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PARALLELISM BETWEEN MOORE-PENROSE INVERSE AND ALUTHGE TRANSFORMATION OF OPERATORS

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In this paper we study some parallelisms between †-Aluthge transform and binormal operators on a Hilbert space via the Moore-Penrose inverse. Moreover, we give some applications of these results on the Lambert multiplication operators acting on $L^2(\Sigma)$.

1. INTRODUCTION AND PRELIMINARIES

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and the range of an operator $T \in B(H)$, respectively. Recall that for $T \in B(H)$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$, and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. Note that $T = |T^*|U = \sqrt{|T^*|}U\sqrt{|T|}$. More generally, $T = |T^*|^p U|T|^{1-p}$ for $p \in (0, 1)$; see e.g. [**13**, Theorem 2.7]. If $T = U|T|$ is the polar decomposition of $T \in B(\mathcal{H})$, then $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation of T. Let CR(H) be the set of all bounded linear operators on H with closed range. For $T \in$ CR(H), the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger$ that satisfies the following:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

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We recall that $T^\dagger$ exists if and only if $T \in$ CR(H). Note that if $T \in$ CR(H), then T^*, $|T|$ and $T^\dagger$ have closed range. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T|$ is invertible. It is a classical fact that the polar decomposition of T^* is $U^*|T^*|$. It is easy to check that $U^*|T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$ are the polar decomposition and Aluthge transformation of $T^\dagger$, respectively. For other important properties of $T^\dagger$ see [**2, 10, 17**].

An operator $T \in B(H)$ is said to be binormal if $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$ for operators A and B. The numerical range $W(T)$ of an operator $T \in B(H)$ is defined by $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. Also, $\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ and $Sp(T)$ denote the numerical radius and spectrum of T, respectively.

Study of Moore-Penrose inverse and Aluthge transformationaton of bounded linear operators has a long history. In this paper, we introduce †-Aluthge transformation which is parallel to Aluthge transformation. Then we investigate some connections and parallelisms between †-Aluthge transformation and binormal operators via the Moore-Penrose inverse. In section 2, firstly, we give a necessary and sufficient condition to the quasinormality of $T^\dagger$. We show that if T is onto, then T^* is quasinormal if and only if $T^\dagger$ is quasinormal. Afterward, we give a formula for $(\widetilde{T^*})^\dagger$ when T is binormal. Also, we prove that T^* is quasinormal if and only if $(T^\dagger)^* = (\widetilde{T^*})^\dagger$, whenever T is onto. Moreover, we briefly discuss some classical results on the spectrum, numerical range and numerical radius via the Moore-Penrose inverse and Aluthge transformation. In section 3, we obtain some applications of these results to the Lambert multiplication operator M_wEM_u on $L^2(\Sigma)$, where E is the conditional expectation operator with respect to a sub-sigma algebra $\mathcal{A} \subseteq \Sigma$. In addition, we determine lower and upper bounds estimates for the numerical range of $(M_wEM_u)^\dagger$.

2. ON SOME CHARACTERIZATIONS OF $T^\dagger$

For any closed subspace M of H, let P_M denote the orthogonal projection onto M. For $T \in$ CR(H), we shall make used the following general properties of T^*, $\widetilde{T}$, $T^\dagger$ and their polar decompositions. For proofs and discussions of these facts see [**10, 9, 12, 20, 22**].

P(1) $\widetilde{T^\dagger} = |T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$;
P(2) For $\lambda > 0$, $\lambda \in Sp(T)$ if and only if $\lambda^{-1} \in Sp(T^\dagger)$;
P(3) $|T^\dagger| = |T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}} = (|T^*|^{\frac{1}{2}})^\dagger$;
P(4) $|T^*|^{\frac{1}{2}}(|T^*|^{\frac{1}{2}})^\dagger = P_{R(|T^*|)} = (|T^*|^{\frac{1}{2}})^\dagger|T^*|^{\frac{1}{2}}$;
P(5) If T is binormal, then $P_{R(|T^*|)}P_{R(|T|)} = P_{R(|T|)}P_{R(|T^*|)}$;

P(6) $U^*P_{R(|T^*|)} = U^* = P_{R(|T|)}U^*$;
P(7) $U^*U = P_{R(|T|)}$ and $UU^* = P_{R(|T^*|)}$;
P(8) $|T^\dagger|^{\frac{1}{2}}P_{R(|T^*|)} = |T^\dagger|^{\frac{1}{2}}$;
P(9) $U^*(|T^*|^\dagger)^{\frac{1}{2}} = (|T|^\dagger)^{\frac{1}{2}}U^*$;
P(10) $UU^*|T^*|^\dagger = |T^*|^\dagger$;
P(11) $(T^\dagger)^* = |T^*|^\dagger U$;
P(12) $|(T^*)^\dagger| = |T|^\dagger$;
P(13) $T \geq 0 \Leftrightarrow T^\dagger \geq 0$;
P(14) $U^*|T^*|$ and $U^*|T^*|^\dagger$ are the polar decompositions of T^* and $T^\dagger$.

Let f be a bounded Borel real-valued function defined in an interval $\mathcal{I} \subseteq \mathbb{R}$. If $T \in B(H)$ is a self-adjoint operator, then by $f(T)$ we mean the self-adjoint operator $\int_{-\infty}^{+\infty} f(\lambda)dE_\lambda$ where E_λ is the spectral resolution of identity corresponding to T. The restriction of f to the set of all self-adjoint operators is called an operator function. For example, for each $q > 0$, $f(x) = x^q$ is an operator function. In addition, in this case, if U is any unitary operator, then $f(U^*TU) = U^*f(T)U$. For more details see [**3, 4**].

Lemma 2.1. *Let $T \in$ CR(H). Then the following assertions hold.*

(i) $(|T|^\dagger)^q = U^*(|T^*|^\dagger)^q U$, for each $q > 0$.

(ii) $T^\dagger$ is quasinormal if and only if $U^*|T^*|^\dagger = |T^*|^\dagger U^*$.

Proof. (i) By P(3), P(10), P(11) and P(14) we have

$$\begin{aligned}(|T|^\dagger)^2 &= |(T^*)^\dagger|^2 = |(T^\dagger)^*|^2 = T^\dagger(T^\dagger)^* \\ &= U^*|T^*|^\dagger|T^*|^\dagger U \\ &= U^*|T^*|^\dagger UU^*|T^*|^\dagger U \\ &= (U^*|T^*|^\dagger U)^2.\end{aligned}$$

Since for each $q > 0$, $f(x) = x^{\frac{q}{2}}$ is an operator function, we obtain $(|T|^\dagger)^q = U^*(|T^*|^\dagger)^q U$.

(ii) It is a classical fact that T is quasinormal if and only if $U|T| = |T|U$ (see for example [**12**, Theorem 3]). Now, the desired conclusion follows from this and P(14). □

Theorem 2.2. *Let $T \in B(H)$ be onto. Then the following statements are equivalent:*

(i) T^* is quasinormal.

(ii) $T^\dagger$ *is quasinormal.*

Moreover, if one of the above statements hold then

(iii) $[(T^\dagger)^*T^\dagger, (T^\dagger)^* + T^\dagger] = 0.$

Proof. (i)⇔(ii) Since $|T^*||T^*|^\dagger|T^*| = |T^*|$, then we have

$$\begin{aligned} T^* \text{is quasinormal} &\Longleftrightarrow U^*|T^*| = |T^*|U^* \\ &\Longleftrightarrow U^*|T^*||T^*|^\dagger|T^*| = |T^*||T^*|^\dagger|T^*|U^* \\ &\Longleftrightarrow |T^*|U^*|T^*|^\dagger|T^*| = |T^*||T^*|^\dagger U^*|T^*| \\ &\Longleftrightarrow |T^*|(U^*|T^*|^\dagger - |T^*|^\dagger U^*)|T^*| = 0. \end{aligned}$$

By hypothesis, $\mathcal{N}(|T^*|) = \mathcal{N}(T^*) = \{0\}$. Hence $(U^*|T^*|^\dagger - |T^*|^\dagger U^*)|T^*| = 0$, and so $U^*|T^*|^\dagger = |T^*|^\dagger U^*$ on $\overline{R(|T^*|^\dagger)}$. On the other hand, $U^*|T^*|^\dagger = |T^*|^\dagger U^*$ on $N(|T^*|^\dagger) = N(U^*)$. Thus, $U^*|T^*|^\dagger = |T^*|^\dagger U^*$ on H. Consequently, by Lemma 2.1(ii), (i)⇔(ii) holds.

Now, it is easy to check that

$$[(T^\dagger)^*T^\dagger, (T^\dagger)^* + T^\dagger] = [(|T^*|^\dagger)^2, |T^*|^\dagger U + U^*|T^*|^\dagger]$$

$$= \{(|T^*|^\dagger)^2U^*|T^*|^\dagger - U^*(|T^*|^\dagger)^3\} + |T^*|^\dagger\{(|T^*|^\dagger)^2U - U(|T^*|^\dagger)^2\}.$$

If (ii) is holds, then by Lemma 2.1(ii) we obtain

$$\begin{aligned} (|T^*|^\dagger)^2U^*|T^*|^\dagger &= U^*(|T^*|^\dagger)^3; \\ (|T^*|^\dagger)^2U &= U(|T^*|^\dagger)^2. \end{aligned}$$

Thus, $[(T^\dagger)^*T^\dagger, (T^\dagger)^* + T^\dagger] = 0.$ □

For more details and applications on condition (iii) in Theorem 2.2 see [**16**].

Lemma 2.3. *If* $T \in \mathrm{CR(H)}$ *is binormal, then* $(\widetilde{T^*})^\dagger = (|T^*|^\dagger)^{\frac{1}{2}}U(|T^*|^\dagger)^{\frac{1}{2}}$.

Proof. Since T is binormal, then we obtain from direct computations that

$$\begin{aligned} \widetilde{T^*}(\widetilde{T^*})^\dagger\widetilde{T^*} &= |T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}U(|T^*|^\dagger)^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}} \\ &= |T^*|^{\frac{1}{2}}U^*P_{R(|T^*|)}UP_{R(|T^*|)}U^*|T^*|^{\frac{1}{2}} && \text{by P(4)} \\ &= |T^*|^{\frac{1}{2}}U^*UP_{R(|T^*|)}U^*|T^*|^{\frac{1}{2}} && \text{by P(6)} \\ &= |T^*|^{\frac{1}{2}}P_{R(|T|)}P_{R(|T^*|)}U^*|T^*|^{\frac{1}{2}} && \text{by P(7)} \\ &= |T^*|^{\frac{1}{2}}P_{R(|T^*|)}P_{R(|T|)}U^*|T^*|^{\frac{1}{2}} && \text{by P(5)} \\ &= |T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}} = \widetilde{T^*} && \text{by P(4), P(6).} \end{aligned}$$

Also,

$$\begin{aligned}
(\widetilde{T^*})^\dagger\widetilde{T^*}(\widetilde{T^*})^\dagger &= (|T^*|^\dagger)^{\frac{1}{2}}U(|T^*|^\dagger)^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}U(|T^*|^\dagger)^{\frac{1}{2}} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}UP_{R(|T^*|)}U^*P_{R(|T^*|)}U(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(4)} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}UP_{R(|T^*|)}U^*U(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(6)} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}UP_{R(|T^*|)}P_{R(|T|)}(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(7)} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}UP_{R(|T|)}P_{R(|T^*|)}(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(5)} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}P_{R(|T^*|)}UP_{R(|T^*|)}(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(6)} \\
&= (|T^*|^\dagger)^{\frac{1}{2}}U(|T^*|^\dagger)^{\frac{1}{2}} = (\widetilde{T^*})^\dagger && \text{by P(4).}
\end{aligned}$$

Similar computations show that

$$(\widetilde{T^*})^\dagger\widetilde{T^*} = UP_{R(|T^*|)}P_{R(|T|)}U^*$$

and

$$\widetilde{T^*}(\widetilde{T^*})^\dagger = P_{R(|T|)}P_{R(|T^*|)}.$$

Hence, $(\widetilde{T^*})^\dagger\widetilde{T^*}$ and $\widetilde{T^*}(\widetilde{T^*})^\dagger$ are self-adjoint operators. This completes the proof. □

Note that if $T \in \mathrm{CR(H)}$ is binormal, then Lemma 2.3 shows that $\widetilde{T^*}$ and so $\widetilde{T}$ have closed range. Moreover, in this case, we have $\widetilde{T}^\dagger = (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}$.

Theorem 2.4. *Let $T \in B(H)$ be onto and binormal. Then T^* is quasinormal if and only if $(T^\dagger)^* = (\widetilde{T^*})^\dagger$.*

Proof. By [**21**, Theorem 10] and Theorem 2.2, T^* is quasinormal if and only if $T^\dagger = \widetilde{T}^\dagger$. Now, taking adjoint of both sides and using Lemma 2.3 and P(1), we obtain $(T^\dagger)^* = (\widetilde{T^*})^\dagger$. □

In the following, we concentrate on the polar decomposition of $(\widetilde{T})^\dagger$. We require the following lemma.

Lemma 2.5. *(i) [**11**, Corollary 1] Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. If T and S are doubly commutative (i.e., $[T,S] = [T,S^*] = 0$), then $TS = UV|TS|$.*

*(ii) [**22**, Proposition 3.9] Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then $\widetilde{T} = U^*UU|\widetilde{T}|$ is also the polar decomposition of $\widetilde{T}$.*

*(iii) [**22**, Theorem 2.1] Let $T = U|T|$ and $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$ be the polar decompositions. Then $\widetilde{T} = VU|\widetilde{T}|$ is also the polar decomposition.*

Theorem 2.6. *Let* $T = U|T| \in \mathrm{CR(H)}$ *and* $(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}} = V|(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}|$ *be the polar decompositions. If* T *is binormal, then* $(\widetilde{T})^\dagger = U^*V|(\widetilde{T})^\dagger|$ *is also the polar decomposition.*

Proof. (i) First we show that $(\widetilde{T})^\dagger = U^*V|(\widetilde{T})^\dagger|$.

$$\begin{aligned}
U^*V|(\widetilde{T})^\dagger| &= U^*V(((\widetilde{T})^\dagger)^*(\widetilde{T})^\dagger) \\
&= U^*V((|T|^\dagger)^{\frac{1}{2}}U(|T|^\dagger)U^*(|T|^\dagger)^{\frac{1}{2}})^{\frac{1}{2}} && \text{by Lemma 2.3} \\
&= U^*V((|T|^\dagger)^{\frac{1}{2}}UU^*(|T^*|^\dagger)UU^*(|T|^\dagger)^{\frac{1}{2}})^{\frac{1}{2}} && \text{by Lemma 2.1(i)} \\
&= U^*V((|T|^\dagger)^{\frac{1}{2}}|T^*|^\dagger(|T|^\dagger)^{\frac{1}{2}})^{\frac{1}{2}} && \text{by P(8),P(10)} \\
&= U^*V|(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}| \\
&= U^*(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}} \\
&= U^*(|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}} \\
&= U^*(|T^*|^\dagger)^{\frac{1}{2}}UU^*(|T|^\dagger)^{\frac{1}{2}} \\
&= (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}} = (\widetilde{T})^\dagger.
\end{aligned}$$

Now, we claim that $N((\widetilde{T})^\dagger) = N(U^*V)$. Since T is binormal, then it is easy to check that $T^\dagger$ is binormal. Thus $N((|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}) = N((|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}) = N(V)$. Then we have

$$\begin{aligned}
U^*Vx = 0 &\Leftrightarrow U^*(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}x = 0 \\
&\Leftrightarrow U^*(|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}x = 0 \\
&\Leftrightarrow U^*(|T^*|^\dagger)^{\frac{1}{2}}UU^*(|T|^\dagger)^{\frac{1}{2}}x = 0 && \text{by P(8)} \\
&\Leftrightarrow (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}x = 0 && \text{by Lemma 2.1(i)} \\
&\Leftrightarrow (\widetilde{T})^\dagger x = 0.
\end{aligned}$$

Lastly, we prove that U^*V is partial isometry. Since $(|T|^\dagger)^{\frac{1}{2}} = U^*U(|T|^\dagger)^{\frac{1}{2}}$ and $(|T^*|^\dagger)^{\frac{1}{2}} = UU^*(|T^*|^\dagger)^{\frac{1}{2}}$ are the polar decompositions of $(|T|^\dagger)^{\frac{1}{2}}$ and $(|T^*|^\dagger)^{\frac{1}{2}}$, respectively then by Lemma 2.5(i) we have

$$(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}} = (|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}} = UU^*U^*U|(|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}|.$$

Then by the uniqueness of the polar decomposition we get that $V = UU^*U^*U$. It follows that

$$\begin{aligned}(U^*V)(U^*V)^*(U^*V) &= (U^*UU^*U^*U)(U^*UU^*U^*U)^*(U^*UU^*U^*U)\\ &= P_{R(|T|)}U^*P_{R(|T|)}UP_{R(|T|)}U^*P_{R(|T|)} && \text{by P(7)}\\ &= U^*P_{R(|T|)}UU^*P_{R(|T|)} && \text{by P(6)}\\ &= U^*P_{R(|T|)}P_{R(|T^*|)}P_{R(|T|)} && \text{by P(6)}\\ &= U^*P_{R(|T^*|)}P_{R(|T|)} && \text{by P(5)}\\ &= P_{R(|T|)}U^*P_{R(|T|)} && \text{by P(6)}\\ &= U^*UU^*U^*U = U^*V.\end{aligned}$$

This completes the proof. □

Corollary 2.7. *Let $T \in \mathrm{CR(H)}$ be binormal and let $T^\dagger = U^*|T^*|^\dagger$ and $(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}} = V|(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}|$ be the polar decompositions. Then the following statements are hold:*

(i) $\widetilde{T^\dagger} = UU^*U^*|\widetilde{T^\dagger}|$ *is the polar decomposition.*

(ii) $(\widetilde{T})^\dagger = U^*U^*U|(\widetilde{T})^\dagger|$ *is the polar decomposition.*

Proof. (i) Since T is binormal, $T^\dagger$ is binormal. Now, the desired conclusion follows by Lemma 2.5(iii).

(ii) Recall that $(|T|^\dagger)^{\frac{1}{2}} = U^*U(|T|^\dagger)^{\frac{1}{2}}$ and $(|T^*|^\dagger)^{\frac{1}{2}} = UU^*(|T^*|^\dagger)^{\frac{1}{2}}$ are the polar decompositions of $(|T|^\dagger)^{\frac{1}{2}}$ and $(|T^*|^\dagger)^{\frac{1}{2}}$, respectively. Then by Lemma 2.5(i) we obtain

$$(|T|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}} = (|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}} = UU^*U^*U|(|T^*|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}|.$$

Thus, by Theorem 2.6, $(\widetilde{T})^\dagger = U^*UU^*U^*U|(\widetilde{T})^\dagger| = U^*U^*U|(\widetilde{T})^\dagger|$. □

In [**19**], Yamazaki introduce the notion of the $*$-Aluthge transformation $\widetilde{T}^{(*)}$ of T by setting $\widetilde{T}^{(*)} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$. Like this notion we introduce †-Aluthge transformation $\widetilde{T}^{(\dagger)}$ of T by setting $\widetilde{T}^{(\dagger)} = (\widetilde{T^\dagger})^\dagger$. Similar computations show that $\widetilde{T}^{(\dagger)} = \widetilde{T}^{(*)}$, whenever $T \in \mathrm{CR(H)}$ is binormal.

Proposition 2.8. *Let $T \in \mathrm{CR(H)}$. Then the following statements hold.*

(i) If T is self-adjoint, then $W(T) \subseteq W(T^\dagger)W(T^2)$.

(ii) If T is onto and T^* is quasinormal, then $W(\widetilde{T^\dagger}) = W(T^\dagger) \subseteq W(U^*)W(|T^*|^\dagger)$.

(iii) If T is binormal, then $W((\widetilde{T})^{(\dagger)}) \subseteq W(U)W(|T^*|)$.

Proof. (i) Let $x \in H$ with $\|x\| = 1$. Then we get that

$$\begin{aligned}\langle Tx, x\rangle &= \langle TT^{\dagger}Tx, x\rangle = \langle T^{\dagger}Tx, Tx\rangle \\ &= \langle T^{\dagger}\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\rangle\langle Tx, Tx\rangle.\end{aligned}$$

It follows that $\langle Tx, x\rangle \in W(T^{\dagger})W(T^2)$, for each $x \in H$ with $\|x\| = 1$.

(ii) By Theorem 2.2 and Lemma 2.1, $T^{\dagger}$ is quasinormal and so $U^*|T^*|^{\dagger} = |T^*|^{\dagger}U^*$. It follows that $U^*(|T^*|^{\dagger})^{\frac{1}{2}} = (|T^*|^{\dagger})^{\frac{1}{2}}U^*$. Then by P(1) and P(14) we have

$$\begin{aligned}\langle T^{\dagger}x, x\rangle &= \langle U^*|T^*|^{\dagger}x, x\rangle = \langle \widetilde{T^{\dagger}}x, x\rangle \\ &= \langle U^*(|T^*|^{\dagger})^{\frac{1}{2}}x, (|T^*|^{\dagger})^{\frac{1}{2}}x\rangle \\ &= \langle U^*\frac{(|T^*|^{\dagger})^{\frac{1}{2}}x}{\|(|T^*|^{\dagger})^{\frac{1}{2}}x\|}, \frac{(|T^*|^{\dagger})^{\frac{1}{2}}x}{\|(|T^*|^{\dagger})^{\frac{1}{2}}x\|}\rangle\langle(|T^*|^{\dagger})x, x\rangle,\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

(iii) Direct replacement shows that

$$\begin{aligned}\langle \widetilde{T}^{(\dagger)}x, x\rangle &= \langle |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}x, x\rangle \\ &= \langle U|T^*|^{\frac{1}{2}}x, |T^*|^{\frac{1}{2}}x\rangle \\ &= \langle U\frac{|T^*|^{\frac{1}{2}}x}{\||T^*|^{\frac{1}{2}}x\|}, \frac{|T^*|^{\frac{1}{2}}x}{\||T^*|^{\frac{1}{2}}x\|}\rangle\langle |T^*|x, x\rangle,\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$. This completes the proof. □

Proposition 2.9. *Let $T \in \mathrm{CR(H)}$. Then the following assertions hold.*

(i) Let $T \in \mathrm{CR(H)}$ be self-adjoint. Then $\omega(T) \leq \omega(T^{\dagger})\|T\|^2$.

(ii) If T is onto and T^* be quasinormal, then

$$\omega(\widetilde{T}^{\dagger}) = \omega(T^{\dagger}) \leq \omega(U^*)\||T^*|^{\dagger}\|.$$

Proof. (i) Since T is self-adjoint and $T = TT^{\dagger}T$, we have

$$\begin{aligned}\omega(T) &= \sup_{\|x\|=1} |\langle TT^{\dagger}Tx, x\rangle| = \sup_{\|x\|=1} |\langle T^{\dagger}Tx, Tx\rangle| \\ &= \sup_{\|x\|=1} |\langle T^{\dagger}\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\rangle|\ \|Tx\|^2 \\ &\leq \omega(T^{\dagger})\|T\|^2.\end{aligned}$$

(ii) Since T^* is quasinormal then by Theorem 2.2, we obtain

$$\begin{aligned}\omega(T^\dagger) &= \sup_{\|x\|=1} |\langle T^\dagger x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle U^*|T^*|^\dagger x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle (|T^*|^\dagger)^{\frac{1}{2}} U^* (|T^*|^\dagger)^{\frac{1}{2}} x, x\rangle| = \omega(\widetilde{T}^\dagger).\end{aligned}$$

Thus,

$$\begin{aligned}\omega(T^\dagger) &= \sup_{\|x\|=1} |\langle T^\dagger x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle U^*|T^*|^\dagger x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle U^*(|T^*|^\dagger)^{\frac{1}{2}} x, (|T^*|^\dagger)^{\frac{1}{2}} x\rangle| \\ &= \sup_{\|x\|=1} |\langle U^* \frac{(|T^*|^\dagger)^{\frac{1}{2}} x}{\|(|T^*|^\dagger)^{\frac{1}{2}} x\|}, \frac{(|T^*|^\dagger)^{\frac{1}{2}} x}{\|(|T^*|^\dagger)^{\frac{1}{2}} x\|}\rangle| \parallel |T^*|^\dagger x\| \\ &\leq \omega(U^*)\| \, |T^*|^\dagger\|.\end{aligned}$$

□

Proposition 2.10. *Let $T \in \mathrm{CR(H)}$. Then the following assertions hold.*

(i) $\omega(\widetilde{T}^{(\dagger)}) \leq \omega(|T^*|) \leq \||T^*|\|$.

(ii) If T is binormal, then $\omega(\widetilde{T^\dagger}) = \omega(\widetilde{T}^\dagger)$ *and* $\|\widetilde{T}^\dagger\| = \|\widetilde{T^\dagger}\|$.

Proof. The first part is easily follow from Proposition 2.8. For the second part, since

$$\begin{aligned}\widetilde{T}^\dagger &= (|T|^\dagger)^{\frac{1}{2}} U^* (|T|^\dagger)^{\frac{1}{2}} && \text{by Lemma 2.3} \\ &= U^*(|T^*|^\dagger)^{\frac{1}{2}} UU^*U^*(|T^*|^\dagger)^{\frac{1}{2}} U && \text{by Lemma 2.1(i)} \\ &= U^*(|T^*|^\dagger)^{\frac{1}{2}} U^*(|T^*|^\dagger)^{\frac{1}{2}} U && \text{by P(8)} \\ &= U^*|T^\dagger|^{\frac{1}{2}} U^*|T^\dagger|^{\frac{1}{2}} U && \text{by P(3)} \\ &= U^*\widetilde{T^\dagger} U && \text{by P(1),}\end{aligned}$$

then

$$\begin{aligned}\omega(\widetilde{T}^\dagger) &= \sup_{\|x\|=1} |\langle U^*\widetilde{T^\dagger} Ux, x\rangle| = \sup_{\|x\|=1} |\langle \widetilde{T^\dagger} Ux, Ux\rangle| \\ &= \sup_{\|x\|=1} |\langle \widetilde{T^\dagger} \frac{Ux}{\|Ux\|}, \frac{Ux}{\|Ux\|}\rangle \|Ux\|^2 \leq \omega(\widetilde{T^\dagger}).\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\widetilde{\widetilde{T^\dagger}} &= (|T^*|^\dagger)^{\frac{1}{2}}U^*(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(1)}\\
&= (|T^*|^\dagger)^{\frac{1}{2}}UU^*U^*(|T^*|^\dagger)^{\frac{1}{2}} && \text{by P(8)}\\
&= U(|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}U^* && \text{by P(9)}\\
&= U\widetilde{T}^\dagger U^* && \text{by Lemma 2.3,}
\end{aligned}$$

then

$$\begin{aligned}
\omega(\widetilde{T^\dagger}) &= \sup_{\|x\|=1} |\langle U\widetilde{T}^\dagger U^*x, x\rangle| = \sup_{\|x\|=1} |\langle \widetilde{T}^\dagger U^*x, U^*x\rangle|\\
&= \sup_{\|x\|=1} |\langle \widetilde{T}^\dagger \frac{U^*x}{\|U^*x\|}, \frac{U^*x}{\|U^*x\|}\rangle| \|U^*x\|^2 \leq \omega(\widetilde{T}^\dagger).
\end{aligned}$$

Moreover, since $\widetilde{T}^\dagger = U^*\widetilde{T^\dagger}U$ and $\widetilde{T^\dagger} = U^*\widetilde{T}^\dagger U$, we obtain $\|\widetilde{T}^\dagger\| = \|\widetilde{T^\dagger}\|$. □

Lemma 2.11. *[**7**, Theorem 2.8] If A be an arbitrary operator and B is normal. Then $Sp(AB) = Sp(BA)$.*

Proposition 2.12. *Let $T \in$ CR(H). Then the following statements hold.*

(i) $(T^*)^\dagger = U|T|^\dagger$ is the polar decomposition.

(ii) If T is binormal, then $Sp(T^\dagger) = Sp(\widetilde{T^\dagger}) = Sp(\widetilde{T}^\dagger)$.

(iii) If T is binormal and $\lambda > 0$, then $\lambda \in Sp(T) \Leftrightarrow \lambda \in Sp((\widetilde{T})^{(\dagger)})$.

Proof. (i) Since $U^*|T^*|^\dagger$ is the unique polar decomposition of $T^\dagger$, then $N(U^*) = N(|T^*|^\dagger)$, and so $N(U^*U) = N(|T^*|^\dagger U)$. Now, by Lemma 2.1(i), P(11) and P(12) we have

$$(T^*)^\dagger = (T^\dagger)^* = |T^*|^\dagger U = UU^*|T^*|^\dagger U = U|T|^\dagger; \tag{2.1}$$

$$N(U) = N(U^*U) = N(|T^*|^\dagger U) = N((T^*)^\dagger) = N(|(T^\dagger)^*|) = N(|T|^\dagger).$$

Therefore, $(T^*)^\dagger = U|T|^\dagger$ is the unique polar decomposition.

(ii) By P(14), $(|T|^\dagger)^{\frac{1}{2}} \geq 0$ and hence it is normal. Thus, by (2.1) and Lemma 2.11 we have

$$\begin{aligned}
Sp(T^\dagger) = Sp(|T|^\dagger U^*) &= Sp((|T|^\dagger)^{\frac{1}{2}}(|T|^\dagger)^{\frac{1}{2}}U^*)\\
&= Sp((|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}) = Sp(\widetilde{T}^\dagger).
\end{aligned}$$

Similarly, we get that

$$\begin{aligned}
Sp(\widetilde{T^\dagger}) &= Sp((|T^*|^\dagger)^{\frac{1}{2}}U^*(|T^*|^\dagger)^{\frac{1}{2}})\\
&= Sp(U^*(|T^*|^\dagger)^{\frac{1}{2}}(|T^*|^\dagger)^{\frac{1}{2}}) = Sp(T^\dagger).
\end{aligned}$$

(iii) If $\lambda > 0$, then by Lemma 2.3 and part (ii) we have

$$\lambda \in Sp(T) \Leftrightarrow \lambda^{-1} \in Sp(T^\dagger) \Leftrightarrow \lambda^{-1} \in Sp(\widetilde{T^\dagger})$$
$$\Leftrightarrow \lambda \in Sp((\widetilde{T^\dagger})^\dagger) = Sp((\widetilde{T})^{(\dagger)}).$$

□

Recall that an operator $T \in \mathrm{CR(H)}$ is an EP operator if and only if $TT^\dagger = T^\dagger T$ [**8**]. If $T = T^\dagger$, then $T^3 = TT^\dagger T = T$ and hence $T^{2n+1} = T$ for all $n \in \mathbb{N}$. On the other hand, if $TT^\dagger = T^\dagger T$ and $T^{2n+1} = T$, then for $n = 3$ we obtain $T^\dagger = T^\dagger TT^\dagger = T^\dagger T^3 T^\dagger$ and thus

$$T^\dagger = T^\dagger TTTT^\dagger = TT^\dagger TTT^\dagger = TT^\dagger TT^\dagger T = TT^\dagger T = T.$$

Now, let $T^{2n+3} = T$. Since T is an EP operator, then

$$T^\dagger = T^\dagger TT^\dagger = T^\dagger T^{2n+3} T^\dagger = T^\dagger T^{2n+1} T^\dagger T^2 = T^\dagger T^2 = TT^\dagger T = T.$$

These observations establish the following proposition.

Proposition 2.13. *Let $T \in \mathrm{CR(H)}$ and $n \in \mathbb{N}$. Then the following statements hold.*

(i) If $T = T^\dagger$, then $T = T^{2n+1}$.

(ii) If $T = T^{2n+1}$ and T is an EP operator, then $T = T^\dagger$.

3. APPLICATIONS TO THE LAMBERT MULTIPLICATION OPERATORS

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each non-negative $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu,$$

where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$

uniquely defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. Put $E = E^{\mathcal{A}}$. The mapping E is a linear orthogonal projection onto $L^2(\mathcal{A})$. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on the properties of E see [**14, 18**].

We shall always take $u \in L^0(\Sigma)$ for which $uf \in \mathcal{D}(E)$ for all $f \in L^2(\Sigma)$. In other words, EM_u is a well-defined operator on $L^2(\Sigma)$. The mapping $T : L^2(\Sigma) \to L^2(\Sigma)$ defined by $T(f) = wE(uf)$ is called Lambert multiplication operator. For other important properties of T see ([**5, 6, 15**]). By [**15**, Proposition 2.1(b)], EM_u is bounded on $L^2(\Sigma)$ if and only if $E(|u|^2) \in L^\infty(\mathcal{A})$. In this case $\|EM_u\| = \|E(|u|^2)\|_\infty^{1/2}$. Now, let $f \in L^2(\Sigma)$. Then

$$\|Tf\|^2 = \int E(|w|^2)|E(uf)|^2 d\mu = \int |E(u(E(|w|^2))^{\frac{1}{2}} f)|^2 d\mu$$

$$= \int |E(M_\upsilon f)|^2 d\mu = \|EM_\upsilon f\|^2, \tag{3.2}$$

where $\upsilon := u(E(|w|^2))^{\frac{1}{2}}$. It follows that $T = M_w EM_u$ is bounded on $L^2(\Sigma)$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T\| = \|E(|w|^2)^{1/2}E(|u|^2)^{1/2}\|_\infty$. Now, let $0 \leq u \in L^0(\Sigma)$ and let $E(u) \geq \delta$ on $S := \sigma(E(u))$. Note that $L^2(\Sigma) = L^2(S) \oplus L^2(S^c)$, where $S^c = X \setminus S$, $L^2(S) = L^2(S, \Sigma_S, \mu_{|S})$ and $\Sigma_S = \{A \cap S : A \in \Sigma\}$. We claim that $T_1 := EM_u$ has closed range. To this end let $f_n, g \in L^2(\Sigma)$ with $\|g\|_2 > 0$ and $T_1 f_n \to g$ in $L^2(\Sigma)$. Since $L^2(S^c) \subseteq N(T)$, then $g = 0$ on S^c and hence $T_1 f_n \to \chi_S g$ in $L^2(\Sigma)$. But $\chi_S g = EM_u(\frac{\chi_S g}{E(u)})$, because $g \in L^2(\mathcal{A})$ and

$$\|\frac{\chi_S g}{E(u)}\|_2 \leq \frac{1}{\delta}\|g\|_2.$$

It follows that $g = \chi_{S^c} g + \chi_S g = 0 + E(\frac{\chi_S ug}{E(u)}) \in R(T_1)$, and so T_1 has closed range. By (3.1), $T \in B(L^2(\Sigma))$ has closed range if and only if $T_1 \in B(L^2(\Sigma))$ has closed range. These observations establish the following proposition.

Proposition 3.14. *Let $T : L^2(\Sigma) \to L^0(\Sigma)$ defined by $T = M_w EM_u$ is a Lambert multiplication operator.*

(i) $T \in B(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T\| = \|E(|w|^2)E(|u|^2)\|_\infty^{1/2}$.

(ii) Let $T \in B(L^2(\Sigma))$, $0 \leq u \in L^0(\Sigma)$ and $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$. If $E(\upsilon) \geq \delta$ on $\sigma(\upsilon)$, then T has closed range.

In what follows, since for each $u \geq 0$, $\sigma(u) \subseteq \sigma(E(u^2))$, we use the notational convention of $\frac{u}{E(u)}$ for $\frac{u}{E(u)}\chi_{\sigma(u)}$. From now on, we assume that $u, w \in \mathcal{D}(E)$ are non-negative, $S = \sigma(E(u^2)) = \sigma(E(u))$ and $T = M_w EM_u \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$.

Let B, C be bounded positive operators on H such that $BC = CB$. Put $A = BC$. Since $f(x) = x^p$ is an operator function, we obtain $A^p = B^pC^p$ for each $p > 0$. In particular, take $B = M_\nu$ and $C = M_{\bar{\omega}}EM_\omega$, where $0 \leq \nu \in L^0(\mathcal{A})$ and $\omega \in L^0(\Sigma)$. A direct computations shows that $C^p = M_{\omega E(|\omega|^2)^{p-1}}EM_\omega$. Consequently, we have the following lemma.

Lemma 3.15. *Let $0 \leq \nu \in L^0(\mathcal{A})$, $\omega \in L^0(\Sigma)$ and let $A := M_{\nu\bar{\omega}}EM_\omega \in B(L^2(\Sigma))$. Then for each $p \in (0, \infty)$, $A^p = M_{\nu^p\bar{\omega}E(|\omega|^2)^{p-1}}EM_\omega$.*

Put

$$A(f) = \frac{u\chi_G}{E(u^2)E(w^2)}E(wf), \quad f \in L^2(\Sigma),\ G = \sigma(E(w)). \tag{3.3}$$

Then by Proposition 3.1, $A \in B(L^2(\Sigma))$. Also, it is easy to check that

$$TAT = T, \quad ATA = A, \quad (TA)^* = TA, \quad (AT)^* = AT.$$

Thus, $A = T^\dagger = M_{\frac{\chi_{S\cap G}}{E(u^2)E(w^2)}}T^*$ and hence A has closed range.

Now, we concentrate on the parts of the polar decomposition T, $T^\dagger$ and their Aluthge transformations. Let $f \in L^2(\Sigma)$. Then we can obtain from direct computations that

$$\begin{aligned}
|T|^2(f) &= \chi_S uE(w^2)E(uf); \\
|T|\ (f) &= u(E(u^2))^{-\frac{1}{2}}(E(w^2))^{\frac{1}{2}}E(uf) && \text{by Lemma 3.2;} \\
|T|^{\frac{1}{2}}(f) &= u(E(u^2))^{-\frac{3}{4}}(E(w^2))^{\frac{1}{4}}E(uf) && \text{by Lemma 3.2;} \\
U\ (f) &= \chi_S w(E(u^2))^{-\frac{1}{2}}(E(w^2))^{-\frac{1}{2}}E(uf) && \text{because } U|T| = T.
\end{aligned}$$

It follows that

$$\widetilde{T}(f) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}(f) = \frac{uE(uw)}{E(u^2)}E(uf), \tag{3.4}$$

for each $f \in L^2(\Sigma)$. Also, we have

$$\begin{aligned}
|T^*|^2(f) &= \chi_S wE(u^2)E(wf); \\
|T^*|\ (f) &= w(E(u^2))^{\frac{1}{2}}(E(w^2))^{-\frac{1}{2}}E(wf) && \text{by Lemma 3.2;} \\
|T^*|^\dagger(f) &= \left(\frac{\chi_S}{E(u^2)(E(w^2))^3}\right)^{\frac{1}{2}}wE(wf) && \text{by (3.2);} \\
U^*(f) &= \left(\frac{\chi_G}{E(u^2)E(w^2)}\right)^{\frac{1}{2}}uE(wf).
\end{aligned}$$

Take $r = \chi_G(E(u^2)E(w^2))^{-1/2}$. Then $U^* = M_r M_u E M_w$, and

$$UU^*U = M_w(M_r)^3 M_{E(u^2)} M_{E(w^2)} E M_u = M_r M_w E M_u = U.$$

Thus, U^* is a partial isometry. By (3.1), $N(M_w E M_u) = N(EM_{u\sqrt{E(w^2)}})$. It follows that

$$N(U) = N(|T|) = N(T);$$
$$N(U^*) = N(|T^*|^\dagger) = N(T^\dagger),$$

and so $T = U|T|$ and $T^\dagger = U^*|T^*|^\dagger$ are the unique polar decompositions.

Theorem 3.16. *Let $T, \widetilde{T} \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$ with $u, w \geq 0$. Then*

(a) $T^\dagger = M_{\frac{u\chi_{\sigma(E(w))}}{E(u^2)E(w^2)}} E M_w$.

(b) $\widetilde{T} = M_{\frac{uE(uw)}{E(u^2)}} E M_u$.

(c) $(\widetilde{T})^\dagger = M_{\frac{u\chi_{\sigma(E(uw))}}{E(u^2)E(uw)}} E M_u$.

(d) $\widetilde{T^\dagger} = M_{\frac{\chi_S wE(uw)}{E(u^2)(E(w^2))^2}} E M_w$.

(e) $\widetilde{T}$, $(\widetilde{T})^\dagger$ *and* $\widetilde{T^\dagger}$ *are self-adjoint.*

Proof. (a) and (b) follows from (3.2) and (3.3).

(c) Take $\nu = \frac{uE(uw)}{E(u^2)}$. Then by (3.3), $\widetilde{T} = M_\nu E M_u$. Moreover, by (3.2) we obtain that $(\widetilde{T})^\dagger = M_{\frac{\chi_{\sigma(\nu)}}{E(u^2)E(\nu^2)}} M_\nu E M_u$, where $\sigma(\nu) = \sigma(u) \cap \sigma(E(uw))$. Therefore,

$$(\widetilde{T})^\dagger(f) = \frac{u\chi_{\sigma(E(uw))}}{E(u^2)E(uw)} E(uf), \qquad f \in L^2(\Sigma).$$

(d) Put $\vartheta = \frac{u\chi_G}{E(u^2)E(w^2)}$. Then by (3.2), $T^\dagger = M_\vartheta E M_w$. Hence

$$\widetilde{T^\dagger}(f) = \frac{\chi_S wE(uw)}{E(u^2)(E(w^2))^2} E(wf), \qquad f \in L^2(\Sigma).$$

(e) It follows from (3.3), (c) and (d). □

Remark 3.17. If we omit the non-negativity hypothesis of u and w in Theorem 3.3, then for every bounded Lambert multiplication $T \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$, $\widetilde{T}$, $\widetilde{T}^\dagger$ and $\widetilde{T^\dagger}$ are always normal operators. Also, by using Theorem 3.3, once again, $\widetilde{T}^\dagger = \widetilde{T^\dagger}$ if and only if $u = w$.

Let $f \in L^2(\Sigma)$. It is easy to see that

$$\begin{aligned} TT^*(f) &= wE(u^2)E(wf); \\ T^*T(f) &= uE(w^2)E(uf); \\ T^*TTT^*(f) &= uE(w^2)E(u^2)E(uw)E(wf); \\ TT^*T^*T(f) &= wE(u^2)E(w^2)E(uw)E(uf). \end{aligned}$$

So, if $u = w$ or $u, w \in L^0(\mathcal{A})$, then $TT^* = T^*T$. Conversely, if T is normal then

$$wE(u^2)E(wf) = uE(w^2)E(uf), \quad f \in L^2(\Sigma). \tag{3.5}$$

Since $\mathcal{A}$ is sigma-finite, there exists $f_0 \in L^2(\mathcal{A})$ with $\sigma(f_0) = X$. By replacing f_0 by f and then taking the conditional expectation E of both sides of (3.4) gives $E(u^2)(E(w))^2 = (E(u))^2E(w^2)$. Thus, $E(u^2) = (E(u))^2$ and $E(w^2) = (E(w))^2$. But, we know that $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^0(\mathcal{A})$. Consequently, $u, w \in L^0(\mathcal{A})$. Moreover, $T = M_wEM_u$ is binormal if and only if $uE(w) = wE(u)$ on $\sigma(E(uw))$.

We recall that T is an EP operator if and only if $T^\dagger T = TT^\dagger$. Since

$$\begin{aligned} T^\dagger T &= M_{\frac{\chi_{S\cap G}}{E(u^2)E(w^2)}} T^*T; \\ TT^\dagger &= M_{\frac{\chi_{S\cap G}}{E(u^2)E(w^2)}} TT^*, \end{aligned}$$

then T is an EP operator on $L^2(\Sigma)$ if and only if T is a normal operator on $L^2(\Sigma_K)$, where $K = S \cap G$. Thus, we have the following result.

Theorem 3.18. *Let $0 \le u, w \in L^0(\Sigma)$ with $u \ne w$ and let $T = M_wEM_u \in B(L^2(\Sigma))$. Then the following assertions hold.*

(i) T is normal if and only if $u, w \in L^0(\mathcal{A})$.

(ii) $T \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$ is an EP operator on $L^2(\Sigma)$ if and only if $u, w \in L^0(\mathcal{A}_K)$, where $K = S \cap G$.

(iii) T is binormal if and only if $uE(w) = wE(u)$ on $\sigma(E(uw))$.

Now, we determine the lower and upper estimates for the numerical range of $T^\dagger$. Let $\mu(X) = 1$ and let $T = M_wEM_u \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$ with $0 \le u, w \in \mathcal{D}(E)$. By (3.2) and definition of $\omega(T^\dagger)$ we have

$$\begin{aligned} \omega(T^\dagger) \ge |\langle T^\dagger 1, 1\rangle| &= |\int_X \frac{\chi_{S\cap G}}{E(u^2)E(w^2)} uE(w)d\mu| \\ &\ge \int_{S\cap G} |\frac{E(u)E(w)}{E(u^2)E(w^2)}|d\mu. \end{aligned}$$

On the other hand, since $L^\infty(\Sigma)\cap L^2(\Sigma)$ is dense in $L^2(\Sigma)$, then by the Hölder and conditional Hölder inequality we get that,

$$\begin{aligned}\omega(T^\dagger) &= \sup_{\|f\|\leq 1}|\langle T^\dagger f, f\rangle| \leq \sup_{\|f\|\leq 1}|\int_X \frac{\chi_{S\cap G}}{E(u^2)E(w^2)}uE(wf)\bar{f}d\mu| \\ &\leq \sup_{\|f\|\leq 1}\int_X |\frac{\chi_{S\cap G}}{E(u^2)E(w^2)}(E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}E(|f|^2)|d\mu \\ &\leq \sup_{\|f\|\leq 1}\int_{S\cap G}\frac{1}{(E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}}E(|f|^2)d\mu \\ &\leq \int_{S\cap G}\frac{d\mu}{(E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}}.\end{aligned}$$

Consequently, we have the following theorem.

Theorem 3.19. *Let $\mu(X)=1$ and let $T=M_wEM_u\in \mathrm{CR}(\mathrm{L}^2(\Sigma))$ with $0\leq u,w\in \mathcal{D}(E)$. Then*

$$\int_{S\cap G}\frac{E(u)E(w)}{E(u^2)E(w^2)}d\mu \leq \omega(T^\dagger)\leq \int_{S\cap G}\frac{d\mu}{\sqrt{E(u^2)E(w^2)}},$$

where $S=\sigma(E(u)), G=\sigma(E(w))$.

Example 3.20. Let $X=[-\frac{1}{2},\frac{1}{2}]$, $d\mu=dx$, Σ be the Lebesgue sets, and let $\mathcal{A}\subseteq\Sigma$ be the σ-algebra generated by the symmetric sets about the origin. Let $0<a\leq\frac{1}{2}$ and $f\in L^2(\Sigma)$. Then

$$\int_{-a}^{a}E(f)(x)dx=\int_{-a}^{a}f(x)dx$$

$$=\int_{-a}^{a}\{\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\}dx=\int_{-a}^{a}\frac{f(x)+f(-x)}{2}dx.$$

Thus, $E(f)(x)=\frac{f(x)+f(-x)}{2}$. Put $u(x)=x+2$, $w(x)=x+3$ and $T=M_wEM_u$. Then $E(u)=2$, $E(w)=3$, $E(u^2)=x^2+4$ and $E(w^2)=x^2+9$. Now, by Proposition 3.1(i) we get that

$$\|T\|=\|\sqrt{(x^2+4)(x^2+9)}\|_\infty=\frac{\sqrt{629}}{4}=6.2699.$$

Moreover, since $u\sqrt{E(w^2)}=(x+2)\sqrt{x^2+9}\geq\frac{9}{2}$, then by Proposition 3.1(ii),

$T \in \mathrm{CR}(\mathrm{L}^2(\Sigma))$. Also, it is easy to check that

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{6dx}{(x^2+4)(x^2+9)} = 0.1618;$$

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{d\mu}{\sqrt{E(u^2)E(w^2)}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(x^2+4)(x^2+9)}} = 0.1642;$$

$$\|T^\dagger\| = \|\frac{1}{\sqrt{E(u^2)E(w^2)}}\|_\infty = \frac{1}{6} = 1.666;$$

$$\|\widetilde{T}\| = \|E(uw)\|_\infty = \frac{25}{4} = 6.250.$$

Thus, $\|\widetilde{T}\| \leq \|T\|$, $\|T^\dagger\| \geq 1/\|T\|$ and by Theorem 3.6 we obtain

$$0.1618 \leq \omega(T^\dagger) \leq 0.1642 \leq 1.666 = \|T^\dagger\|.$$

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