SOME EQUIVALENT METRICS FOR BOUNDED NORMAL OPERATORS

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Abstract. Some stronger and equivalent metrics are defined on $\mathcal{M}$, the set of all bounded normal operators on a Hilbert space H and then some topological properties of $\mathcal{M}$ are investigated.

Keywords: Hilbert space; normal operator; equivalent metrics; composition operator

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1. Introduction and preliminaries

Let H be a separable, infinite dimensional, complex Hilbert space with inner product $\langle\,,\rangle$ and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on H. The problem of the topological structure of $\mathcal{C}(H)$, the subsets of closed and densely defined linear operators on H has been considered starting with the paper by Cordes and Labrousse [2]; see also [7]. They prove that the metric distance between two densely defined unbounded operators A and B may be taken as $\|(I+AA^*)^{-1}-(I+BB^*)^{-1}\|$. As the authors show, this metric defines the same topology for bounded operators as the ordinary metric $\|A-B\|$. For $A\in\mathcal{C}(H)$, let α denote the contraction defined as $\alpha(T)=A(1+A^*A)^{-1/2}$. Kaufman [5] studies a metric δ on $\mathcal{C}(H)$ defined as $\delta(A,B)=\|\alpha(A)-\alpha(B)\|$ and then the author discusses connections between δ-convergence and strong-operator-topology convergence. Also, he shows that this metric is stronger than the gap metric d (see [4], page 197) and not equivalent to it. In [6], Kittaneh presents quantitative improvements of the result of Kaufman [5] concerning equivalence of three metrics on the space of bounded linear operators on a Hilbert space. In [1], Benharrat and Messirdi defined some new strictly stronger metrics than the gap metric d and characterized the closure with respect to these metrics of the subset $\mathcal{B}(H)$ of bounded elements of $\mathcal{C}(H)$.

Let $\mathcal{M}$ be the subset of bounded normal operators in $\mathcal{B}(H)$, $A \in \mathcal{M}$ and let $0 < a < \|A\|^{-1}$. In this paper, by motivation of the above mentioned results, we shall replace $1 + A^*A$ with $I + a^2A^*A + a^4(A^*)^2A^2 + \ldots$, and then we obtain some analogous results on topological properties of $\mathcal{M}$.

In Section 2, we show that $K_a(A) := \sum\limits_{n=0}^{\infty} a^{2n}A^{*n}A^n$ is positive, invertible and then we obtain the relation between the operators $K_a(A)$, $K_a^{-1}(A)$ and $(K_a(A))^{-1/2}$ in the case when A is normal. Moreover, we introduce some special types of metrics on normal operators in $\mathcal{B}(H)$ and then we compare the topologies induced by these metrics.

In Section 3, inspired by definition of bisecting for $A \in \mathcal{C}(H)$ in [8], we define $\widetilde{A}_a$ for $A \in \mathcal{M}$. Then using $\widetilde{A}_a$ and the metrics defined in Section 2, we introduce the $F_1, \ldots, F_4$ maps on $\mathcal{M}$ with different metrics into $\mathcal{M}$ with the aid of usual operator norm. Then we will proceed on investigating the continuity of these maps. At the end, as an example we determine $K_a(C_\varphi)$, $R_a(C_\varphi)$, $S_a(C_\varphi)$, $(\widetilde{C}_\varphi)_a$ for $C_\varphi \in \mathcal{M}$, where $C_\varphi(f) = f \circ \varphi$ is the composition operator on $L^2(\Sigma)$.

2. Stronger and equivalent metrics on $\mathcal{M}$

For $A \in \mathcal{B}(H)$, let A^*, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $r(A)$ and $\|A\|$ denote the adjoint, the null space, the range, the spectral radius and the usual operator norm of A, respectively. Note that $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{1/n} \leqslant \|A\|$ and that the equality holds if A is normal. A is called positive if $\langle Ax, x\rangle \geqslant 0$ holds for every $x \in H$ in which case we write $A \geqslant 0$. For an operator $A \in \mathcal{B}(H)$ let $0 < a < (r(A))^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum\limits_{n=0}^{\infty} a^{2n}A^{*n}A^n$. The definition of $K_a(A)$ is due to Gilfeather [3], Lambert and Petrovic [9].

Lemma 2.1. *Let $A \in \mathcal{B}(H)$. Then $0 \leqslant K_a(A) \in \mathcal{B}(H)$ and $K_a(A)$ is invertible with $\|K_a^{-1}(A)\| \leqslant 1$.*

Proof. Since $\lim\limits_{n\to\infty} \|a^{2n}A^{*n}A^n\|^{1/n} < (r(A))^{-2} \lim\limits_{n\to\infty} \|A^n\|^{2/n} = 1$, so the infinite series $K_a(A)$ converges absolutely. Also, for all $x \in H$ we have

$$\langle K_a(A)(x), x\rangle = \sum_{n=0}^{\infty} a^{2n}\|A^n(x)\|^2 \geqslant 0.$$

Thus,

$$\bigl\|\sqrt{K_a(A)(x)}\bigr\|^2 = \langle K_a(A)(x), x\rangle = \|x\|^2 + \sum_{n=1}^{\infty} a^{2n}\|A^n(x)\|^2 \geqslant \|x\|^2,$$

and so

$$R(\sqrt{K_a(A)}) = \overline{R(\sqrt{K_a(A)})} = N(\sqrt{K_a(A)})^{\perp} = H.$$

It follows that $\sqrt{K_a(A)}$ and hence $K_a(A)$ is invertible. Now, replacing x by $(K_a(A))^{-1/2}(x)$ we obtain $\|(K_a(A))^{-1/2}(x)\| \leqslant \|x\|$. This implies that

$$\frac{1}{\|K_a(A)\|} \leqslant \frac{1}{\|\sqrt{K_a(A)}\|^2} \leqslant 1.$$

□

For $A \in \mathcal{B}(H)$ set $R_a(A) = (K_a(A))^{-1}$ and $S_a(A) = \sqrt{R_a(A)}$. Then by Lemma 2.1, $R_a(A)$ and $S_a(A) = (K_a(A))^{-1/2}$ are positive and $S_a(A)$ is a contraction.

Moreover, when A is a normal operator, i.e. $AA^* = A^*A$, then $R_a(A) = R_a(A^*)$, $AR_a(A) = R_a(A)A$ and $A^*R_a(A) = R_a(A)A^*$.

Recall that for $A \in \mathcal{C}(H)$, the fundamental properties of $R_A = (I + A^*A)^{-1}$ and $S_A = (I + A^*A)^{-1/2}$ have been investigated by many authors, e.g. [2], [1]. In the following lemma we obtain a relationship between the concepts of $R_a(A)$ and $S_a(A)$ when $A \in \mathcal{B}(H)$ is a normal operator.

Lemma 2.2. *Let $A \in \mathcal{B}(H)$ be a normal operator and let $n \in \mathbb{N} \cup \{0\}$. Then the following assertions hold.*

(a) $A^n R_a(A) = R_a(A)A^n$;
(b) $A^n S_a(A) = S_a(A)A^n$;
(c) $S_a(A)(K_a(A) - I)S_a(A) = I - R_a(A)$;
(d) $\sqrt{K_a(A) - I} = a|A|(S_a(A))^{-1}$;
(e) $R_a(A) = I - a^2|A|^2$;
(f) $\mathcal{N}(S_a(A)) \cap \mathcal{N}(A) = \{0\}$.

Proof. (a) Since A is normal, from direct computations we obtain that

$$\begin{aligned} A^n K_a(A) &= A^n(I + a^2A^*A + a^4(A^*)^2A^2 + \ldots) \\ &= A^n + a^2A^nA^*A + a^4A^n(A^*)^2A^2 + \ldots \\ &= (I + a^2AA^* + a^4A^2(A^*)^2 + \ldots)A^n = K_a(A^*)A^n = K_a(A)A^n. \end{aligned}$$

Therefore, the inverse of $K_a(A)$ is also commute with all A^n.

(b) Since $A^nR_a(A) = R_a(A)A^n$, it follows that $A^nP(R_a(A)) = P(R_a(A))A^n$, where P is any polynomial. Now let $\{P_m\}$ be a sequence of polynomials converging uniformly to a continuous function g. Then for each $x, y \in H$ we have

$$\begin{aligned} \langle A^n g(R_a(A))(x), y\rangle &= \lim_{m\to\infty} \langle P_m(R_a(A))(x), (A^n)^*y\rangle \\ &= \lim_{m\to\infty} \langle P_m(R_a(A))A^n(x), y\rangle \quad \text{(by part (a))} \\ &= \langle g(R_a(A))A^n(x), y\rangle. \end{aligned}$$

Thus, $A^n g(R_a(A)) = g(R_a(A))A^n$. Let g be a square root function. Consequently, $A^n\sqrt{R_a(A)} = \sqrt{R_a(A)}A^n$, and so $A^n S_a(A) = S_a(A)A^n$.

(c) Since $R_a(A) = S_a^2(A)$, then

$$\begin{aligned} I - R_a(A) &= (R_a^{-1}(A) - I)R_a(A) = a^2A^*AR_a(A) + a^4(A^*)^2A^2R_a(A) + \dots \\ &= a^2A^*S_a(A)S_a(A)A + a^4(A^*)^2S_a(A)S_a(A)A^2 + \dots \\ &= a^2A^*S_a(A)AS_a(A) + a^4(A^*)^2S_a(A)A^2S_a(A) + \dots \\ &= \sum_{n=1}^{\infty} a^{2n}(A^*)^nS_a(A)A^nS_a(A) = S_a(A)(K_a(A) - I)S_a(A). \end{aligned}$$

(d) Normality of A implies that

$$K_a(A) - I = a^2A^*A(I + a^2A^*A + a^4(A^*)^2A^2 + \dots) = a^2|A|^2K_a(A).$$

Thus, $\sqrt{K_a(A) - I} = a|A|\sqrt{K_a(A)} = a|A|(S_a(A))^{-1}$.

(e) It follows from (c) and (d).

(f) It suffices to show that $\|S_a(A)u\|^2 + \|a|A|u\|^2 = \|u\|^2$ for all $u \in H$. For this, let $u \in H$. Then by (e) we have

$$\begin{aligned} \|S_a(A)u\|^2 + \|a|A|u\|^2 &= \langle S_a(A)u, S_a(A)u\rangle + \langle a|A|u, a|A|u\rangle \\ &= \langle u, R_a(A)u\rangle + \langle u, a^2|A|^2u\rangle \\ &= \langle u, R_a(A)u\rangle + \langle u, (I - R_a(A))u\rangle = \langle u, u\rangle = \|u\|^2. \end{aligned}$$

□

Lemma 2.3 ([2]). *Let A be closed. Then*

$$\Pi_{\mathcal{G}(A)} = \begin{bmatrix} R_A & A^*R_A^* \\ AR_A & I - R_A^* \end{bmatrix},$$

where $\Pi_{\mathcal{G}(A)}$ denotes the orthogonal projection onto $\mathcal{G}(A) = \{(x, Ax)\colon\ x \in \mathcal{D}(A)\}$.

Now inspired by matrix $\Pi_{\mathcal{G}(A)}$, we define $\Pi_a(A) \in \mathcal{B}(H \otimes H)$ for $A \in \mathcal{M}$:

$$\Pi_a(A) = \begin{bmatrix} R_a(A) & a|A|S_a(A) \\ a|A|S_a(A) & I - R_a(A) \end{bmatrix}.$$

In [1], Benharrat and Messirdi introduced metrics $g_G(T, S)$, $p_G(T, S)$, $q_G(T, S)$ and $\Sigma_G(T, S)$ for $S, T \in \mathcal{C}(H_1, H_2)$ and a positive bijection $G \in \mathcal{L}^+(H_1)$.

Now, inspired by these metrics we define special types of metrics on $\mathcal{M}$:

$$\begin{aligned}
d^{[1]}_{(a,b)}(A,B) &= \|\Pi_a(A) - \Pi_b(B)\|;\\
d^{[2]}_{(a,b)}(A,B) &= \sqrt{\|R_a(A) - R_b(B)\|^2 + \|a|A|S_a(A) - b|B|S_b(B)\|^2};\\
d^{[3]}_{(a,b)}(A,B) &= \|a|A| - b|B|\|;\\
d^{[4]}_{(a,b)}(A,B) &= \sqrt{2\|a|A| - b|B|\|^2 + 2\|S_a(A) - S_b(B)\|^2},
\end{aligned}$$

where $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$ are arbitrary but fixed numbers, whenever A and B are nonzero elements of $\mathcal{M}$. Note that $d^{[3]} \leqslant d^{[4]}$. Hence, the topology induced from the metric $d^{[4]}$ on $\mathcal{M}$ is stronger than that induced from $d^{[3]}$.

Lemma 2.4 ([6]).

(a) *If* $A, B \in \mathcal{B}(H)$ *are positive, then*

$$\|A - B\| \leqslant \sqrt{\|A^2 - B^2\|}.$$

(b) *If* $T \in \mathcal{B}(H \oplus H)$ *and*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then $\|T\|^2 \leqslant \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2$.

It was proved that in [1] the topology induced from the metric $g_G(T,S)$ on $\mathcal{C}(H_1, H_2)$ is strictly stronger than that induced from $p_G(T,S)$. But the following proposition proves that the metrics $d^{[1]}$ and $d^{[2]}$ on $\mathcal{M}$ generate the same topology.

Proposition 2.5. *The topology induced from the metric* $d^{[1]}$ *on* $\mathcal{M}$ *is equivalent to the topology induced from* $d^{[2]}$ *on* $\mathcal{M}$.

Proof. Let $A, B \in \mathcal{M}$. Evidently, $d^{[2]}_{(a,b)}(A,B) \leqslant d^{[1]}_{(a,b)}(A,B)$. On the other hand, by Lemma 2.4 (b) we have

$$\|\Pi_a(A) - \Pi_b(B)\|^2 \leqslant 2\|R_a(A) - R_b(B)\|^2 + 2\|a|A|S_a(A) - b|B|S_b(B)\|^2.$$

Thus, $d^{[1]}_{(a,b)}(A,B) \leqslant \sqrt{2} d^{[2]}_{(a,b)}(A,B)$. □

Lemma 2.6. *Let* A *and* B *be two nonzero elements of* $\mathcal{B}(H)$. *Then*

$$\left\| \frac{A}{\|A\|} - \frac{B}{\|B\|} \right\| \leqslant \frac{2\|A - B\|}{\|A\|}.$$

P r o o f. Since $\|B\| - \|A\|$ is not greater than $\|A - B\|$, so

$$\|B\|\|A\|\Big\|\frac{A}{\|A\|} - \frac{B}{\|B\|}\Big\| \leqslant \|B\|\|A - B\| + \|B\|(\|B\| - \|A\|) \leqslant 2\|B\|\|A - B\|.$$

The result follows. $\square$

Now, let A and B be two nonzero normal elements of $\mathcal{B}(H)$. Then $r(A) = \|A\|$ and $r(B) = \|B\|$. For $0 < \alpha < 1$ put $a_\alpha = \alpha\|A\|^{-1}$ and $b_\alpha = \alpha\|B\|^{-1}$. By Lemma 2.6 we obtain

$$\|a_\alpha A - b_\alpha B\| = \Big\|\frac{\alpha A}{\|A\|} - \frac{\alpha B}{\|B\|}\Big\| \leqslant \frac{2\alpha\|A - B\|}{\|A\|}.$$

In the following theorem, we show that $d^{[i]}_{(a_\alpha, b_\alpha)} < \|\cdot\|$ for $i = 3, 4$ on $\mathcal{M}$. This is why, in the study carried out by Benharrat and Messirdi, it was found that the restriction of the metric $q_G(T, S)$ to $\mathcal{L}(H_1, H_2)$ is equivalent to the operator norm.

Theorem 2.7. *The topology induced from the operator norm on $\mathcal{M}$ is strictly stronger than that induced from $d^{[i]}_{(a_\alpha, b_\alpha)}$ for $i = 3, 4$ on $\mathcal{M}$.*

P r o o f. Let $A, B \in \mathcal{M}$. Let $A \neq 0$ and $B = 0$. Then by Lemma 2.4 (a) we have

$$\|S_{a_\alpha}(A) - I\| = \|\sqrt{I - a_\alpha^2|A|^2} - I\| \leqslant \sqrt{\|a_\alpha^2|A|^2\|} \leqslant a_\alpha\|A\|$$

and $\|a_\alpha|A|\| = a_\alpha\|A\|$. It follows that $d^{[3]}_{(a_\alpha, b_\alpha)}(A, 0) = a_\alpha\|A\|$ and

$$d^{[4]}_{(a_\alpha, b_\alpha)}(A, 0) = \sqrt{2(\|a_\alpha|A|\|)^2 + 2\|S_{a_\alpha}(A) - I\|^2} \leqslant 2a_\alpha\|A\|.$$

Now, let A and B be two nonzero elements of $\mathcal{M}$. Then by Lemma 2.4 (a) and Lemma 2.6 we have

$$\begin{aligned} d^{[3]}_{(a_\alpha, b_\alpha)}(A, B) &= \|a_\alpha|A| - b_\alpha|B|\| \leqslant \sqrt{\|a_\alpha^2 A^*A - b_\alpha^2 B^*B\|} \\ &\leqslant \sqrt{\|a_\alpha A^* - b_\alpha B^*\|\|a_\alpha A\| + \|b_\alpha B^*\|\|a_\alpha A - b_\alpha B\|} \\ &= \sqrt{(\|a_\alpha A\| + \|b_\alpha B\|)\|a_\alpha A - b_\alpha B\|} \\ &\leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|}\sqrt{\frac{2\alpha\|A - B\|}{\|A\|}}. \end{aligned}$$

Also, since

$$\begin{aligned} \|S_{a_\alpha}(A) - S_{b_\alpha}(B)\| &= \|\sqrt{I - a_\alpha^2|A|^2} - \sqrt{I - b_\alpha^2|B|^2}\| \\ &\leqslant \sqrt{\|(I - a_\alpha^2|A|^2) - (I - b_\alpha^2|B|^2)\|} \\ &= \sqrt{\|a_\alpha^2 A^*A - b_\alpha^2 B^*B\|} \leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|}\sqrt{\frac{2\alpha\|A - B\|}{\|A\|}}, \end{aligned}$$

we get that

$$d^{[4]}_{(a_\alpha,b_\alpha)}(A,B) \leqslant \sqrt{4(\|a_\alpha A\| + \|b_\alpha B\|)\frac{2\alpha\|A-B\|}{\|A\|}}.$$

This completes the proof. □

Recall that in the study carried out by Benharrat and Messirdi in [1], it was proved that the topology induced from the metric $q_G(T,S)$ on $\mathcal{C}(H_1,H_2)$ is strictly stronger than that induced from $g_G(T,S)$. However, in the following theorem we show that $d^{[1]} \cong d^{[3]}$.

Theorem 2.8. *The topology induced from the metric $d^{[1]}$ on $\mathcal{M}$ is equivalent to the topology induced from to the metric $d^{[3]}$ on $\mathcal{M}$.*

Proof. Let $A,B \in \mathcal{M}$. Then by Lemma 2.4 (a) and the definition of $d^{[i]}$ for $i=1,3$ we have

$$\begin{aligned}
d^{[3]}_{(a,b)}(A,B) &= \|a|A| - b|B|\| = \|a|A|S_a(A)S_a^{-1}(A) - b|B|S_b(B)S_b^{-1}(B)\| \\
&\leqslant \|a|A|S_a(A) - b|B|S_b(B)\|\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\|S_a^{-1}(A) - S_b^{-1}(B)\| \\
&\leqslant d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| + \|b|B|S_b(B)\|\sqrt{\|S_a^{-2}(A) - S_b^{-2}(B)\|} \\
&= d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A) - R_b^{-1}(B)\|} \\
&= d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A)(R_a(A) - R_b(B))R_b^{-1}(B)\|} \\
&\leqslant d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A)\|}\sqrt{\|R_b^{-1}(B)\|d^{[1]}_{(a,b)}(A,B)}.
\end{aligned}$$

Conversely, by Lemma 2.2 (e) and Lemma 2.4 (a) we obtain

$$\begin{aligned}
\|R_a(A) - R_b(B)\| &= \|(I - R_a(A)) - (I - R_b(B))\| = \|a^2|A|^2 - b^2|B|^2\| \\
&\leqslant \|a|A| - b|B|\|(\|a|A|\| + \|b|B|\|) = d^{[3]}_{(a,b)}(A,B)(\|a|A|\| + \|b|B|\|)
\end{aligned}$$

and

$$\begin{aligned}
\|a|A|S_a(A) - b|B|S_b(B)\| &\leqslant \|a|A| - b|B|\|\|S_a(A)\| + \|b|B|\|\|S_a(A) - S_b(B)\| \\
&\leqslant \|a|A| - b|B|\| + \|b|B|\|\|\sqrt{R_a(A)} - \sqrt{R_b(B)}\| \\
&\leqslant d^{[3]}_{(a,b)}(A,B) + \|b|B|\|\sqrt{\|R_a(A) - R_b(B)\|} \\
&\leqslant d^{[3]}_{(a,b)}(A,B) + \|b|B|\|\sqrt{d^{[3]}_{(a,b)}(A,B)(\|a|A|\| + \|b|B|\|)}.
\end{aligned}$$

But

$$(d^{[1]}_{(a,b)}(A,B))^2 \leqslant 2\|R_a(A)-R_b(B)\|^2+2\|a|A|S_a(A)-b|B|S_b(B)\|^2.$$

This completes the proof. □

3. Some operator transformations

The following lemma will be used in this section to obtain a new operator transform.

Lemma 3.1. *Let $A \in \mathcal{B}(H)$ be a normal operator. Then*

$$\|(I+S_a(A))^{-1}\| \leqslant 1.$$

Proof. For all $x \in H$ we have

$$\begin{aligned}\big\|\sqrt{(I+S_a(A))(x)}\big\|^2 &= \big\langle \sqrt{I+S_a(A)}(x), \sqrt{I+S_a(A)}x\big\rangle \\ &= \langle (I+S_a(A))(x), x\rangle = \langle x,x\rangle + \langle (S_a(A))x, x\rangle \geqslant \|x\|^2,\end{aligned}$$

and $R\big(\sqrt{I+S_a(A)}\big) = N\big(\sqrt{I+S_a(A)}\big)^{\perp} = H$. Thus, $\sqrt{I+S_a(A)}$ and hence $I+S_a(A)$ is invertible. Now, replacing x by $\sqrt{I+S_a(A)}(x)$ we obtain

$$\|\sqrt{I+S_a(A)}(x)\| \leqslant \|x\|.$$

It follows that

$$\big\|(I+S_a(A))^{-1}\big\| \leqslant \big\|\sqrt{I+S_a(A)}\big\|^2 \leqslant 1.$$

□

Definition 3.2. For $A \in \mathcal{M}$ and $0 < a < \|A\|^{-1}$ the bisecting of A, in the sense of Lambert and Petrovic, is the operator $\widetilde{A}_a$ defined as

$$\widetilde{A}_a = a|A|(I+S_a(A))^{-1}.$$

The bisecting of A was originally introduced in [8] by Labrousse in order to study closed operators. By Lemma 3.1, $I+S_a(A)$ is invertible and so $\widetilde{A}_a$ as a positive operator is well defined. Moreover, $\|\widetilde{A}_a\| \leqslant \|a|A|\|\|(I+S_a(A))^{-1}\| \leqslant 1$.

Now we consider the maps

$$\begin{aligned}
F_1 &\colon (\mathcal{M}, \|\cdot\|) \to (\mathcal{M}, \|\cdot\|), \quad A \to (I + S_a(A))^{-1};\\
F_2 &\colon (\mathcal{M}, \|\cdot\|) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a;\\
F_3 &\colon (\mathcal{M}, d^{[3]}) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a;\\
F_4 &\colon (\mathcal{M}, d^{[4]}) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a.
\end{aligned}$$

We note that in $(\mathcal{M}, \|\cdot\|)$, $\|\cdot\|$ is the norm of H. We pose the following question:

For which operators $A \in \mathcal{M}$ is the map F_i continuous?

Theorem 3.3. *The maps F_1, F_2, F_3 and F_4 are continuous.*

Proof. Let $A \in \mathcal{M}$ and $\|A\| \to 0$. By Theorem 2.7 and Lemma 3.1 we obtain

$$\begin{aligned}
\|F_1(A) - F_1(0)\| &= \|(I + S_{a_\alpha}(A))^{-1} - (I + I)^{-1}\|\\
&\leqslant \|(I + S_{a_\alpha}(A))^{-1}\| \|I + S_{a_\alpha}(A) - 2I\| \|(2I)^{-1}\|\\
&\leqslant \|S_{a_\alpha}(A) - I\| \leqslant a_\alpha \|A\| \to 0.
\end{aligned}$$

Now, let A and B be two nonzero elements of $\mathcal{M}$ and $\|A - B\| \to 0$. We show that $\|F_1(A) - F_1(B)\| \to 0$. Again by Theorem 2.7 and Lemma 3.1, if $\|A - B\| \to 0$, we have

$$\begin{aligned}
\|F_1(A) - F_1(B)\| &= \|(I + S_{a_\alpha}(A))^{-1} - (I + S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \|(I + S_{a_\alpha}(A))^{-1}\| \|S_{a_\alpha}(A) - S_{b_\alpha}(B)\| \|(I + S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{\frac{2\alpha\|A - B\|}{\|A\|}} \to 0.
\end{aligned}$$

Thus, F_1 is continuous.

Let $A \in \mathcal{M}$ and $\|A\| \to 0$. By Lemma 3.1 we have

$$\|F_2(A) - F_2(0)\| = \|\widetilde{A}_a - \widetilde{0}\| = \|a_\alpha |A| (I + S_{a_\alpha}(A))^{-1}\| \leqslant \|a_\alpha |A|\| = a_\alpha \|A\| \to 0.$$

Now, let A and B be two nonzero elements of $\mathcal{M}$ and $\|A - B\| \to 0$. Then from Theorem 2.7 we obtain

$$\begin{aligned}
\|F_2(A) - F_2(B)\| &= \|\widetilde{A}_a - \widetilde{B}_b\| = \|a_\alpha |A| (I + S_{a_\alpha}(A))^{-1} - b_\alpha |B| (I + S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha^2 A^* A - b_\alpha^2 B^* B\|}\, \|(I + S_{a_\alpha}(A))^{-1}\|\\
&\quad + \sqrt{\|b_\alpha B^* B\|}\, \|(I + S_{a_\alpha}(A))^{-1} - (I + S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{\frac{2\alpha\|A - B\|}{\|A\|}} \big(1 + \sqrt{\|b_\alpha B^* B\|}\big) \to 0.
\end{aligned}$$

This implies that F_2 is continuous.

Let $A \in \mathcal{M}$ such that $d^{[3]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|\| \to 0$. Then we have

$$\|F_3(A) - F_3(0)\| = \|\widetilde{A}_a - \widetilde{0}\| = \|a|A|(I + S_a(A))^{-1} - 0\| \leqslant \|a|A|\| \to 0.$$

Let A and B be two nonzero elements of $\mathcal{M}$ and $d^{[3]}_{(a,b)}(A,B) \to 0$. Then

$$\|a|A| - b|B|\| \to 0.$$

Again by Theorem 2.7 and definition of $d^{[3]}$ we have

$$\begin{aligned}\|F_3(A) - F_3(B)\| = \|\widetilde{A}_a - \widetilde{B}_b\| &= \|a|A|(I + S_a(A))^{-1} - b|B|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\|\|(I + S_a(A))^{-1}\| \\ &\quad + \|b|B|\|\|(I + S_a(A))^{-1}\|\|S_a(A) - S_b(B)\|\|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\sqrt{\|a^2|A|^2 - b^2|B|^2\|} \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\sqrt{\|a|A|\| + \|b|B|\|}\sqrt{\|a|A| - b|B|\|} \to 0.\end{aligned}$$

Thus, F_3 is also continuous.

Let $A \in \mathcal{M}$ and $d^{[4]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|\| \to 0$. Then

$$\begin{aligned}\|F_4(A) - F_4(0)\| = \|\widetilde{A}_a - \widetilde{0}\| &= \|a|A|(I + S_a(A))^{-1} - 0\| \\ &\leqslant \|a|A|\|\|(I + S_a(A))^{-1}\| \leqslant \|a|A|\| \to 0.\end{aligned}$$

Let $A, B \in \mathcal{M}$ such that $d^{[4]}_{(a,b)}(A,B) \to 0$. Then $\|a|A| - b|B|\| \to 0$ and $\|S_a(A) - S_b(B)\| \to 0$. Then we have

$$\begin{aligned}\|F_4(A) - F_4(B)\| = \|\widetilde{A}_a - \widetilde{B}_b\| &= \|a|A|(I + S_a(A))^{-1} - b|B|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\|\|(I + S_a(A))^{-1}\| \\ &\quad + \|b|B|\|\|(I + S_a(A))^{-1}\|\|S_a(A) - S_b(B)\|\|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\|S_a(A) - S_b(B)\| \to 0.\end{aligned}$$

Consequently, $\|F_4(A) - F_4(B)\| \to 0$ as $d^{[4]}_{(a,b)}(A,B) \to 0$. □

Definition 3.4. If $A, B \in \mathcal{M}$, $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$. The Cordes-Labrousse transform with respect to the pair (A,B) is the operator $V^{(a,b)}_{A,B}$ given by

$$V^{(a,b)}_{A,B} = S_a(A)S_b(B) + (a|A|)(b|B|).$$

We will write $V^{(a,b)}_{A,B}$ simply as $V_{A,B}$ for fixed elements A and B when no confusion can arise. Since A and B are normal operators then $V^*_{A,B} = V_{B,A}$. Also, $V_{A,A} = R_a(A) + a^2|A|^2 = R_a(A) + I - R_a(A) = I$.

The proof of the following proposition is similar in spirit to [2], Lemma 5.3.

Lemma 3.5. *Let $A, B \in \mathcal{M}$ and let $x \in H$. Then the following assertions hold.*

(a) $\big|\|V_{A,B}(x)\|^2 - \|x\|^2\big| \leqslant \|x\|^2 d^{[2]}_{(a,b)}(A,B)$;

(b) $\|V_{A,B}(x)\|^2 \geqslant (1 - (d^{[2]}_{(a,b)}(A,B))^2)\|x\|^2$;

(c) *If* $d^{[2]}_{(a,b)}(A,B) < 1$, *then* $V_{A,B}$ *is invertible.*

Example 3.6. Let (X, Σ, μ) be a complete σ-finite measure space. Let $\varphi\colon X \to X$ be a non-singular measurable point transformation, which means the measure $\mu \circ \varphi^{-1}$, defined by $\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1}(B))$ for all $B \in \Sigma$, is absolutely continuous with respect to μ (we write $\mu \circ \varphi^{-1} \ll \mu$). It follows that $\mu \circ \varphi^{-n} \ll \mu$ for every $n \in \mathbb{N}$. Then by Radon-Nikodym theorem there exists a unique non-negative Σ-measurable function h_n on X with $h_n = \mathrm{d}\mu \circ \varphi^{-n}/\mathrm{d}\mu$. Put $h_1 = h$. Now, let C_φ defined by $C_\varphi(f) = f \circ \varphi$ be a composition operator on $L^2(\Sigma)$. Note that $C_\varphi \in B(L^2(\Sigma))$ if and only if $h \in L^\infty(\Sigma)$ and in this case $\|C_\varphi\| = \|h\|_\infty^{1/2}$. Also it is a classical fact that $C_\varphi \in B(L^2(\Sigma))$ is normal if and only if $\varphi^{-1}(\Sigma) = \Sigma$ and $h \circ \varphi = h$ (see [10]). Let $\mathcal{M} = \{C_\varphi \in B(L^2(\Sigma))\colon C_\varphi \text{ is normal}\}$. Let $C_\varphi \in B(L^2(\Sigma))$ and $f \in L^2(\Sigma)$. Then we have

$$\begin{aligned}\langle C_\varphi^{*^n} C_\varphi^n f, f\rangle &= \langle C_\varphi^n f, C_\varphi^n f\rangle = \|C_\varphi^n f\|^2 = \|C_{\varphi^n} f\|^2 \\ &= \|M_{\sqrt{h_n}} f\|^2 = \langle M_{\sqrt{h_n}} f, M_{\sqrt{h_n}} f\rangle = \langle M_{h_n} f, f\rangle,\end{aligned}$$

where M_{h_n} is the multiplication operator. So, $C_\varphi^{*^n} C_\varphi^n = M_{h_n}$. In particular, if $C_\varphi \in \mathcal{M}$, then $C_\varphi^{*^n} C_\varphi^n = (C_\varphi^* C_\varphi)^n = (M_h)^n = M_{h^n}$, and so $h_n = h^n$ for each $n \in \mathbb{N}$. Let $0 < a < \|h\|_\infty^{-1/2} = \|C_\varphi\|^{-1} = r(C_\varphi)^{-1}$. Then

$$K_a(C_\varphi) = \sum_{n=0}^{\infty} a^{2n} C_\varphi^{*^n} C_\varphi^n = \sum_{n=0}^{\infty} M_{a^{2n} h^n} = (I - M_{a^2 h})^{-1}.$$

Hence

$$\begin{aligned} R_a(C_\varphi) = K_a(C_\varphi)^{-1} = I - M_{a^2 h}, \quad S_a(C_\varphi) = R_a\sqrt{C_\varphi} = M_{\sqrt{1 - a^2 h}}, \\ (\widetilde{C}_\varphi)_a = a|C_\varphi|(I + S_a(C_\varphi))^{-1} = M_{\sqrt{a^2 h}/(1 + \sqrt{1 - a^2 h})}. \end{aligned}$$

Now, for $i = 1, 2$ let $C_{\varphi_i} \in \mathcal{M}$ and $h_i = (\mathrm{d}\mu \circ \varphi_i^{-1})/\mathrm{d}\mu$. Then we have

$$V_{C_{\varphi_1}, C_{\varphi_2}} = S_a(C_{\varphi_1}) S_b(C_{\varphi_2}) + (a|C_{\varphi_1}|)(b|C_{\varphi_2}|) = M_{\sqrt{(1 - a^2 h_1)(1 - b^2 h_2)} + \sqrt{a^2 b^2 h_1 h_2}}.$$

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