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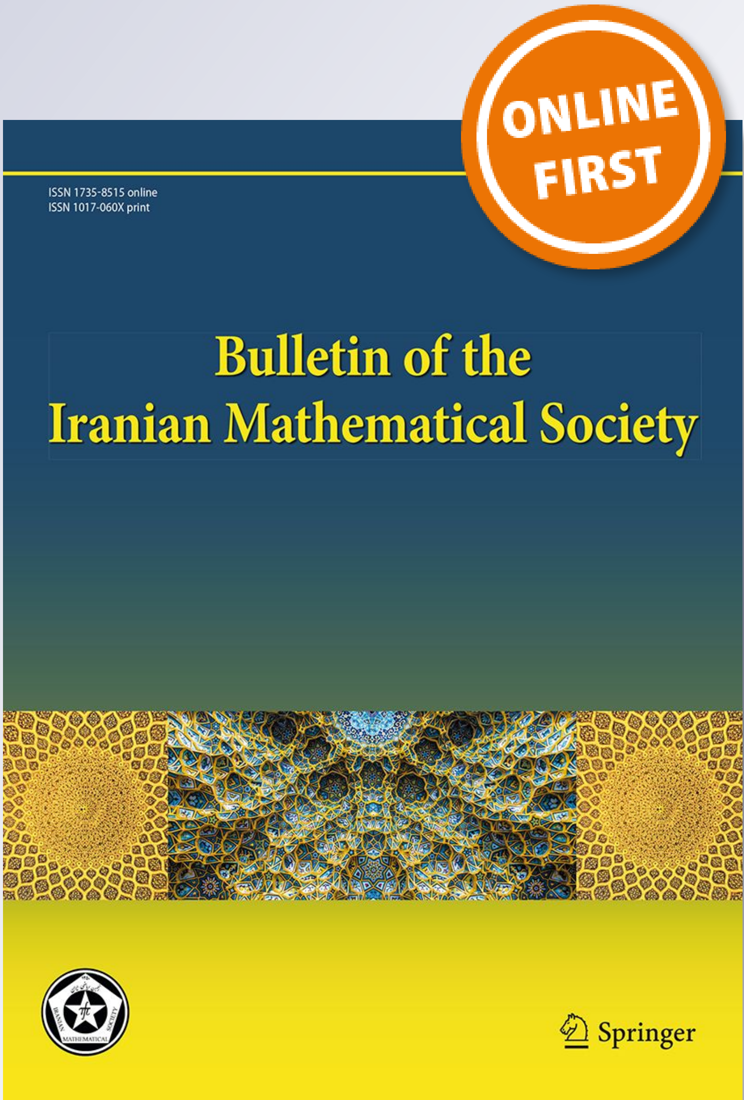

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ORIGINAL PAPER

Weighted Composition Lambert-Type Operators via Matrix Representation

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Abstract

In this note, we discuss matrix theoretic characterizations for weighted composition Lambert-type operators of the form $T_\varphi := M_w E M_u C_\varphi$ in some operator classes on $\ell^2(\mathbb{N}_0)$, such as quasinormal, hyponormal, binormal, n-hyponormal, A-class and $*$-A-classes. Also, polar decomposition, Aluthge and mean transform of T_φ will be investigated.

Keywords Aluthge transformation · Mean transform · Polar decomposition · Matrix representation · A-class operator

Mathematics Subject Classifiation 47B20 · 47B38

1 Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. All sets and function statements are to be interpreted as being valid almost everywhere with respect to μ. For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon–Nikodym theorem, there exists a unique

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$\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f \mathrm{d}\mu = \int_A E^{\mathcal{A}}(f) \mathrm{d}\mu,$$

where A is any $\mathcal{A}$-measurable set for which $\int_A f \mathrm{d}\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} : L^2(\Sigma) \rightarrow L^2(\mathcal{A})$ uniquely defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. We shall henceforth find it convenient to write $E^{\mathcal{A}}$ simply as E. The mapping E is a linear orthogonal projection onto $L^2(\mathcal{A})$. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. The role of this operator is important in this note. For more details on the properties of E, see [7,11]. In this note, we will restrict ourselves to the Hilbert space $\ell^2(\mathbb{N}_0) = L^2(\mathbb{N}_0, 2^{\mathbb{N}_0}, \mu)$, where μ is the counting measure on $2^{\mathbb{N}_0}$. Put $\mathcal{A}_0 = \varphi^{-1}(2^{\mathbb{N}_0})$. It is easy to check that for each $f \in B(\ell^2(\mathbb{N}_0))$ and $k \in \mathbb{N}_0$, we have (see [9])

$$E^{\mathcal{A}_0}(f)(k) = \frac{\sum_{n \in \varphi^{-1}(\varphi(k))} f_n}{\sum_{n \in \varphi^{-1}(\varphi(k))} 1}.$$

Let φ be a nonsingular measurable transformation from X into X; that is, $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ and write $\mu \circ \varphi^{-1} \ll \mu$. Let h be the Radon–Nikodym derivative $\mathrm{d}\mu \circ \varphi^{-1}/\mathrm{d}\mu$. The composition operator $C_\varphi : L^2(\Sigma) \rightarrow L^0(\Sigma)$ induced by φ is given by $C_\varphi(f) = f \circ \varphi$, for each $f \in L^2(\Sigma)$. Here, the non-singularity of φ guarantees that C_φ is well defined. A good reference for information on (weighted) composition operators on measurable function spaces is [1] and the monograph [12]. Now, take $u, w \in \mathcal{D}(E)$. Then the triple (u, w, φ) induces a weighted composition Lambert-type operator T_φ from $L^2(\Sigma)$ into $L^0(\Sigma)$ defined by $T_\varphi = M_w E M_u C_\varphi$, where M_w and M_u are multiplication operators, E is a conditional expectation operator and C_φ is a composition operator. Weighted composition Lambert-type operators on $L^p(\Sigma)$ spaces were initially introduced in [3]. These type of operators are a generalization of the Lambert operators, weighted Lambert operators and the classical composition operators on measurable function spaces. If $hE^{\mathcal{A}_0}(E(|u|^2)E(|w|^2)) \circ \varphi^{-1} \in \ell^\infty(\mathbb{N}_0)$, then T_φ is bounded on $\ell^2(\mathbb{N}_0)$ (see [3]). Throughout this paper, we assume that $u\mathcal{R}(C_\varphi) \subseteq \mathcal{D}(E)$, $w \in \mathcal{D}(E)$, $E = E^{\mathcal{A}}$, φ is non-singular and $T_\varphi = M_w E M_u C_\varphi = M_w E W$, where $W = M_u C_\varphi$, $\mathcal{R}(C_\varphi)$ denotes the range of C_φ.

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $B(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. Let α denote a weight sequence, $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$, where it is without loss of generality to assume these are all positive. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with standard basis $e_0, e_1, \ldots$, is defined by $W_\alpha(e_k) = \alpha_k e_{k+1}$ for all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $T \in B(\mathcal{H})$, let $T = U|T|$ be the polar decomposition of T. We set $T = [T]$, where $[T]$ denotes the matrix representation of T. The Aluthge transform $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. The mean transform $\widehat{T}$ of T is defined by $\widehat{T} = \frac{1}{2}(T + \widetilde{T}^D)$, where $\widetilde{T}^D$ denotes the Duggal transform $\widetilde{T}^D$ of T given by $\widetilde{T}^D = |T|U$. The mean transform $\widehat{T}$ is more convenient than $\widetilde{T}$ in practical use (see [10]). A good reference for information on partial normality classes of operators is the monograph [6].

In Sect. 2, we discuss matrix theoretic characterizations for weighted composition Lambert-type operators of the form $T_\varphi = M_w E M_u C_\varphi$ in some operator classes on $\ell^2(\mathbb{N}_0)$ such as, quasinormal, hyponormal, binormal, n-hyponormal, A-class and $*$-A-classes. Also, polar decomposition, Aluthge and mean transform of T_φ will be investigated. Our characterizations are based on the matrix representation of T_φ. The class of weighted composition Lambert-type operators includes the two well-known classes of operators, namely, the class of weighted composition operators and the weighted Lambert-type operators whenever $E = I$ and φ is identity transform, respectively. Most of these operator classes for these special cases have been characterized (see, e.g., [2,4,7,8]) without using the matrix representation with a relatively complex proof.

2 Main Results

Let $\{e_n\}_{n\in\mathbb{N}_0}$ be an orthornormal basis for $\ell^2(\mathbb{N}_0)$ and let $u \in \ell^2(\mathbb{N}_0)$ with $u_0 = 0$ and $u(n) = u_n \geq 0$ for all $n \in \mathbb{N}$. Define $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ as

$$\varphi(n) = \begin{cases} 0 & n = 0, 1, \\ n-1 & n \geq 2. \end{cases}$$

Then for each $f \in \ell^2(\mathbb{N}_0)$, we have $Wf = (0, u_1 f_0, u_2 f_1, \ldots)$, where $W = M_u C_\varphi$ is a weighted composition operator induced by the pair (u, φ) and $f(n) = f_n$. Thus, the matrix representation of the forward weighted shift W can now be written as:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \ldots \\ u_1 & 0 & 0 & 0 \ldots \\ 0 & u_2 & 0 & 0 \ldots \\ 0 & 0 & u_3 & 0 \ldots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}.$$

Then for $u \in l^\infty(\mathbb{N}_0)$, we have

$$W^* = \begin{pmatrix} 0 & u_1 & 0 & 0 & \ldots \\ 0 & 0 & u_2 & 0 & \ldots \\ 0 & 0 & 0 & u_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Recall that for $T \in B(\mathcal{H})$, the C^*-algebras of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry, i.e., $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. Then the parts of the polar decomposition U_W, $|W|$ for W are given by

$$|W| = \begin{pmatrix} u_1 & 0 & 0 & 0 & \dots \\ 0 & u_2 & 0 & 0 & \dots \\ 0 & 0 & u_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check that $UU^*U = U$. Hence, U is a partial isometry. Moreover, the matrix representation $\widetilde{W} = |W|^{\frac{1}{2}}U|W|^{\frac{1}{2}}$, the Aluthge transformation of W, is obtained as follows:

$$\widetilde{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ (u_1u_2)^{\frac{1}{2}} & 0 & 0 & 0 & \dots \\ 0 & (u_2u_3)^{\frac{1}{2}} & 0 & 0 & \dots \\ 0 & 0 & (u_3u_4)^{\frac{1}{2}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we define the mean transform of T by $\widehat{W} = \frac{1}{2}(U|W|+|W|U) = \frac{1}{2}(W+W^D)$, then we get that

$$\widehat{W} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \frac{1}{2}(u_1+u_2) & 0 & 0 & \dots \\ 0 & \frac{1}{2}(u_2+u_3) & 0 & \dots \\ 0 & 0 & \frac{1}{2}(u_3+u_4) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $r, s \in \mathbb{N}$. Define a non-singular measurable transformation ψ on $\mathbb{N}_0$ such that $\psi^{-1}(\{0\}) = \{0, 1\}$ and

$$\psi^{-1}(\{2k\}) = \{(k-1)(r+s)+r+i+1 : 1 \le i \le s\}, \quad k = 1, 2, 3, \dots$$
$$\psi^{-1}(\{2k-1\}) = \{(k-1)(r+s)+i+1 : 1 \le i \le r\}, \quad k = 1, 2, 3, \dots .$$

Put $\mathcal{A}_{r,s} = \psi^{-1}(2^{\mathbb{N}_0}) = \{\{0, 1\}, \{2, \dots, r+1\}, \{r+2, \dots, r+s+1\}, \{r+s+2, \dots, 2r+s+1\}, \{2r+s+2, \dots, 2r+2s+1\}, \dots\}$. Then,

$$E^{\mathcal{A}_{r,s}}(e_i)(k) = \frac{\sum_{j\in\psi^{-1}(\psi(k))} e_i(j)}{\sum_{j\in\psi^{-1}(\psi(k))} 1}.$$

The matrix of conditional expectation operator $E^{\mathcal{A}_{r,s}}$ can now be written in block matrix form as

$$E^{\mathcal{A}_{r,s}} = \begin{pmatrix} A_{1,1} & & & O & \\ & A_{2,2} & & & \\ & & \ddots & & \\ & & & A_{n,n} & \\ & O & & & \ddots \end{pmatrix},$$

where $A_{i,j} = 0$ for $i \neq j$,

$$A_{1,1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and for $k = 1, 2, 3, \ldots$, we have

$$A_{2k,2k} = \begin{pmatrix} \frac{1}{r} & \cdots & \frac{1}{r} \\ \vdots & \vdots & \vdots \\ \frac{1}{r} & \cdots & \frac{1}{r} \end{pmatrix}, \qquad A_{2k+1,2k+1} = \begin{pmatrix} \frac{1}{s} & \cdots & \frac{1}{s} \\ \vdots & \vdots & \vdots \\ \frac{1}{s} & \cdots & \frac{1}{s} \end{pmatrix}.$$

To avoid tedious calculations, from now on, we will consider the case where $r = s = 1$ and take $E^{\mathcal{A}_{1,1}} = E$. In this case, we have

$$E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and then

$$EW = \begin{pmatrix} \frac{1}{2}u_1 & 0 & 0 & \ldots \\ \frac{1}{2}u_1 & 0 & 0 & \ldots \\ 0 & u_2 & 0 & \ldots \\ 0 & 0 & u_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, let $w = \{w_n\}_{n=0}^{\infty} \in l^{\infty}(\mathbb{N}_0)$ be a sequence of real numbers. Then,

$$M_w = \begin{pmatrix} w_0 & 0 & 0 & 0 & \ldots \\ 0 & w_1 & 0 & 0 & \ldots \\ 0 & 0 & w_2 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and hence matrix T_φ can be represented by

$$T_\varphi = M_w E W = \begin{pmatrix} \frac{1}{2}u_1 w_0 & 0 & 0 & \dots \\ \frac{1}{2}u_1 w_1 & 0 & 0 & \dots \\ 0 & u_2 w_2 & 0 & \dots \\ 0 & 0 & u_3 w_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in B(l^2(\mathbb{N}_0)). \tag{2.1}$$

Thus,

$$T_\varphi^* = \begin{pmatrix} \frac{1}{2}u_1 w_0 & \frac{1}{2}u_1 w_1 & 0 & 0 & \dots \\ 0 & 0 & u_2 w_2 & 0 & \dots \\ 0 & 0 & 0 & u_3 w_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It follows that

$$T_\varphi^* T_\varphi = \begin{pmatrix} \frac{1}{4}(u_1 w_0)^2 + \frac{1}{4}(u_1 w_1)^2 & 0 & 0 & \dots \\ 0 & (u_2 w_2)^2 & 0 & \dots \\ 0 & 0 & (u_3 w_3)^2 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.2}$$

and

$$T_\varphi T_\varphi^* = \begin{pmatrix} \frac{1}{4}(u_1 w_0)^2 & \frac{1}{4}(u_1 w_0 u_1 w_1) & 0 & 0 & \dots \\ \frac{1}{4}(u_1 w_0 u_1 w_1) & \frac{1}{4}(u_1 w_1)^2 & 0 & 0 & \dots \\ 0 & 0 & (u_2 w_2)^2 & 0 & \dots \\ 0 & 0 & 0 & (u_3 w_3)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.3}$$

Then,

$$|T_\varphi| = \begin{pmatrix} \frac{1}{2}\sqrt{(u_1 w_0)^2 + (u_1 w_1)^2} & 0 & 0 & \dots \\ 0 & u_2|w_2| & 0 & \dots \\ 0 & 0 & u_3|w_3| & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $T_\varphi = U_\varphi |T_\varphi|$ be the polar decomposition of T_φ. Then we obtain

$$U_\varphi = \begin{pmatrix} \frac{u_1 w_0}{\sqrt{(u_1 w_0)^2+(u_1 w_1)^2}} & 0 & 0 & \cdots \\ \frac{u_1 w_1}{\sqrt{(u_1 w_0)^2+(u_1 w_1)^2}} & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check that U_φ is a partial isometry, i.e., $U_\varphi U_\varphi^* U_\varphi = U_\varphi$. Put $\lambda := \frac{1}{2}(u_1|w_1|) + \frac{u_1 u_2 |w_2 w_3|}{\sqrt{(u_1 w_0)^2+(u_1 w_1)^2}}$. Then we get that

$$\widehat{T_\varphi} = \frac{1}{2} \begin{pmatrix} u_1|w_0| & 0 & 0 & \cdots \\ \lambda & 0 & 0 & \ldots \\ 0 & u_2|w_2| + u_3|w_3| & 0 & \ldots \\ 0 & 0 & u_3|w_3| + u_4|w_4| & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.4}$$

Consequently,

$$(\widehat{T_\varphi})^* \widehat{T_\varphi} = \frac{1}{4} \begin{pmatrix} (u_1 w_0)^2 + \lambda^2 & 0 & 0 & \cdots \\ 0 & (u_2|w_2| + u_3|w_3|)^2 & 0 & \cdots \\ 0 & 0 & (u_3|w_3| + u_4|w_4|)^2 & \ldots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\widehat{T_\varphi}(\widehat{T_\varphi})^* = \frac{1}{4} \begin{pmatrix} (u_1 w_0)^2 & \lambda u_1|w_0| & 0 & \cdots \\ \lambda u_1|w_0| & \lambda^2 & 0 & \ldots \\ 0 & 0 & (u_2|w_2| + u_3|w_3|)^2 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.5}$$

These observations establish the following theorem.

Theorem 2.1 *Let $T_\varphi \in B(l^2(\mathbb{N}_0))$ and let $\lambda = \frac{1}{2}(u_1|w_1|) + \frac{u_1 u_2 |w_2 w_3|}{\sqrt{(u_1 w_0)^2+(u_1 w_1)^2}}$. Then the following assertions hold.*

(a) *T_φ is partial isometry, i.e., $T_\varphi T_\varphi^* T_\varphi = T_\varphi$ if and only if $u_1 w_0 \neq 0$, $(u_1 w_0)^2 + (u_1 w_1)^2 = 4$ and, for each $n \geq 1$, $u_n |w_{n+1}| \neq 0$ and $(u_{n+1} w_{n+1})^2 = 1$.*
(b) *$\widehat{T_\varphi} = T_\varphi$ if and only if $2(u_2|w_2|) = \sqrt{(u_1 w_0)^2 + (u_1 w_1)^2}$ and for each $n \geq 2$, $u_n|w_n| = u_{n+1}|w_{n+1}|$.*

(c) *T_φ is hyponormal if and only if $w_0 w_1 \leq 0$, $4(u_2 w_2)^2 \geq (u_1 w_1)^2$ and for each $n \geq 3$, $(u_n w_n)^2 \geq (u_{n-1} w_{n-1})^2$.*

(d) *$\widehat{T_\varphi}$ is hyponormal if and only if $\lambda u_1 w_0 \leq 0$, $(u_2 w_2 + u_3 w_3)^2 \geq \lambda^2$ and for each $n \geq 3$, $(u_n w_n + u_{n+1} w_{n+1})^2 \geq (u_{n-1} w_{n-1} + u_n w_n)^2$.*

(e) *The matrix form of the Aluthge transformation of T_φ is*

$$\widetilde{T_\varphi} = \begin{pmatrix} \frac{1}{2}(u_1|w_0|) & 0 & 0 & \cdots \\ \frac{(u_1|w_1|)\sqrt{u_2|w_2|}}{\sqrt{2}((u_1 w_0)^2+(u_1 w_1)^2)^{\frac{1}{4}}} & 0 & 0 & \cdots \\ 0 & \sqrt{(u_2|w_2|)(u_3|w_3|)} & 0 & \cdots \\ 0 & 0 & \sqrt{(u_3|w_3|)(u_4|w_4|)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof Since

$$T_\varphi = \begin{pmatrix} \frac{1}{2}u_1 w_0 & 0 & 0 & \cdots \\ \frac{1}{2}u_1 w_1 & 0 & 0 & \cdots \\ 0 & u_2 w_2 & 0 & \cdots \\ 0 & 0 & u_3 w_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$T_\varphi^* T_\varphi = \begin{pmatrix} \frac{1}{4}(u_1 w_0)^2 + \frac{1}{4}(u_1 w_1)^2 & 0 & 0 & \cdots \\ 0 & (u_2 w_2)^2 & 0 & \cdots \\ 0 & 0 & (u_3 w_3)^2 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then

$$T_\varphi T_\varphi^* T_\varphi = \begin{pmatrix} \frac{1}{8}(u_1 w_0)^3 + \frac{1}{8}(u_1 w_0)(u_1 w_1)^2 & 0 & 0 & \cdots \\ \frac{1}{8}(u_1 w_0)^2(u_1 w_1) + \frac{1}{8}(u_1 w_1)^3 & 0 & 0 & \cdots \\ 0 & (u_2 w_2)^3 & 0 & \cdots \\ 0 & 0 & (u_3 w_3)^3 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By the above relations (a) holds. The proofs of the other implications are similar by relations (2.1), (2.2), (2.3), (2.4), (2.5). □

Recall that an operator $T \in B(H)$ is quasinormal if $[T_\varphi, T_\varphi^* T_\varphi] = 0$ and T is binormal if $[T_\varphi^* T_\varphi, T_\varphi T_\varphi^*] = 0$. For each $n \in \mathbb{N}$, if $(T^*T)^n \geq (TT^*)^n$, T is called n-hyponormal operator. T is an A-class operator if $|T^2| \geq |T|^2$ and T is a $*$-A-class if $|T^2| \geq |T^*|^2$.

By using (2.2) and (2.3), T is quasinormal if and only if $(u_1w_0)^2 + (u_1w_1)^2 = 4(u_2w_2)^2 = 4(u_3w_3)^2 = \cdots = 4(u_nw_n)^2$ for all $n \in \mathbb{N}$. Moreover, by (2.2) and (2.3), we obtain

$$T_\varphi^* T_\varphi T_\varphi T_\varphi^* = \begin{pmatrix} M_1 & M_2 & 0 & 0 & \dots \\ M_3 & M_4 & 0 & 0 & \dots \\ 0 & 0 & (u_2w_2)^2(u_3w_3)^2 & 0 & \dots \\ 0 & 0 & 0 & (u_3w_3)^2(u_4w_4)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$T_\varphi T_\varphi^* T_\varphi^* T_\varphi = \begin{pmatrix} M_1 & M_3 & 0 & 0 & \dots \\ M_2 & M_4 & 0 & 0 & \dots \\ 0 & 0 & (u_2w_2)^2(u_3w_3)^2 & 0 & \dots \\ 0 & 0 & 0 & (u_3w_3)^2(u_4w_4)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} M_1 &= \frac{1}{16}\{(u_1w_0)^2 + (u_1w_1)^2\}(u_1w_0)^2; \\ M_2 &= \frac{1}{16}(u_1w_0)(u_1w_1)\{(u_1w_0)^2 + (u_1w_1)^2\}; \\ M_3 &= \frac{1}{4}(u_1w_0)(u_1w_1)(u_2w_2)^2; \\ M_4 &= \frac{1}{4}(u_1w_1)^2(u_2w_2)^2. \end{aligned}$$

Then, $T_\varphi^* T_\varphi T_\varphi T_\varphi^* = T_\varphi T_\varphi^* T_\varphi^* T_\varphi$ if and only if $M_2 = M_3$. Now, by direct calculations we have

$$T_\varphi^2 = \begin{pmatrix} \frac{1}{4}(u_1w_0)^2 & 0 & 0 & \dots \\ \frac{1}{4}(u_1w_0)(u_1w_1) & 0 & 0 & \dots \\ \frac{1}{2}(u_1w_1)(u_2w_2) & 0 & 0 & \dots \\ 0 & (u_2w_2)(u_3w_3) & 0 & \dots \\ 0 & 0 & (u_3w_3)(u_4w_4) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$(T_\varphi^*)^2 = \begin{pmatrix} \frac{1}{4}(u_1w_0)^2 & \frac{1}{4}(u_1w_0)(u_1w_1) & \frac{1}{2}(u_1w_1)(u_2w_2) & 0 & \dots \\ 0 & 0 & 0 & (u_2w_2)(u_3w_3) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$|T_\varphi^2|^2 = (T_\varphi^*)^2(T_\varphi)^2 = \begin{pmatrix} A_1 & 0 & 0 & 0 \dots \\ 0 & (u_2w_2)^2(u_3w_3)^2 & 0 & 0 \dots \\ 0 & 0 & (u_3w_3)^2(u_4w_4)^2 & 0 \dots \\ 0 & 0 & 0 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots \ \vdots \end{pmatrix},$$

where $A_1 = \frac{1}{16}(u_1w_0)^4 + \frac{1}{16}(u_1w_0)^2(u_1w_1)^2 + \frac{1}{4}(u_1w_1)^2(u_2w_2)^2$. Thus,

$$|T_\varphi^2| = \begin{pmatrix} \sqrt{A_1} & 0 & 0 & 0 \dots \\ 0 & \sqrt{(u_2w_2)^2(u_3w_3)^2} & 0 & 0 \dots \\ 0 & 0 & \sqrt{(u_3w_3)^2(u_4w_4)^2} & 0 \dots \\ 0 & 0 & 0 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots \ \vdots \end{pmatrix}.$$

Then T_φ is an A-class operator if and only if

$$\sqrt{A_1} \geq \frac{1}{4}(u_1w_0)^2 + \frac{1}{4}(u_1w_1)^2;$$
$$\sqrt{(u_2w_2)^2(u_3w_3)^2} \geq (u_2w_2)^2;$$
$$\sqrt{(u_3w_3)^2(u_4w_4)^2} \geq (u_3w_3)^2.$$

These observations establish the following theorem.

Theorem 2.2 *Let $T_\varphi \in B(l^2(\mathbb{N}_0))$. Then the following assertions hold.*

(a) *T_φ is quasinormal iff $(u_1w_0)^2 + (u_1w_1)^2 = 4(u_nw_n)^2$ for each $n \geq 2$.*
(b) *T_φ is binormal iff $u_1^2w_0w_1\{(u_1w_0)^2 + (u_1w_1)^2\} = 4(u_1^2w_0w_1)(u_2w_2)^2$.*
(c) *T_φ is 2-hyponormal iff $w_0w_1 \leq 0$, $16(u_2w_2)^4 \geq (u_1w_1)^2\{(u_1w_0)^2 + (u_1w_1)^2\}$ and for each $n \geq 3$, $(u_nw_n)^4 \geq (u_{n-1}w_{n-1})^4$.*
(d) *T_φ is 3-hyponormal iff $w_0w_1 \leq 0$, $64(u_2w_2)^6 \geq (u_1w_1)^2\{(u_1w_0)^2 + (u_1w_1)^2\}^2$ and for each $n \geq 3$, $(u_nw_n)^6 \geq (u_{n-1}w_{n-1})^6$.*
(e) *T_φ is an A-class operator iff $4(u_2w_2)^2 \geq (u_1w_0)^2 + (u_1w_1)^2$ and for each $n \geq 3$, $(u_nw_n)^2 \geq (u_{n-1}w_{n-1})^2$.*
(f) *T_φ is a $*$-A-class operator iff $w_0w_1 \leq 0$, $16(u_2w_2)^2(u_3w_3)^2 \geq (u_1w_1)^4$ and for each $n \geq 3$, $(u_nw_n)^2(u_{n+1}w_{n+1})^2 \geq (u_{n-1}w_{n-1})^4$.*

Example 2.3 (i) Let $u_n = \{0, 0, 1, 0, 1, 1, 1, \dots\}$ and $w_n = \{0, 2, 0, 1, 0, 0, \dots\}$. Then it is easy to check that T_φ is hyponormal, binormal, A-class and $*$-A-class operator, but it is neither quasinormal nor partial isometry. Moreover, $\widehat{T_\varphi}$ is also hyponormal.

(ii) Let $u_n = \{0, 1, 0, 1, 1, 1, 1, \dots\}$ and $w_n = \{1, 0, 1, 0, 0, \dots\}$. Then T_φ is hyponormal, binormal, $*$-A-class operator, but it is not quasinormal and partial isometry and A-class operator. In this case, $\widehat{T_\varphi}$ is also hyponormal.

Remark 2.4 Estaremi in [5] proved that $\widetilde{T_\varphi}$ is always normal whenever φ is an identity map. Now, let φ be a not identity map. Direct computations show that

$$T_\varphi^* T_\varphi f = h\{E_\varphi(\overline{u}E(|w|^2))E(uf \circ \varphi)\} \circ \varphi^{-1}.$$

But it is sometimes difficult to obtain $|T_\varphi|$. For showing this, we consider only the case $\varphi^{-1}(\Sigma) \subseteq \mathcal{A}$. Put $v = wE(u)$. In this case, $T_\varphi = M_v C_\varphi$ is a weighted composition operator. Let $V|T_\varphi|$ be the polar decomposition of T_φ. It is easy to check that $|T_\varphi|^{\frac{1}{2}} = M_{\sqrt[4]{J}}$ and $V = M_{\sqrt{J\circ\varphi}} T_\varphi$, where $J = hE_\varphi(|w|^2|E(w)|^2) \circ \varphi^{-1}$. Thus,

$$\widetilde{T_\varphi} = |T_\varphi|^{\frac{1}{2}} V |T_\varphi|^{\frac{1}{2}} = M_{\frac{\sqrt[4]{J}}{\sqrt{J\circ\varphi}}} T_\varphi M_{\sqrt[4]{J}}.$$

We recall that T_φ is normal if and only if $\widetilde{T_\varphi} = T_\varphi$ (see e.g. [6]). So, in this case, if $J = 1$ on X or T_φ is normal, then so is $\widetilde{T_\varphi}$. But, in general $\widetilde{T_\varphi}$ is not normal.

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