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Lambert conditional type operators on $L^2(\Sigma)$

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ABSTRACT
In this paper, we discuss matrix theoretic characterizations for weighted conditional type operators in some operator classes on $L^2(\Sigma)$ such as self-adjoint, normal, quasi-normal and positive operator classes. In addition, some necessary and sufficient conditions are given for such operators to have reducing subspaces.

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1. Introduction and preliminaries

Let (X, Σ, μ) be a sigma-finite measure space and let $\mathcal{A}$ be a sigma-finite subalgebra of Σ. The space $L^2(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$ and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $E : L^2(\Sigma) \to L^2(\mathcal{A})$ be the conditional expectation operator, so that for $f \in L^2(\Sigma)$, $E(f)$ is the unique $\mathcal{A}$-measurable function such that

$$\int_A f \, \mathrm{d}\mu = \int_A E^{\mathcal{A}}(f) \, \mathrm{d}\mu$$

for all $A \in \mathcal{A}$. As an operator on $L^2(\Sigma)$, $E := E^{\mathcal{A}}$ is a positive and orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. This operator will play a major role in our work. A detailed discussion and verification of most of the properties may be found in [1–4]. Those properties of E used in our discussion are summarized below. In all cases, we assume that $f, g, fg \in \mathcal{D}(E)$.

- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$.

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- $\sigma(E(|f|))$ is the smallest $\mathcal{A}$-measurable set containing $\sigma(f)$.
- (Conditional Cauchy–Schwarz) $|E(fg)|^2 \le E(|f|^2)E(|g|^2)$, where $f, g \in L^0(\Sigma)$ are finite-valued functions.

The products of conditional expectation and multiplication operators appear more often in the service of the study of other operators rather than being the object of study in and of themselves. Weighted Lambert conditional operators in $L^2(\Sigma)$-spaces turn out to be interesting objects of measure and operator theory. The class of these operators includes unweighted conditional operators [5], multiplication operators, integral operators and their adjoints. Throughout the paper, we assume that the measure spaces under consideration are complete and that the corresponding Lambert conditional operators are densely defined. This enables us to use the conditional expectation $E = E^{\mathcal{A}}$ with respect to the sigma-finite subalgebra $\mathcal{A}$ of Σ and to regard a Lambert conditional operator $T = T_{w,u}$ as the products M_wEM_u of the operator M_w and M_u of multiplications by w and u and the conditional expectation operator E.

In Section 2, using the matrix representation, complete measure-theoretic characterizations are given for self-adjoint, normal, quasi-normal and positive weighted Lambert conditional operators in $L^2(\Sigma)$ space.

2. Characterizations

Let $w, u \in \mathcal{D}(E)$, the domain of E. Then, the mapping $T : L^2(\Sigma) \supseteq \mathcal{D}(T) \to L^2(\Sigma)$ given by $T(f) = wE(uf)$ for $f \in \mathcal{D}(T) = \{f \in L^2(\Sigma) : T(f) \in L^2(\Sigma)\}$ is well-defined and linear. Such an operator is called a Lambert conditional operator induced by the pair (w, u). Let $K := E(|u|^2)E(|w|^2)$ be a finite-valued function; that is $\mu(K_\infty) = 0$, where $K_\infty = \{x \in X : K(x) = \infty\}$. Put $\mathrm{d}\nu = (1 + K)\,\mathrm{d}\mu$ and take $f \in L^2(X, \Sigma, \nu)$. Then, by conditional Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|T(f)\|_\mu^2 &= \int_X |wE(uf)|^2\,\mathrm{d}\mu = \int_X E(|w|^2)|E(uf)|^2\,\mathrm{d}\mu \\ &\le \int_X E(|u|^2)E(|w|^2)E(|f|^2)\,\mathrm{d}\mu \\ &= \int_X K|f|^2\,\mathrm{d}\mu \le \|f\|_\nu^2 < \infty.\end{aligned}$$

Thus, $L^2(X, \Sigma, \nu) \subseteq \mathcal{D}(T)$. Now, let $f \in L^2(\Sigma)$. We can assume that $f_{|K_\infty} = 0$. Put $F_n = \{x \in X : K(x) \le n\}$ for $n \in \mathbb{N}$. Then, $F_n \nearrow F := \{x \in X : K(x) < \infty\}$, $\|\chi_{F_n} f\|_\nu^2 \le (1 + n)\|f\|_\mu^2 < \infty$, and by Lebesgue's dominated convergence theorem $\int_X |f - \chi_{F_n} f|\,\mathrm{d}\mu \to 0$ as $n \to \infty$. Therefore, $L^2(X, \Sigma, \nu)$ is dense in $L^2(\Sigma)$ and so T is a densely defined operator on $L^2(\Sigma)$.

It was shown in [6] that $T = M_wEM_u$ is bounded in $L^2(\Sigma)$ if and only if $K \in L^\infty(\Sigma)$. In this case, $\|T\|^2 = \|K\|_\infty$ and $L^2(X, \Sigma, \nu) = \mathcal{D}(T) = L^2(\Sigma)$. For further information on conditional type operators, see e.g. [3,5,7–12]. From now on, to avoid the repetition, we gather the following assumptions which will be used frequently throughout this paper.

The triplet (X, Σ, μ) is a complete sigma-finite measure space, $\mathcal{A} \subseteq \Sigma$ is complete sigma-finite, $w, u, uw \in \mathcal{D}(E)$, $E = E^{\mathcal{A}}$ and $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$, the algebra of bounded linear operators on $L^2(\Sigma)$.

Relative to the direct sum decomposition $L^2(\Sigma) = \mathcal{R}(E) \oplus \mathcal{N}(E)$, any element f of $L^2(\Sigma)$ can be written uniquely as $f = f_1 + f_2$ where $f_1 = E(f) \in L^2(\mathcal{A})$ and $f_2 = f - E(f) \in \mathcal{N}(E)$. Note that $\mathcal{R}(E) = L^2(\mathcal{A}), \sigma(f_1) \in \mathcal{A}$ and $E(f_2) = 0$. In case $\mathcal{A} = \{\emptyset, X\}$ with $\mu(X) = 1$, $E(|f_2|^2)$ is called the variance of $|f|$. It is worth nothing that $f = 0$ whenever $f \geq 0$ and $E(f) = 0$. So, there is no strictly positive element in $\mathcal{N}(E)$. However, we have the following simple but useful fact.

Lemma 2.1: *Let $f \in L^2(\Sigma)$. Then, $E(|f|^2) = |f_1|^2 + E(|f_2|^2)$.*

Proof: Knowing that for each $f \in L^2(\Sigma)$, $E(f_1) = f_1$ and $E(\bar{f_2}) = \overline{E(f_2)} = 0$ we have $E(|f|^2) = E(|f_1|^2) + f_1 E(\bar{f_2}) + \bar{f_1} E(f_2) + E(|f_2|^2) = |f_1|^2 + E(|f_2|^2)$. ■

Corollary 2.2: *The following statements hold:*

(a) $E(|f_2|^2) = 0$ *if and only if* $f \in L^2(\mathcal{A})$;
(b) $\sigma(E(|f|^2)) = \sigma(f_1) \cup \sigma(E(|f_2|^2))$;
(c) $\sigma(f_i) \subseteq \sigma(E(|f|^2))$, *for* $i = 1, 2$.

Let $f, g \in L^0(\Sigma)$ be finite-valued functions. Then, by the conditional Cauchy–Schwarz inequality, we have $\sigma(E(fg)) \subseteq \sigma(E(|f|^2)) \cap \sigma(E(|g|^2))$. Moreover, if $A \in \Sigma$ with $\sigma(f) \subseteq A$, then $f\chi_A = f$. Then, we have the following corollary.

Corollary 2.3: *Let $f, g \in \mathcal{D}(E)$ and $S = \sigma(E(|f|^2))$. Then,*

(a) $f_i \chi_S = f_i$, *for* $i = 1, 2$;
(b) $E(fg)\chi_S = E(fg)$.

Relative to the direct sum decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, the matrix form of each $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$ is

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} ET|_{L^2(\mathcal{A})} & ET|_{\mathcal{N}(E)} \\ (I - E)T|_{L^2(\mathcal{A})} & (I - E)T|_{\mathcal{N}(E)} \end{bmatrix},$$

where for $f \in L^2(\mathcal{A})$,

$$T_1(f) = E(w)E(u)(f) = M_{E(w)E(u)}(f);$$
$$T_3(f) = wE(u)f - E(w)E(u)f = M_{E(u)(w-E(w))}(f)$$

and for $f \in \mathcal{N}(E)$,

$$T_2(f) = E(w)E(uf) = M_{E(w)} E M_u(f);$$
$$T_4(f) = wE(uf) - E(w)E(uf) = M_{w-E(w)} E M_u(f).$$

Replacing $E(f)$ and $f - E(f)$ by f_1 and f_2 in the above, respectively, we obtain

$$T = \begin{bmatrix} M_{w_1u_1} & EM_{w_1u_2} \\ M_{w_2u_1} & M_{w_2}EM_{u_2} \end{bmatrix}. \tag{1}$$

Note that we have used the fact that the expression EM_u in T_4 can be rewritten in the following way:

$$EM_u(f) = E(u_1f + u_2f) = u_1E(f) + E(u_2f) = EM_{u_2}(f), \ f \in \mathcal{N}(E).$$

From (1) we obtain

$$T^* = \begin{bmatrix} M_{\overline{w_1u_1}} & EM_{\overline{w_2u_1}} \\ M_{\overline{w_1u_2}} & M_{\bar{u}_2}EM_{\bar{w}_2} \end{bmatrix}. \tag{2}$$

The following corollary follows directly from (1) and (2).

Corollary 2.4: *T is self-adjoint if and only if the followings hold.*

(a) $\overline{w_1u_1} = w_1u_1$;
(b) $\overline{w_2u_1} = w_1u_2$;
(c) $w_2E(u_2f) = \bar{u}_2E(\bar{w}_2f), \ \forall f \in \mathcal{N}(E)$.

Our next aim is to replace above conditions by the minimal ones.

Lemma 2.5: $\overline{E(w)u} = wE(u)$ *if and only if* $\overline{w_1u_1} = w_1u_1$ *and* $\overline{w_1u_2} = w_2u_1$. *Moreover, in this case* $\bar{u}_1w_2 = \bar{u}_2w_1$.

Proof: Since $w_1u_1 \in L^2(\mathcal{A})$ and for $1 \le i \neq j \le 2$, $w_iu_j \in \mathcal{N}(E)$ we have

$$\begin{aligned} \overline{E(w)u} = wE(u) &\Longleftrightarrow \bar{w}_1(\bar{u}_1 + \bar{u}_2) = (w_1 + w_2)u_1 \\ &\Longleftrightarrow \overline{w_1u_1} + \overline{w_1u_2} = w_1u_1 + w_2u_1 \\ &\Longleftrightarrow \overline{w_1u_1} = w_1u_1, \ \ \overline{w_1u_2} = w_2u_1. \end{aligned} \tag{3}$$

Moreover, the last equality in (3) implies that

$$\bar{w}_1u_2\chi_{\sigma(u_1)} = \bar{w}_1u_2; \tag{4}$$

$$\bar{u}_1w_2\chi_{\sigma(w_1)} = \bar{u}_1w_2. \tag{5}$$

By multiplying the sides of (3) we obtain

$$(\bar{w}_1\bar{u}_1)(w_2u_1) = (w_1u_1)(\bar{w}_1\bar{u}_2) \xrightarrow[(5)]{(4)} \bar{u}_1w_2 = \bar{u}_2w_1. \qquad \blacksquare$$

Lemma 2.6: *The following equalities are equivalent:*

(a) $\bar{u}_2E(\bar{w}_2f) = w_2E(u_2f), \ \forall f \in \mathcal{N}(E)$,
(b) $\bar{w}_2E(|u_2|^2) = u_2E(u_2w_2)$,

(c) $\bar{u}_2 E(|w_2|^2) = w_2 E(u_2 w_2)$,

and in this case $\overline{w_2 u_2} = w_2 u_2$ *and* $E(|w_2|^2)E(|u_2|^2) = (E(u_2w_2))^2$.

Proof: (a) $\Rightarrow$ (b) Since $\mathcal{A}$ is sigma-finite, there exists $\{A_n\}_n \subseteq \mathcal{A}$ such that $X = \cup_n A_n$, $A_n \subseteq A_{n+1}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. In this case, $A_n \nearrow X$ and so $\chi_{A_n} \nearrow \chi_X$. Put $f = \bar{u}_2 \sqrt{E(|w_2|^2)}\, \chi_{A_n}$. Then,

$$\|f\|^2 = \int_{A_n} |u_2|^2 E(|w_2|^2)\,\mathrm{d}\mu \leq \|T\|^2 \mu(A_n) < \infty,$$

so that $f \in L^2(\Sigma) \cap \mathcal{N}(E)$. By hypothesis and using Corollary 2.3, we have

$$\begin{aligned}
\text{(a)} &\Longrightarrow \bar{w}_2 E(\bar{u}_2 u_2 \sqrt{E(|w_2|^2)}\, \chi_{A_n}) = u_2 E(u_2 w_2 \sqrt{E(|w_2|^2)}\, \chi_{A_n}) \\
&\Longrightarrow \bar{w}_2 E(|u_2|^2)\chi_{A_n} = u_2 E(u_2 w_2)\chi_{A_n}, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

It follows that $\bar{w}_2 E(|u_2|^2) = u_2 E(u_2 w_2)$ as $n \to \infty$.

(b) $\Rightarrow$ (c) Using Corollary 2.3, we have

$$\begin{aligned}
\text{(b)} &\xrightarrow[E]{\times \bar{u}_2} \overline{E(u_2 w_2)} E(|u_2|^2) = E(|u_2|^2) E(u_2 w_2) \\
&\Longrightarrow \overline{E(u_2 w_2)} = E(u_2 w_2)
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\text{(b)} &\xrightarrow[E]{\times w_2} E(|w_2|^2)E(|u_2|^2) = (E(u_2 w_2))^2 \\
&\Longrightarrow E(|w_2|^2)\chi_G = \frac{(E(u_2 w_2))^2 \chi_G}{E(|u_2|^2)}, \quad G = \sigma(E(|u_2|^2)).
\end{aligned} \tag{7}$$

Then, we have

$$\begin{aligned}
\bar{u}_2 E(|w_2|^2) = \bar{u}_2 E(|w_2|^2)\chi_G &= \bar{u}_2 \frac{(E(u_2 w_2))^2}{E(|u_2|^2)} \chi_G, \quad \text{by (7)} \\
&= w_2 E(|u_2|^2) \frac{E(u_2 w_2)}{E(|u_2|^2)} \chi_G \quad \text{by (b) and (6)} \\
&= w_2 E(u_2 w_2).
\end{aligned}$$

(c) $\Rightarrow$ (a) Put $G = \sigma(E(|w_2|^2)$. Multiplying both sides of (c) by u_2 and $\bar{w}_2$, respectively, and then taking the conditional expectation E of both sides equation we obtain

$$\bar{u}_2 E(|w_2|^2) = w_2 E(u_2 w_2) \xrightarrow[E]{\times u_2} E(|u_2|^2)E(|w_2|^2) = (E(u_2 w_2))^2. \tag{8}$$

$$\begin{aligned}
\bar{u}_2 E(|w_2|^2) = w_2 E(u_2 w_2) &\xrightarrow[E]{\times \bar{w}_2} \overline{E(u_2 w_2)} E(|w_2|^2) = E(|w_2|^2) E(u_2 w_2) \\
&\xrightarrow[\mathit{Cor.\ 2.3}]{(8)} \overline{E(u_2 w_2)} = E(u_2 w_2).
\end{aligned} \tag{9}$$

Moreover, we obtain

$$\begin{aligned}\bar{u}_2E(|w_2|^2) = w_2E(u_2w_2) &\xrightarrow{\times u_2} |u_2|^2E(|w_2|^2) = (u_2w_2)E(u_2w_2)\\ &\xrightarrow{(9)} \overline{u_2w_2} = u_2w_2.\end{aligned}$$

These imply that

$$\begin{aligned}\bar{u}_2E(|w_2|^2) = w_2E(u_2w_2) &\xrightarrow{(9)} u_2E(|w_2|^2) = \bar{w}_2E(u_2w_2) \quad \text{(by conjugation)}\\ &\xrightarrow[E]{\times f} E(|w_2|^2)E(u_2f) = E(u_2w_2)E(\bar{w}_2f)\\ &\xrightarrow{\times \bar{u}_2} (\bar{u}_2E(|w_2|^2))E(u_2f) = \bar{u}_2E(u_2w_2)E(\bar{w}_2f)\\ &\xrightarrow{(b)} w_2E(u_2w_2)E(u_2f) = \bar{u}_2E(u_2w_2)E(\bar{w}_2f)\\ &\xrightarrow[Cor.2.3]{(8)} \bar{u}_2E(\bar{w}_2f) = w_2E(u_2f), \quad \forall f \in \mathcal{N}(E).\end{aligned}$$

This completes the proof. ■

Lemma 2.7: *If $\bar{u}E(uw) = wE(|u|^2)$, then $E(\bar{u})w = \bar{u}E(w)$.*

Proof: Put $S = \sigma(E(|u|^2))$. Then, $\bar{u}\chi_S = \bar{u}$ and by assumption $w\chi_S = (\bar{u}E(uw)/E(|u|^2))\chi_S$. It follows that

$$\begin{aligned}E(\bar{u})w = E(\bar{u})w\chi_S &= E(\bar{u})\frac{\bar{u}E(uw)}{E(|u|^2)}\chi_S\\ &= \frac{E(\bar{u}E(uw))}{E(|u|^2)}\bar{u}\chi_S = \frac{E(wE(|u|^2))}{E(|u|^2)}\bar{u}\chi_S = \bar{u}E(w).\end{aligned}$$

■

Lemma 2.8: *The following statements are equivalent:*

(a) $\bar{u}E(uw) = wE(|u|^2)$;
(b) $\bar{u}_2E(u_2w_2) = w_2E(|u_2|^2)$ *and* $\bar{u}_1w_2 = \bar{u}_2w_1$.

Proof: Direct computations show that $\bar{u}E(uw) = wE(|u|^2)$ if and only if

$$(\bar{u}_1 + \bar{u}_2)(u_1w_1 + E(u_2w_2)) = (w_1 + w_2)(|u_1|^2 + E(|u_2|^2)).$$

By Lemma 2.7, this is equivalent to $\bar{u}_2E(u_2w_2) = w_2E(|u_2|^2)$ and $\bar{u}_1w_2 = \bar{u}_2w_1$. ■

By Corollary 2.4 and the previous lemmas, we have the following theorem.

Theorem 2.9: *The Lambert conditional operator $T \in \mathcal{B}(L^2(\Sigma))$ is self-adjoint if and only if the followings hold:*

(a) $\overline{E(w)u} = wE(u)$;
(b) $\bar{u}E(uw) = wE(|u|^2)$.

Moreover, in this case, the conditional Cauchy–Schwarz inequality for u,w turns into equality, i.e. $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.

Example 2.10: Suppose Σ is the sigma-algebra of Lebesgue-measurable sets in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and μ is the area measure in $\mathbb{D}$. For fix $n \in \mathbb{N}$, let $\mathcal{A} = \mathcal{A}(\varphi)$ be the sub-sigma algebra of Σ generated by $\{(z^n)^{-1}(U) : U \subset \mathbb{D} \text{ is open}\}$ and $E = E^{\mathcal{A}}$. Put $c_z = \{\zeta : \zeta^n = z^n\}$ for each $z \in \mathbb{D}$. Then, by [9, Example 2.5(ii)], we have

$$E(f)(z) = \frac{1}{n} \sum_{\zeta \in c_z} f(\zeta), \quad f \in L^2(\Sigma),\ z \in \mathbb{D}.$$

Let $u, h \in L^\infty(\mathbb{D})$ with $\bar{h} = h$ and take $w(z) = \bar{u}(z)h(z^n)$. Then, by [6, Theorem 2.1(a)], $T = M_w E M_u$ is a bounded operator in $L^2(\Sigma)$. Moreover, we have

$$\begin{aligned}
(\overline{E(w)u})(z) &= \frac{\bar{u}(z)}{n} \sum_{\zeta \in c_z} u(\zeta)h(\zeta^n) \\
&= \frac{\bar{u}(z)h(z^n)}{n} \sum_{\zeta \in c_z} u(\zeta) \\
&= (wE(u))(z); \\
(\bar{u}E(uw))(z) &= \frac{\bar{u}(z)}{n} \sum_{\zeta \in c_z} u(\zeta)w(\zeta) \\
&= \frac{\bar{u}(z)h(z^n)}{n} \sum_{\zeta \in c_z} |u(\zeta)|^2 \\
&= (wE(|u|^2))(z), \quad z \in \mathbb{D}.
\end{aligned}$$

Then, by Theorem 2.9, T is self-adjoint. On the other hand, direct computation shows that

$$(T^* f)(z) = \frac{\bar{u}(z)h(z^n)}{n} \sum_{\zeta \in c_z} u(\zeta)f(\zeta) = (Tf)(z), \quad f \in L^2(\Sigma),\ z \in \mathbb{D}.$$

In view of (1) and (2) we have

$$T^*T = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix},$$

where

$$\begin{aligned} S_1 &= M_{|w_1|^2|u_1|^2} + EM_{|w_1|^2|u_1|^2} \\ &= M_{|u_1|^2(|w_1|^2+E(|w_2|^2)} \\ &= M_{|u_1|^2E(|w|^2)}; \\ S_2 &= EM_{|w_1|^2\bar{u}_1u_2} + EM_{|w_2|^2\bar{u}_1}EM_{u_2} \\ &= EM_{|w_1|^2\bar{u}_1u_2+E(|w_2|^2)\bar{u}_1u_2} \\ &= EM_{\bar{u}_1u_2E(|w|^2)}; \\ S_3 &= M_{|w_1|^2\bar{u}_2u_1} + M_{\bar{u}_2}EM_{|w_2|^2u_1} \\ &= M_{\bar{u}_2u_1}M_{(|w_1|^2+E(|w_2|^2)} \\ &= M_{\bar{u}_2u_1E(|w|^2)}; \\ S_4 &= M_{\bar{u}_2}EM_{|w_1|^2u_2} + M_{\bar{u}_2}EM_{|w_2|^2}EM_{u_2} \\ &= M_{\bar{u}_2E(|w|^2)}EM_{u_2}. \end{aligned}$$

Thus,

$$T^*T = \begin{bmatrix} M_{|u_1|^2E(|w|^2)} & EM_{\bar{u}_1u_2E(|w|^2)} \\ M_{\bar{u}_2u_1E(|w|^2)} & M_{\bar{u}_2E(|w|^2)}EM_{u_2} \end{bmatrix}. \tag{10}$$

We conclude similarly that

$$TT^* = \begin{bmatrix} M_{|w_1|^2E(|u|^2)} & EM_{\bar{w}_2w_1E(|u|^2)} \\ M_{\bar{w}_1w_2E(|u|^2)} & M_{w_2E(|u|^2)}EM_{\bar{w}_2} \end{bmatrix}. \tag{11}$$

The following corollary follows directly from (10) and (11).

Corollary 2.11: *T is normal if and only if the followings hold:*

(a) $|w_1|^2E(|u|^2) = |u_1|^2E(|w|^2)$;
(b) $\bar{w}_1w_2E(|u|^2) = \bar{u}_2u_1E(|w|^2)$;
(c) $w_2E(|u|^2)E(\bar{w}_2f) = \bar{u}_2E(|w|^2)E(u_2f)$, $\forall f \in \mathcal{N}(E)$.

Proposition 2.12: $\bar{u}_1w_2 = \bar{u}_2w_1$ *if and only if the following equalities hold:*

(a) $|u_1|^2E(|w|^2) = |w_1|^2E(|u|^2)$;
(b) $\bar{u}_2u_1E(|w|^2) = w_2\bar{w}_1E(|u|^2)$.

Proof: Let $\bar{u}_1w_2 = \bar{u}_2w_1$. Then, we have

$$\begin{aligned} |u_1|^2E(|w|^2) &= |u_1|^2(|w_1|^2 + E(|w_2|^2) = |u_1|^2|w_1|^2 + E(|u_1w_2|^2) \\ &= |u_1|^2|w_1|^2 + E(|u_2w_1|^2) = |w_1|^2(|u_1|^2 + E(|u_2|^2)) \\ &= |w_1|^2E(|u|^2) \end{aligned}$$

and

$$\begin{aligned}\bar{u}_1 w_2 = \bar{u}_2 w_1 &\Rightarrow (\bar{u}_1 w_2)(\bar{w}_1 E(|u|^2) = \bar{u}_2 |w_1|^2 E(|u|^2)\\ &\Rightarrow \bar{u}_1 w_2 \bar{w}_1 E(|u|^2) = \bar{u}_2 |u_1|^2 E(|w|^2).\end{aligned}$$

But $\bar{w}_1 E(|u|^2)\chi_{\sigma(u_1)} = \bar{w}_1 E(|u|^2)$, by (a), because $\sigma(\bar{w}_1 E(|u|^2)) = \sigma(|u_1|^2 E(|w|^2)) \subseteq \sigma(u_1)$. Hence, $w_2 \bar{w}_1 E(|u|^2) = \bar{u}_2 u_1 E(|w|^2)$.

Conversely, let (a) and (b) hold. From (a), we have

$$w_2 \bar{u}_1 \chi_{\sigma(w_1)} = w_2 \bar{u}_1. \tag{12}$$

Multiplying both sides of (b) by $\bar{u}_1$, we obtain $\bar{u}_2 |u_1|^2 E(|w|^2) = \bar{u}_1 (w_2 \bar{w}_1 E(|u|^2))$. Thus, $\bar{u}_2 |w_1|^2 E(|u|^2) = \bar{u}_1 (w_2 \bar{w}_1 E(|u|^2))$ and so $\bar{u}_1 w_2 = \bar{u}_2 w_1$, by (12). This completes the proof. ■

Lemma 2.13: *The following equalities are equivalent:*

(a) $wE(|u|^2)E(\bar{w}f) = \bar{u}E(|w|^2)E(uf)$, $\forall f \in L^2(\Sigma)$;
(b) $\bar{w}E(|u|^2) = uE(\overline{uw})$;
(c) $\bar{u}E(|w|^2) = wE(\overline{uw})$.

Proof: For $A_n \in L^0(\mathcal{A})$ with $\mu(A_n) < \infty$, put $f = w\sqrt{E(|u|^2)}\chi_{A_n}$ in (a). Then, precisely the same calculation as that shown in the proof of Lemma 2.6 yields that $f \in L^2(\Sigma)$. Therefore, (b) holds by Corollary 2.3. Now, let (b) hold. First, note that

$$\begin{aligned}\bar{w}E(|u|^2) = uE(\overline{uw}) &\xrightarrow[E]{\times w} E(|w|^2)E(|u|^2) = |E(uw)|^2\\ &\Longrightarrow E(|w|^2)\chi_{E(|u|^2)} = \frac{|E(uw)|^2}{E(|u|^2)}\chi_{E(|u|^2)}.\end{aligned} \tag{13}$$

Then, by Lemma 2.3, we have

$$\begin{aligned}\bar{u}E(|w|^2) = \bar{u}E(|w|^2)\chi_{E(|u|^2)} &\overset{(13)}{=} \frac{\bar{u}|E(uw)|^2}{E(|u|^2)}\chi_{E(|u|^2)}\\ &= \frac{(\bar{u}E(uw))(E(\overline{uw}))}{E(|u|^2)}\chi_{E(|u|^2)}\\ &\overset{(b)}{=} wE(\overline{uw})\chi_{E(|u|^2)}\\ &= wE(\overline{uw}).\end{aligned}$$

This proves (b) $\Rightarrow$ (c). The above argument with u replaced by w shows that (c) implies (b). Now, let (b) holds. Then, we have

$$\begin{aligned}\bar{w}E(|u|^2) = uE(\overline{uw}) &\xrightarrow[E]{\times f} E(|u|^2)E(\bar{w}f) = E(\overline{uw})E(uf)\\ &\xrightarrow[(c)]{\times w} wE(|u|^2)E(\bar{w}f) = \bar{u}E(|w|^2)E(uf).\end{aligned}$$

This completes the proof. ■

Theorem 2.14: *The Lambert conditional operator $T \in \mathcal{B}(L^2(\Sigma))$ is normal if and only if $\bar{u}E(uw) = wE(|u|^2)$. In this case, $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.*

Proof: By Lemma (2.13), it is clear. ■

Corollary 2.15: *Let $A \in \mathcal{A}$. Then, the following equalities are equivalent:*

(a) $wE(|u|^2)E(\bar{w}f)\chi_A = \bar{u}E(|w|^2)E(uf)\chi_A$, $\forall f \in L^2(\Sigma)$;
(b) $\bar{w}E(|u|^2)\chi_A = uE(\overline{uw})\chi_A$;
(c) $\bar{u}E(|w|^2)\chi_A = wE(\overline{uw})\chi_A$.

Put $S = \sigma(u_1) \cup \sigma(w_1)$ and $S^c = X \setminus S$. It follows that $S^c \in \mathcal{A}$, $u\chi_{S^c} = u_2\chi_{S^c}$ and $w\chi_{S^c} = w_2\chi_{S^c}$. Then, by Corollary 2.15, we have the following corollary.

Corollary 2.16: *Let $A \in \mathcal{A}$. Then, the following equalities are equivalent:*

(a) $w_2E(|u|^2)E(\bar{w}_2f)\chi_{S^c} = \bar{u}_2E(|w|^2)E(u_2f)\chi_{S^c}$, $\forall f \in L^2(\Sigma)$;
(b) $\bar{w}_2E(|u|^2)\chi_{S^c} = u_2E(\overline{u_2w_2})\chi_{S^c}$;
(c) $\bar{u}_2E(|w|^2)\chi_{S^c} = w_2E(\overline{u_2w_2})\chi_{S^c}$.

Proposition 2.17: *Let $S = \sigma(u_1) \cup \sigma(w_1)$ and $\bar{u}_1w_2 = \bar{u}_2w_1$. Then, we have*

(a) $\bar{u}E(uw)\chi_S = wE(|u|^2)\chi_S$;
(b) $\bar{u}_2E(|w|^2)E(u_2f)\chi_S = w_2E(|u|^2)E(\bar{w}_2f)\chi_S$, $\forall f \in L^2(\Sigma)$.

Proof: (a) First, note that

$$\begin{aligned} E(\bar{u})w = \bar{u}E(w) &\Longleftrightarrow \bar{u}_1(w_1 + w_2) = (\bar{u}_1 + \bar{u}_2)w_1 \\ &\Longleftrightarrow \bar{u}_1w_2 = \bar{u}_2w_1. \end{aligned}$$

Also, the equality $E(\bar{u})w = \bar{u}E(w)$ implies that

$$\bar{u}\chi_{\sigma(w_1)} = \frac{E(\bar{u})w}{E(w)}\chi_{\sigma(w_1)}; \tag{14}$$

$$w\chi_{\sigma(u_1)} = \frac{\bar{u}E(w)}{E(\bar{u})}\chi_{\sigma(u_1)}. \tag{15}$$

From hypothesis and (14), we obtain

$$\begin{aligned} \bar{u}E(uw)\chi_{\sigma(w_1)} &= \frac{E(\bar{u})w}{E(w)}E(uw)\chi_{\sigma(w_1)} \\ &= \frac{w}{E(w)}E(u(E(\bar{u})w))\chi_{\sigma(w_1)} \\ &= \frac{w}{E(w)}E(u(\bar{u}E(w))\chi_{\sigma(w_1)} \\ &= wE(|u|^2)\chi_{\sigma(w_1)}. \end{aligned} \tag{16}$$

Similarly, we obtain

$$\bar{u}E(uw)\chi_{\sigma(u_1)} = wE(|u|^2)\chi_{\sigma(u_1)}. \tag{17}$$

Now, from (16) and (17), we have $\bar{u}E(uw)\chi_S = wE(|u|^2)\chi_S$. This proves (a).

(b) Let $f \in L^2(\Sigma)$. Then, we have

$$\begin{aligned}
\bar{u}_1 w_2 = \bar{u}_2 w_1 &\xRightarrow[E]{\times f} \bar{w}_1 E(u_2 f) = u_1 E(\bar{w}_2 f) \\
&\xRightarrow{w_2 E|u|^2} \bar{w}_1 w_2 E(|u|^2)E(u_2 f) = u_1 w_2 E(|u|^2)E(\bar{w}_2 f).
\end{aligned}$$

Now, by using Lemma 2.12(b), we obtain

$$\bar{u}_2 E(|w|^2)E(u_2 f)\chi_{\sigma(u_1)} = w_2 E(|u|^2)E(\bar{w}_2 f)\chi_{\sigma(u_1)}. \tag{18}$$

The above argument with u replaced by w and using Lemma 2.12(a) show that

$$\bar{u}_2 E(|w|^2)E(u_2 f)\chi_{\sigma(w_1)} = w_2 E(|u|^2)E(\bar{w}_2 f)\chi_{\sigma(w_1)}. \tag{19}$$

Now, the desired conclusion in (b) follows from (18) and (19). ■

Example 2.18: Let $X = \{1, 2, 3, 4\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/4$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{\{1, 2\}, \{3, 4\}\}$. The $L^2(\Sigma)$ space under consideration is $\mathbb{C}^4$ and relative to the standard orthonormal basis,

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix};$$

and for w and u corresponding to (w_1, w_2, w_3, w_4) and (u_1, u_2, u_3, u_4), respectively, we have

$$\begin{aligned}
T &= \begin{bmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & w_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & u_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{w_1 u_1}{2} & \frac{w_1 u_2}{2} & 0 & 0 \\ \frac{w_2 u_1}{2} & \frac{w_2 u_2}{2} & 0 & 0 \\ 0 & 0 & \frac{w_3 u_3}{2} & \frac{w_3 u_4}{2} \\ 0 & 0 & \frac{w_4 u_3}{2} & \frac{w_4 u_4}{2} \end{bmatrix}.
\end{aligned}$$

Now, put $u = (1, 2, i, 2i)$ and $w = (i, 2i, 1, 2)$. Then,

$$T = \begin{bmatrix} \frac{i}{2} & i & 0 & 0 \\ i & 2i & 0 & 0 \\ 0 & 0 & \frac{i}{2} & i \\ 0 & 0 & i & 2i \end{bmatrix}, \text{ and so } T^* = \begin{bmatrix} -\frac{i}{2} & -i & 0 & 0 \\ -i & -2i & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & -i \\ 0 & 0 & -i & -2i \end{bmatrix}.$$

On the other hand,

$$\begin{aligned} \bar{u}E(uw) &= (1, 2, -i, -2i)E(i, 4i, i, 4i) \\ &= \left(\frac{5i}{2}, 5i, \frac{5}{2}, 5\right) = wE(|u|^2). \end{aligned}$$

Thus, T is not self-adjoint but it is a normal operator, by Theorem 2.14. Note that if in the above we take $w = -\bar{u}$, then T will be a self-adjoint operator but not a positive one.

Recall that T is said to be quasi-normal if $T(T^*T) = (T^*T)T$. In view of (1) and (2), we have

$$T(T^*T) = (E(|w|^2)E(|u|^2)) \begin{bmatrix} M_{w_1u_1} & EM_{w_1u_2} \\ M_{w_2u_1} & M_{w_2}EM_{u_2} \end{bmatrix}; \tag{20}$$

$$(T^*T)T = (E(|w|^2)E(uw)) \begin{bmatrix} M_{|u_1|^2} & EM_{\bar{u}_1u_2} \\ M_{\bar{u}_2u_1} & M_{\bar{u}_2}EM_{u_2} \end{bmatrix}. \tag{21}$$

The following corollary follows directly from (20) and (21).

Corollary 2.19: *T is quasi-normal if and only if the followings hold:*

(a) $E(|w|^2)E(|u|^2)w_1u_1 = E(|w|^2)E(uw)|u_1|^2$;
(b) $E(|w|^2)E(|u|^2)w_1u_2 = E(|w|^2)E(uw)\bar{u}_1u_2$;
(c) $E(|w|^2)E(|u|^2)w_2u_1 = E(|w|^2)E(uw)\bar{u}_2u_1$;
(d) $E(|w|^2)E(|u|^2)w_2E(u_2f) = E(|w|^2)E(uw)\bar{u}_2E(u_2f), \quad \forall f \in \mathcal{N}(E)$.

In the following, we discuss some simple consequences of items in Corollary 2.19 and suppose that T is quasi-normal.

$$\text{(a)} \iff E(|u|^2)w_1u_1 = E(uw)|u_1|^2 \iff E(|u|^2)w_1\chi_{\sigma(u_1)} = E(uw)\bar{u}_1\chi_{\sigma(u_1)};$$

$$\text{(b)} \iff E(|u|^2)w_1u_2 = E(uw)\bar{u}_1u_2 \iff E(|u|^2)w_1\chi_{\sigma(u_2)} = E(uw)\bar{u}_1\chi_{\sigma(u_2)};$$

$$\text{(c)} \iff E(|u|^2)w_2u_1 = E(uw)\bar{u}_2u_1 \iff E(|u|^2)w_2\chi_{\sigma(u_1)} = E(uw)\bar{u}_2\chi_{\sigma(u_1)}.$$

Now, set $f = \bar{u}_2\sqrt{E|w|^2}\chi_{A_n}$ in (d) in which $\mu(A_n) < \infty$. Using the same method as in the proof of Lemma 2.6, we have

$$\begin{aligned} \text{(d)} &\iff E(|u|^2)w_2E(u_2f) = E(uw)\bar{u}_2E(u_2f) \\ &\iff E(|u|^2)w_2\chi_{\sigma(E(|u_2|^2))} = E(uw)\bar{u}_2\chi_{\sigma(E(|u_2|^2))}. \end{aligned}$$

Theorem 2.20: *Let $T \in B(L^2(\Sigma))$. Then, T is quasi-normal if and only if $wE(|u|^2) = \bar{u}E(uw)$.*

Proof: If $wE(|u|^2) = \bar{u}E(uw)$ holds, then T is normal so T is quasi-normal.

Conversely, suppose (a), (b), (c) and (d) in Corollary 2.19 hold. Then, we have

$$\begin{aligned}
\text{(b)} &\Longrightarrow E(|u|^2)w_1\chi_{\sigma(u_2)}|u_2|^2 = E(uw)\bar{u}_1\chi_{\sigma(u_2)}|u_2|^2\\
&\Longrightarrow E(|u|^2)w_1|u_2|^2 = E(uw)\bar{u}_1|u_2|^2\\
&\Longrightarrow E(|u|^2)w_1E(|u_2|^2) = E(uw)\bar{u}_1E(|u_2|^2)\\
&\Longrightarrow E(|u|^2)w_1\chi_{\sigma(E(|u_2|^2))} = E(uw)\bar{u}_1\chi_{\sigma(E(|u_2|^2))}\\
&\xrightarrow{\text{(a)}} E(|u|^2)w_1\chi_{\sigma(u_1)\cup\sigma(E(|u_2|^2))} = E(uw)\bar{u}_1\chi_{\sigma(u_1)\cup\sigma(E(|u_2|^2))}\\
&\Longrightarrow E(|u|^2)w_1 = E(uw)\bar{u}_1.
\end{aligned}$$

In the same manner, from (c) and (d) we obtain $E(|u|^2)w_2 = E(uw)\bar{u}_2$. Thus, $wE(|u|^2) = \bar{u}E(uw)$. ■

We say that $\lambda \in \mathbb{C}$ belongs to the essential range of a measurable function f if for each neighbourhood G of λ, $\mu(f^{-1}(G)) > 0$. Our next task is about the spectra. For a bounded linear operator T, spec(T) denote its spectrum. Let M_u be a bounded multiplication operator on $L^2(\Sigma)$. It is well-known fact that

$$\text{spec}(M_u) = \{\lambda \in \mathbb{C} : \nexists c > 0,\ \text{s.t. } |u(x) - \lambda| \geq c \ \text{ a.e.}\}$$

Herron [2] proved that the spectrum of an unweighted bounded conditional operator EM_u is the essential range of $E(u)$. In the following, we determine the spectrum of bounded weighted conditional operators on $L^2(\Sigma)$.

Theorem 2.21: *Let $\mathcal{A} \neq \Sigma$ and $T = M_wEM_u \in \mathcal{B}(L^2(\Sigma))$. Then,*

$$\textit{spec}(T) \setminus \{0\} = \textit{ess range}(E(uw)) \setminus \{0\}.$$

Proof: First, assume that $\lambda \notin$ *ess range*$(E(uw))$ and $\lambda \neq 0$. We show that $T - \lambda I$ is invertible. Let $f \in \mathcal{N}(T - \lambda I)$. Then, $wE(uf) = \lambda f$. Multiplying both sides of this equation by u and then taking E we obtain $E(uw)E(uf) = \lambda E(uf)$. Then, $E(uf)(E(uw) - \lambda) = 0$. But, by hypothesis, $E(uw) \neq \lambda$. Thus, $E(uf) = 0$, and so $f = 0$. To show that $T - \lambda I$ is surjective, let $g = g_1 + g_2 \in L^2(\Sigma)$ be given. We show that there exists an $L^2(\Sigma)$ function f such that $(T - \lambda I)f = g$. For this, define

$$\begin{aligned}
f_1 &= \frac{w_1E(ug) - g_1E(uw) + \lambda g_1}{\lambda(E(uw) - \lambda)},\\
f_2 &= \frac{w_2E(ug)}{\lambda(E(uw) - \lambda)} - \frac{g_2}{\lambda}.
\end{aligned}$$

Since T is bounded, $\|g_i\|_2 \le \|g\|_2$ and $\|(E(uw)-\lambda)^{-1}\|_\infty \le c$, then $f_i \in L^2(\Sigma)$ for $i=1,2$. Moreover, f_1 is $\mathcal{A}$-measurable, $f_2 \in \mathcal{N}(E)$ and

$$wE(uf_1) = \frac{wE(uw_1)E(ug) - wE(ug_1)E(uw) + \lambda wE(ug_1)}{\lambda(E(uw)-\lambda)};$$
$$wE(uf_2) = \frac{wE(uw_2)E(ug) - wE(ug_2)E(uw) + \lambda wE(ug_2)}{\lambda(E(uw)-\lambda)};$$
$$\lambda f = \lambda(f_1+f_2) = \frac{wE(ug) - gE(uw) + \lambda g}{E(uw)-\lambda}.$$

It follows that $T(f) = wE(uf) = wE(ug)/(E(uw)-\lambda)$, and hence $(T-\lambda)f = g$.

Conversely, suppose $\lambda \notin \mathrm{spec}(T)$. Put $W=w$ and $U = u/(\lambda(E(uw)-\lambda))$ and define a linear operator S on $L^2(\Sigma)$ as $S = M_W E M_U - 1/\lambda I$. We claim that $\lambda \notin$ *ess range*$(E(uw))$ if and only if S is bounded. Suppose $\lambda \notin$ *ess range*$(E(uw))$. Then, for all $f \in L^2(\Sigma)$, we have

$$\|S(f)\|_2 \le \|wE\left(\frac{uf}{\lambda(E(uw)-\lambda)}\right)\|_2 + \frac{1}{|\lambda|}\|f\|_2$$
$$\le \frac{1}{|\lambda|}\left(\frac{1}{c}\|E(|u|^2)E(|w|^2)\|_\infty^{1/2} + 1\right)\|f\|_2.$$

Now, let S be bounded. Then, $\|ES(f)\|_2 \le \|S\|\,\|f\|_2$ for all $f \in L^2(\Sigma)$. In particular, for any $f \in L^2(\mathcal{A})$, we have

$$ES(f) = \frac{E(w)E(u) - E(uw) + \lambda}{\lambda(E(uw)-\lambda)} f.$$

Put $g = wE(u) - E(uw) + \lambda$. Then, $g \ne 0$ and $ES = M_\varphi$ is a bounded multiplication operator on $L^2(\mathcal{A})$ with multiplier $\varphi = E(g)/(\lambda(E(uw)-\lambda))$. Hence, $\varphi \in L^\infty(\mathcal{A})$. On the other hand, since T is bounded, then $0 \ne E(g) \in L^\infty(\mathcal{A})$. Therefore, $1/(E(uw)-\lambda) \in L^\infty(\mathcal{A})$. This implies that $\lambda \notin$ *ess range*$(E(uw))$. Now it is easy to check that $S(T-\lambda) = (T-\lambda)S = I$. Consequently, $S = (T-\lambda)^{-1}$. This completes the proof. ∎

Corollary 2.22: *Let $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$ and let $\lambda \notin spec(T)$. Then,*

$$(M_w E M_u - \lambda I)^{-1} = M_W E M_U - \frac{1}{\lambda} I,$$

where $W=w$ and $U = u/(\lambda(E(uw)-\lambda))$.

In the following, we characterize the positivity of the weighted conditional operators on $L^2(\Sigma)$. Recall that T is said to be positive if $\langle Tf, f\rangle \ge 0$ for all $f \in L^2(\Sigma)$.

Lemma 2.23: *Let $0 \le g \in L^0(\mathcal{A})$ and $T = M_{g\bar{u}} E M_u \in \mathcal{B}(L^2(\Sigma))$. Then, $T \ge 0$.*

Proof: Put $S = \sigma(E(|u|^2))$, $w = \sqrt{g/E(|u|^2)}\,\bar{u}\chi_S$ and define $T_1 = M_w E M_u$. Then, it is easy to check that $T = T_1^* T_1$, and so $T \ge 0$. ∎

Theorem 2.24: *The bounded weighted conditional operator T on $L^2(\Sigma)$ is positive if and only if the followings hold:*

(a) $E(uw) \geq 0$;
(b) $\bar{u}E(uw) = wE(|u|^2)$.

Proof: Let $T \geq 0$. Then, by definition, T is self-adjoint and $\operatorname{spec}(T) \subseteq [0, \infty)$. Using Theorems 2.9 and 2.21, (a) and (b) hold. Conversely, let (a) and (b) hold. Put $S = \sigma(E(|u|^2))$ and $g = (E(uw)/E(|u|^2))\chi_S$. From (b), $g\bar{u} = w\chi_S$. Moreover, for all $f \in L^2(\Sigma)$, we have $M_{g\bar{u}}EM_u(f) = w\chi_S E(uf) = wE(uf) = T(f)$. Now, the desired conclusion follows from Lemma 2.23. ■

In particular, when $w = 1$, we have the following corollary:

Corollary 2.25: *Let $T = EM_u \in \mathcal{B}(L^2(\Sigma))$. Then, the followings hold:*

(a) *T is self-adjoint if and only if $\bar{u} = u \in L^\infty(\mathcal{A})$;*
(b) *T is normal if and only if $u \in L^\infty(\mathcal{A})$;*
(c) *T is positive if and only if $0 \leq u \in L^\infty(\mathcal{A})$.*

Example 2.26: (a) Let $X = [-1, 1]$, $\mathrm{d}\mu = (1/2)\mathrm{d}x$, Σ the Lebesgue sets, and $\mathcal{A}$ the sigma subalgebra of Σ consisting of sets symmetric about the origin. It is easy to check that $E(f)(x) = (f(x) + f(-x))/2$. Put $u = x^2 + x^3$ and $w = x^4 + x^5$. Then, $\|T\| = 2$, $E(uw) = x^6 + x^8$ and $uE(uw) = x^8(x^2 + 1)(x + 1) = wE(|u|^2)$. Then, by Theorem 2.24, $T = M_w EM_u$ is a bounded positive on $L^2(\Sigma)$. Note that, if we take $W = (x^2 + x)(x^2 + 1)^{-1/2}$, $U = x^2 + x^3$ and $T_1 = M_W EM_U$. Then, $\|T_1\| = \sqrt{2}$ and $T_1^* T_1 = M_{x^4+x^5} EM_{x^2+x^3} = T$.

(b) Let $X = [0, 1]$, $\mathrm{d}\mu = \mathrm{d}x$ and Σ be the Lebesgue sets. Let $P = \{A_n : n \in \mathbb{N}\}$ be a countable partition of X into disjoint sets with $0 < \mu(A_n) < \infty$. Let $\mathcal{A}$ be the sigma subalgebra generated by P. Then, $E(f) = \sum_n (1/\mu(A_n)\int_{A_n} f\,\mathrm{d}\mu)\chi_{A_n}$ (see [4]). Especially, let $\mathcal{A} = \{\emptyset, X\}$. Then, $E(f) = \int_0^1 f(x)\mathrm{d}x$. Put $u = 4x^3$ and $w = 7x^3$. Then, $\|T\| = 4$, $E(uw) = 4$ and $uE(uw) = 16x^2 = wE(|u|^2)$. Then, by Theorem 2.24, T is a bounded positive operator on $L^2(\Sigma)$. On the other hand, by a direct computation, we have $T(f) = 28x^3 \int_0^1 x^3 f(x)\,\mathrm{d}x$ and

$$\langle T(f), f\rangle = 28\left(\int_0^1 x^3 f(x)\mathrm{d}x\right)^2 \geq 0, \quad f \in L^2(\Sigma).$$

Recall that for $T \in B(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{1/2}U|T|^{1/2}$.

Proposition 2.27: *Let $0 \leq g \in L^0(\mathcal{A})$ and $T = M_{g\bar{u}}EM_u \in \mathcal{B}(L^2(\Sigma))$. Then, $\sqrt{T} = M_{k\bar{u}}EM_u$, where $S = \sigma(E(|u|^2)$ and $k = \sqrt{(g/E(|u|^2))}\chi_S$.*

Proof: First, note that T is positive by Theorem 2.24. Since $k\bar{u}E(|u|^2k) = g\bar{u}$ and $\bar{u}\chi_S = \bar{u}$, we have

$$(M_{k\bar{u}}EM_u)^2 = M_{k\bar{u}E(|u|^2k)}EM_u = T.$$

This completes the proof. ■

Corollary 2.28: *Let $T = M_wEM_u \in \mathcal{B}(L^2(\Sigma))$ and let $U|T|$ be its polar decomposition and $S = \sigma(E(|u|^2)$ and $G = \sigma(E(|w|^2)$. Then, the followings hold:*

(a) $|T| = M_{g\bar{u}}EM_u$, *where* $g = \sqrt{(E(|w|^2)/E(|u|^2))}\chi_S$;
(b) $\sqrt{|T|} = M_{p\bar{u}}EM_u$, *where* $p = \sqrt[4]{(E(|w|^2)/(E(|u|^2))^3)}\chi_S$;
(c) $T = U|T|$, *where* $U = M_{g'}EM_u$ *and* $g' = w\chi_{S\cap G}/\sqrt{E(|u|^2)E(|w|^2)}$;
(d) $\tilde{T} = M_{g''\bar{u}}EM_u$, *where* $g'' = (E(uw)/E(|u|^2))\chi_S$.

Corollary 2.29: *Let $T = M_wEM_u \in \mathcal{B}(L^2(\Sigma))$. Then, the following statements hold:*

(a) $\tilde{T}$ *is self-adjoint if and only if* $\overline{E(uw)} = E(uw)$;
(b) $\tilde{T}$ *is normal;*
(c) $\tilde{T}$ *is positive if and only if* $E(uw) \geq 0$.

Recall that a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant for an operator $T \in \mathcal{B}(\mathcal{H})$ whenever $T\mathcal{M} \subseteq \mathcal{M}$. If $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp$ are both invariant for T, then we say that $\mathcal{M}$ reduces T. Now using (1), the closed subspace $L^2(\mathcal{A})$ of $L^2(\Sigma)$ is an invariant subspace of T if and only if the lower left corner's entry of the matrix representation of T is the zero operator on its domain. But, in this case, we have

$$\begin{aligned} M_{w_2u_1} = 0 &\iff w_2u_1f = 0, \ \ \forall f \in L^2(\mathcal{A}) \\ &\iff w_2u_1 = (w - E(w))E(u) = 0 \\ &\iff wE(u) \in L^0(\mathcal{A}) \\ &\iff w\chi_S \in L^0(\mathcal{A}), \ S = \sigma(E(u)). \end{aligned}$$

By the same argument on the lower left corner's entry of matrix representation (2) of T^*, we have the following result.

Proposition 2.30: *Let $T = M_wEM_u \in \mathcal{B}(L^2(\Sigma))$. Then, the following statements hold:*

(a) $L^2(\mathcal{A})$ *is an invariant subspace of* T *if and only if* $w\chi_{\sigma(E(u))} \in L^0(\mathcal{A})$.
(b) $L^2(\mathcal{A})$ *is an invariant subspace of* T^* *if and only if* $u\chi_{\sigma(E(w))} \in L^0(\mathcal{A})$.
(c) $u\chi_{\sigma(E(w))} \in L^0(\mathcal{A})$ *if and only if* $EM_{w_1u_2}|_{\mathcal{N}(E)} = 0$.
(d) $L^2(\mathcal{A})$ *is a reducing subspace of* T *if and only if* $w\chi_{\sigma(E(u))}$ *and* $u\chi_{\sigma(E(w))}$ *are* $\mathcal{A}$-*measurable functions.*

Corollary 2.31: *Let $T = M_wEM_u \in \mathcal{B}(L^2(\Sigma))$. Then, $L^2(\mathcal{A})$ is a reducing subspace of $\tilde{T}$ if and only if $u\chi_{\sigma(E(uw))}$ is an $\mathcal{A}$-measurable function.*

Put $\mathcal{K} = \{M_w E M_u : u, w, uw \in \mathcal{D}(E),\ E(|u|^2)E(|w|^2) \in L^\infty(\Sigma)\}$.

In the following, we describe the general form of self-adjoint, normal, quasi-normal and positive elements of $\mathcal{K}$.

Theorem 2.32: *Let $T = M_w E M_u \in \mathcal{K}$. Then, the following statements hold:*

(a) *T is self-adjoint if and only if $T = M_{g\bar{u}} E M_u$ for some $\bar{g} = g \in L^0(\mathcal{A})$.*
(b) *T is normal if and only if $T = M_{g\bar{u}} E M_u$ for some $g \in L^0(\mathcal{A})$.*
(c) *T is normal if and only if T is quasi-normal.*
(d) *T is positive if and only if $T = M_{g\bar{u}} E M_u$ for some $0 \leq g \in L^0(\mathcal{A})$.*

Proof: We only give proof of (b). Repeating the same argument and using the previous results give the other parts.

First, let $T = M_{g\bar{u}} E M_u$ for some $g \in L^0(\mathcal{A})$. Then,

$$\bar{u}E(uw) = \bar{u}E(ug\bar{u}) = g\bar{u}E(|u|^2) = wE(|u|^2).$$

Thus, by 2.14, T is normal. Conversely, let $T = M_w E M_u \in \mathcal{K}$ be normal. Then, $\bar{u}E(uw) = wE(|u|^2)$, and so

$$w\chi_S = \frac{E(uw)}{E(|u|^2)}\bar{u}\chi_S, \quad S = \sigma(E(|u|^2)).$$

Put $g = (E(uw)/E(|u|^2))\chi_S \in L^0(\mathcal{A})$. Since $E(uf)\chi_S = E(uf)$, then we have

$$\begin{aligned} M_w E M_u(f) = w\chi_S E(uf) &= \frac{E(uw)}{E(|u|^2)}\bar{u}\chi_s E(uf) \\ &= g\bar{u}E(uf) = M_{g\bar{u}} E M_u. \end{aligned}$$

This completes the proof. ∎

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