} P_{\mathcal{R}(|T|)} P_{\mathcal{R}(|T^*|)} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} P_{\mathcal{R}(|T^*|)} P_{\mathcal{R}(|T|)} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} P_{\mathcal{R}(|T^*|)} U^* |T^\dagger|^{\frac{1}{2}} \\ &= |T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}} = \widetilde{T}^\dagger. \end{aligned}$$

By similar computations we have  $\widetilde{T}^\dagger \widetilde{T}^{(*)} = P_{\mathcal{R}(|T|)} P_{\mathcal{R}(|T^*|)}$  and  $\widetilde{T}^{(*)} \widetilde{T}^\dagger = U P_{\mathcal{R}(|T^*|)} U^*$ . Hence  $\widetilde{T}^\dagger \widetilde{T}^{(*)}$  and  $\widetilde{T}^{(*)} \widetilde{T}^\dagger$  are self-adjoint operators. Thus,  $\widetilde{T}^{(t)} = (\widetilde{T}^\dagger)^\dagger = \widetilde{T}^{(*)}$ .  $\square$

The so-called reverse order law, which is one of the most important properties of the Moore-Penrose inverse that has been deeply studied, states that under which condition the equation  $(T_1 T_2)^\dagger = T_2^\dagger T_1^\dagger$  holds (see [6]). In the following theorem we show that the reverse order law holds for  $|T|$  and  $|T^*|$  whenever  $T \in CR(\mathcal{H})$  is binormal.

**Lemma 2.7.** *Let  $T \in CR(\mathcal{H})$  be a binormal operator. Then  $(|T||T^*|)^\dagger = |T^*|^\dagger |T|^\dagger$ .*

*Proof.* Since  $T$  is a binormal operator, so by a modification of [5, Theorem 2],  $P_{\mathcal{R}(|T^*|)} P_{\mathcal{R}(|T|)} = P_{\mathcal{R}(|T|)} P_{\mathcal{R}(|T^*|)}$ . Put  $S = |T^*|^\dagger |T|^\dagger$ . Then we have,

$$\begin{aligned} |T||T^*|S|T||T^*| &= |T|P_{\mathcal{R}(|T^*|)}P_{\mathcal{R}(|T|)}|T^*| \\ &= |T|P_{\mathcal{R}(|T|)}P_{\mathcal{R}(|T^*|)}|T^*| \\ &= |T||T^*|, \end{aligned}$$

$$\begin{aligned} S|T||T^*|S &= |T^*|^\dagger P_{\mathcal{R}(|T|)} P_{\mathcal{R}(|T^*|)} |T|^\dagger \\ &= |T^*|^\dagger P_{\mathcal{R}(|T^*|)} P_{\mathcal{R}(|T|)} |T|^\dagger \\ &= S, \end{aligned}$$

$$\begin{aligned} S|T||T^*| &= |T^*|^\dagger P_{\mathcal{R}(|T|)} |T^*| \\ &= |T^*|^\dagger |T^*| P_{\mathcal{R}(|T|)} \\ &= P_{\mathcal{R}(|T^*|)} P_{\mathcal{R}(|T|)}, \end{aligned}$$

and

$$\begin{aligned} |T||T^*|S &= |T|P_{\mathcal{R}(|T^*|)} |T|^\dagger \\ &= P_{\mathcal{R}(|T^*|)} |T||T|^\dagger \\ &= P_{\mathcal{R}(|T^*|)} P_{\mathcal{R}(|T|)}. \end{aligned}$$

Consequently,  $(|T||T^*|)^\dagger = S$ .  $\square$

**Theorem 2.8.** *Let  $T \in CR(\mathcal{H})$ . Then  $T$  is binormal if and only if  $T^\dagger$  is so.*

*Proof.* Suppose  $TT^*$  commutes with  $T^*T$ . Then by Lemma 2.7, we have  $|T^*|^\dagger |T|^\dagger = |T|^\dagger |T^*|^\dagger$ . Since  $|T|^\dagger = |(T^*)^\dagger| = |(T^\dagger)^*|$ , it follows that  $|T^\dagger| |(T^\dagger)^*| = |(T^\dagger)^*| |T^\dagger|$ . Conversely, since  $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$  and  $(T^\dagger)^\dagger = T$ , then  $T^\dagger \in \text{CR}(\mathcal{H})$  and so the converse also holds.

□

In [7], the authors obtained the polar decomposition of  $\widetilde{T} = VU|\widetilde{T}|$ , when  $T = U|T|$  and  $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$ . In the following theorem we obtain the polar decomposition of the powers of  $\widetilde{T}$ .

**Theorem 2.9.** *Let  $n \in \mathbb{N}$  and  $T \in B(\mathcal{H})$ . Then  $(\widetilde{T})^n = |T|^{\frac{1}{2}}T^{n-1}U|T|^{\frac{1}{2}}$ . Moreover, if  $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$  is the polar decomposition, then  $(\widetilde{T})^n = W_{n-1}U|(\widetilde{T})^n|$  is the polar decomposition of  $(\widetilde{T})^n$ .*

*Proof.* By direct computations we obtain that

$$\begin{aligned} U|(\widetilde{T})^n| &= U|(\widetilde{T})^n|U^*U \\ &= U(|T|^{\frac{1}{2}}U^*(T^{n-1})^*|T|T^{n-1}U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U \\ &= (U|T|^{\frac{1}{2}}U^*(T^{n-1})^*|T|T^{n-1}U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}U \\ &= (|T^*|^{\frac{1}{2}}(T^{n-1})^*|T|T^{n-1}|T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= (||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|^2)^{\frac{1}{2}} \\ &= ||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|U. \end{aligned}$$

Also, by the polar decomposition  $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$  we get that

$$\begin{aligned} W_{n-1}U|(\widetilde{T})^n| &= W_{n-1}U|(\widetilde{T})^n|UU^* \\ &= W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|U \\ &= |T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U \\ &= |T|^{\frac{1}{2}}T^{n-1}U|T|^{\frac{1}{2}}U \\ &= (\widetilde{T})^n. \end{aligned}$$

Now, we show that the kernel condition  $\mathcal{N}((\widetilde{T})^n) = \mathcal{N}(W_{n-1}U)$  holds. Let  $f \in \mathcal{N}(W_{n-1}U)$ . Since by hypothesis  $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}} = W_{n-1}||T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}|$  is the polar decomposition and  $\mathcal{N}(W_{n-1}) = \mathcal{N}(|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}})$ , we obtain that  $(\widetilde{T})^n(f) = |T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U(f) = 0$ . Hence  $\mathcal{N}(W_{n-1}U) \subseteq \mathcal{N}((\widetilde{T})^n)$ . Let  $f \in \mathcal{N}((\widetilde{T})^n)$ . Then  $|T|^{\frac{1}{2}}T^{n-1}|T^*|^{\frac{1}{2}}U(f) = 0$ , and so  $W_{n-1}U(f) = 0$ . Consequently,  $\mathcal{N}((\widetilde{T})^n) \subseteq \mathcal{N}(W_{n-1}U)$ . □

**Corollary 2.10.** *Let  $n \in \mathbb{N}$  and  $T \in B(\mathcal{H})$ . Then  $\widetilde{T}$  is centered if and only if for each positive integer  $n$ ,  $W_{n-1}U = (W_0U)^n$ . In particular, if  $T$  is a centered operator, then  $\widetilde{T}$  is centered if and only if  $W_{n-1}U = U^n$ .*

*Proof.* By the definition  $\widetilde{T}$  is centered if and only if for each positive integer  $n$ ,  $W_{n-1}U = (W_0U)^n$ . If  $T$  is centered then  $T$  is binormal, and so  $W_0 = I$ . Thus we conclude that  $\widetilde{T}$  is centered if and only if  $W_{n-1}U = U^n$ . □

**Lemma 2.11.** *Let  $T = U|T| \in B(\mathcal{H})$  be a quasinormal operator. Then the following assertions hold.*

- (i)  $|T|^{\frac{1}{2}}T = T|T|^{\frac{1}{2}}$ .
- (ii)  $|T^2| = |T|^2$ .
- (iii)  $|T|^n U|T| = U|T|^{n+1}$ , for each  $n \in \mathbb{N}$ .
- (iv)  $|T^*|^{\frac{1}{2}}|T|^{\frac{3}{2}} = |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}}$ .
- (v)  $T$  is centered so the partial isometry part in  $T^n$  is  $U^n$ .
- (vi)  $(T^n)^*T^n = (T^*T)^n$  and hence  $|T^n| = |T|^n$ , for each  $n \in \mathbb{N}$ .

*Proof.* (i) From  $|T|^2T = T^*TT = TT^*T = T|T|^2$ , we deduce that  $|T|T = T|T|$  and hence  $|T|^{\frac{1}{2}}T = T|T|^{\frac{1}{2}}$ .

(ii) Since  $T^*T^*TT = T^*TT^*T$ , so  $(T^*)^2T^2 = (T^*T)^2$  and hence  $|T^2|^2 = (|T|^2)^2$ . Consequently,  $|T^2| = |T|^2$ .

(iii) For  $n = 2$  we have  $|T|^2U|T| = |T||T|U|T| = |T|U|T|^2 = U|T|^3$ . Let for positive integer  $n$ ,  $|T|^nU|T| = U|T|^{n+1}$ . Then

$$\begin{aligned} |T|^{n+1}U|T| &= |T||T|^nU|T| \\ &= |T|U|T|^n \\ &= U|T|^{n+1}. \end{aligned}$$

(iv) Since  $T^*|T| = |T|T^*$ , so  $TT^*|T| = T|T|T^* = |T|TT^*$ . This means  $|T^*|^2|T| = |T||T^*|^2$  and hence  $|T^*||T| = |T||T^*|$ . Therefore,  $|T^*||T|^3 = |T||T^*||T|^2 = |T|^3|T^*|$  and so  $|T^*|^{\frac{1}{2}}|T|^{\frac{3}{2}} = |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}}$ .  $\square$

**Theorem 2.12.** Let  $T = U|T| \in B(\mathcal{H})$  be a quasinormal operator. Suppose that for each  $n \in \mathbb{N}$ ,  $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = W_n||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}|$  is the polar decomposition. Then  $W_n = U^n$ .

*Proof.* By using Lemma 2.11, we have

$$\begin{aligned} |T^*|^{\frac{1}{2}}(T^n)^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} &= |T^*|^{\frac{1}{2}}(T^n)^*|T|T^n|T^*|^{\frac{1}{2}} \\ &= |T^*|^{\frac{1}{2}}(T^n)^*T^n|T||T^*|^{\frac{1}{2}} \\ &= |T^*|^{\frac{1}{2}}(T^*T)^n|T||T^*|^{\frac{1}{2}} \\ &= |T^*|^{\frac{1}{2}}|T|^{2n+1}|T^*|^{\frac{1}{2}} \\ &= |T^*|^{\frac{1}{2}}|T|^{\frac{2n+1}{2}}|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}} \\ &= (|T^*|^{\frac{1}{2}}|T|^{\frac{2n+1}{2}})^2. \end{aligned}$$

This implies that  $||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}| = |T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$ . Also we have  $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = T^n|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = U^n|T|^n|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = U^n|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$ . Now, if  $|T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}} = W_n||T|^{\frac{1}{2}}T^n|T^*|^{\frac{1}{2}}| = W_n|T|^{\frac{2n+1}{2}}|T^*|^{\frac{1}{2}}$  is the polar decomposition, then we obtain  $W_n = U^n$ .  $\square$

**Corollary 2.13.** If  $T \in B(\mathcal{H})$  is quasinormal, then  $\widetilde{T}$  is centered.

**Example 2.14.** Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space and let  $\varphi : X \rightarrow X$  be a measurable transformation such that  $\mu \circ \varphi^{-1}$  is absolutely continuous with respect to  $\mu$ . Put  $h = d\mu \circ \varphi^{-1}/d\mu$ . Let  $C_\varphi : f \mapsto f \circ \varphi$  be a bounded composition operator on  $L^2(\Sigma)$  with polar decomposition  $C_\varphi = U|C_\varphi|$ . It is easy to see that  $U = M_{1/\sqrt{h \circ \varphi}}C_\varphi$  and  $|C_\varphi| = M_{\sqrt{h}}$ . It follows that for each  $f \in L^2(\Sigma)$ ,

$$\widetilde{C}_\varphi(f) = \sqrt[4]{\frac{h}{h \circ \varphi}} f \circ \varphi.$$

Note that  $h \circ \varphi > 0$  almost everywhere. Now, if  $C_\varphi \in CR(L^2(\Sigma))$  then it is easy to check that

$$C_\varphi^\dagger(f) = \chi_{\sigma(h)}E(f) \circ \varphi^{-1}, \quad f \in L^2(\Sigma),$$

where  $E$  is the conditional expectation operator with respect to  $\varphi^{-1}(\Sigma)$ . We can write  $C_\varphi^\dagger = M_{\frac{\chi_{\sigma(h)}}{h}}C_\varphi^*$ , where  $C_\varphi^*(f) = hE(f) \circ \varphi^{-1}$ . For more details on composition and conditional expectation operators see [2] and references

therein. Let  $|C_\varphi|^{\frac{1}{2}}|C_\varphi^*|^{\frac{1}{2}} = V||C_\varphi|^{\frac{1}{2}}|C_\varphi^*|^{\frac{1}{2}}|$  be the polar decomposition of  $|C_\varphi|^{\frac{1}{2}}|C_\varphi^*|^{\frac{1}{2}}$ . Then straightforward calculations show that

$$V(f) = \frac{\sqrt[4]{h}}{\sqrt{E(\sqrt{h} \circ \varphi)}} f;$$

$$||C_\varphi|^{\frac{1}{2}}|C_\varphi^*|^{\frac{1}{2}}|(f) = \sqrt[4]{h \circ \varphi} \sqrt{E(\sqrt{h})} E(f).$$

Hence by Proposition 2.4,  $\widetilde{C}_\varphi = VU||C_\varphi|^{\frac{1}{2}}|C_\varphi^*|^{\frac{1}{2}}|$  is the polar decomposition of  $\widetilde{C}_\varphi$ . In [4], Embry-Wardrop and Lambert proved that the composition operator  $C_\varphi \in B(L^2(\Sigma))$  is centered if and only if  $h$  is  $\Sigma_\infty$ -measurable, where  $\Sigma_\infty = \bigcap_{n=1}^\infty \varphi^{-n}(\Sigma)$ . Now, by this fact and Corollary 2.3,  $C_\varphi^+$  is centered if and only if  $h$  is  $\Sigma_\infty$ -measurable. Recall that  $C_\varphi$  is quasinormal if and only if  $h = h \circ \varphi$  (see [10]). So for each  $n \in \mathbb{N}$ ,  $h = h \circ \varphi^n$  and so  $h \in \Sigma_\infty$ . In this case we have  $\widetilde{C}_\varphi = C_\varphi$ , and hence  $\widetilde{C}_\varphi$  is centered.

### Acknowledgment

The authors would like to acknowledge the valuable contribution of the referee regarding the improvement of the quality of the paper.

### References

- [1] A. Aluthge, On  $p$ -hyponormal operators for  $0 < p < 1$ , Integral Equations and Operator Theory 13 (1990), 307–315.
- [2] C. Burnap, I. Jung and A. Lambert, Separating partial normality classes with composition operators, Journal of Operator Theory 53 (2005), 381–397.
- [3] S. L. Campbell, Linear operators for which  $T^*T$  and  $TT^*$  commute, Proceedings of the American Mathematical Society 34 (1972), 177–180.
- [4] M. Embry-Wardrop and A. Lambert, Measurable transformations and centred composition operators, Proceedings of the Royal Irish Academy. Section A 90 (1990), 165–172.
- [5] T. Furuta, On the polar decomposition of an operator, Acta Scientiarum Mathematicarum (Szeged) 46 (1983), 261–268.
- [6] T. N. E. Greville, Note on the generalized inverse of a matrix product, SIAM Review 8 (1966), 518–521.
- [7] M. Ito, T. Yamazaki and M. Yanagida, On the polar decomposition of the Aluthge transformation and related results, Journal of Operator Theory 51 (2004), 303–319.
- [8] B. B. Morrel and P. S. Muhly, Centered operators, Studia Mathematica 51 (1974), 251–263.
- [9] V. Paulsen, C. Pearcy, S. Petrović, On centered and weakly centered operators, Journal of Functional Analysis 128 (1995), 87–101.
- [10] R. K. Singh and B. S. Komal, Composition operators, Bulletin of the Australian Mathematical Society 18 (1978), 439–446.
- [11] T. Yamazaki, Parallelisms between Aluthge transformation and powers of operators, Acta Scientiarum Mathematicarum (Szeged) 67 (2001), 809–820.