Received: 15 February 2018 | Revised: 14 February 2019 | Accepted: 5 March 2019

DOI: 10.1002/mana.201800066

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Operators whose ranges are contained in the null space of conditional expectations

M. R. Jabbarzadeh | **M. H. Sharifi**

Faculty of Mathematical Sciences, University of Tabriz, 5166615648 Tabriz, Iran

Correspondence
M. R. Jabbarzadeh, Faculty of Mathematical Sciences, University of Tabriz, 5166615648, Tabriz, Iran.
Email: mjabbar@tabrizu.ac.ir

Funding information
Research Council of Tabriz University

Abstract

The goal of this article is to study the algebra $\mathfrak{D}_\varphi$ of a conditional type operators on $L^2(\Sigma)$ such that each of members of $\mathfrak{D}_\varphi$ has its range contained in the kernel of a conditional expectation E. We present characterizations of this algebra in terms of $(\varphi_0)_0$-type sub-sigma algebras of Σ.

KEYWORDS

composition operators, conditional expectation, Lie–Banach algebra, multipliers

MSC (2010)

47B20, 47B38

1 | INTRODUCTION AND PRELIMINARIES

Let (X, Σ, μ) be a complete probability space and let $\mathcal{B}$ be a complete sub-sigma algebra of Σ. The space $L^2(X, \mathcal{B}, \mu|_{\mathcal{B}})$ is abbreviated by $L^2(\mathcal{B})$ and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. When reference is made to the support of a measurable function f, we infer the choice of a representative f_0 from the equivalence class, f, of almost everywhere defined functions. We then choose the support of f to be $S_f = \{x \in X : f_0(x) \neq 0\}$. A $\mathcal{B}$-atom of the measure μ is an element $B \in \mathcal{B}$ with $\mu(B) > 0$ such that for each $A \in \mathcal{B}$, if $A \subseteq B$ then either $\mu(A) = 0$ or $\mu(A) = \mu(B)$.

For a sub-sigma algebra $\mathcal{B} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{B}$ is the mapping $f \to E^{\mathcal{B}} f$, defined for all μ-measurable nonnegative f where $E^{\mathcal{B}} f$, by the Radon–Nikodym theorem, is the unique finite-valued $\mathcal{B}$-measurable function satisfying

$$\int_B f\, d\mu = \int_B E^{\mathcal{B}}(f)\, d\mu, \quad \text{for all} \quad B \in \mathcal{B}.$$

Let $u \in L^0(\Sigma)$ be real-valued and consider the set $B_u = \{x \in X : E(u^+)(x) = E(u^-)(x) = \infty\}$, where $u^+ = \max\{f, 0\}$ and $u^- = \max\{-f, 0\}$. The function u is said to be conditionable with respect to $\mathcal{B}$ if $\mu(B_u) = 0$. Put $E(u) = E(u^+) - E(u^-)$. If $u = u_1 + iu_2 \in L^0(\Sigma)$, then u is said to be conditionable if u_1 and u_2 are conditionable. In this case we set $E(u) = E(u_1) + iE(u_2)$. This defines a linear operator $E : \mathcal{D}(E) \to L^0(\mathcal{B}) \subseteq L^0(\Sigma)$, where the domain $\mathcal{D}(E)$ of E is defined by $\mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$. It follows that $\mathcal{D}(E)$ contains $\{L^p(\Sigma) : 1 \leq p \leq \infty\}$ (see [12,13]). As an operator on $L^2(\Sigma)$, $E^{\mathcal{B}}$ is an orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{B})$. In general, the conditional expectation $E^{\mathcal{B}}$ is used to relate and connect Σ-measurable functions with $\mathcal{B}$-measurable functions. If there is no possibility of confusion we write $E(f)$ in place of $E^{\mathcal{B}}(f)$. This operator will play a major role in our work. A detailed discussion and verification of most of properties may be found in [8,10,12,13,18,22,23]. Those properties of E used in our discussion are summarized below. In all cases we assume that $f, g \in \mathcal{D}(E)$, where $\mathcal{D}(E)$ denotes the domain of E.

- If g is $\mathcal{B}$-measurable, then $E(fg) = E(f)g$.
- $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $S_{E(|f|)}$ is the smallest $\mathcal{B}$-measurable set containing S_f.
- (Conditional variance) $E\left(|f - E(f)|^2\right) = E\left(|f|^2\right) - |E(f)|^2$, where $f \in L^2(\Sigma)$.
- (Conditional Cauchy–Schwarz) $|E(fg)|^2 \leq E\left(|f|^2\right)|E(|g|^2)|$, where $f, g \in L^2(\Sigma)$.

Let φ be a nonsingular measurable transformation from X into X; that is, $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. Let h be the finite-valued Radon–Nikodym derivative $d\mu \circ \varphi^{-1}/d\mu$. The densely defined composition operator $C_\varphi : L^2(\Sigma) \supseteq \mathcal{D}\left(C_\varphi\right) \to L^2(\Sigma)$ induced by φ is given by $C_\varphi(f) = f \circ \varphi$, for each $f \in \mathcal{D}\left(C_\varphi\right) = \left\{f \in L^2(\Sigma) : f \circ \varphi \in L^2(\Sigma)\right\}$. Here, the non-singularity of φ guarantees that C_φ is well defined and closed (see [2,6]). A recent monograph related to unbounded composition operators is [3]. Let $f \in L^1(\Sigma)$. Then by the change of variables formula we have $\int_{\varphi^{-1}(B)} f \circ \varphi \, d\mu = \int_B hf \, d\mu$, for all $B \in \Sigma$. Consequently, C_φ maps $\mathcal{D}\left(C_\varphi\right)$ boundedly into itself, if and only if $h \in L^\infty(\Sigma)$, and in this case, $\mathcal{D}\left(C_\varphi\right) = L^2(\Sigma)$ and $\|C_\varphi\|^2 = \|h\|_\infty$. Put $E_\varphi = E^{\varphi^{-1}(\Sigma)}$. Then for each $f \in L^1(\Sigma)$, there exists a Σ-measurable function g such that $E_\varphi(f) = g \circ \varphi$. We can assume that $S_g \subseteq S_h$. In this case g is unique. We then write $g = E_\varphi(f) \circ \varphi^{-1}$, though we make no assumption regarding the invertibility of φ (see [7]). A result of Hoover, Lambert and Quinn [14] shows that the adjoint C_φ^* of a bounded composition operator C_φ on $L^2(\Sigma)$ is given by $C_\varphi^*(f) = hE_\varphi(f) \circ \varphi^{-1}$. For information on bounded composition operators on measurable function spaces, we refer readers to [5,14,18] and the monograph [24].

Let $B\left(L^2(\Sigma)\right)$ denote the algebra of all bounded linear operators on $L^2(\Sigma)$. For each operator $T \in B\left(L^2(\Sigma)\right)$, the null-space and the range of T are denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. For $\{u, f\} \subseteq \mathcal{D}(E)$, we define $D_u(f) = E(u)f - uE(f)$. A measurable function $u \in \mathcal{D}(E)$ for which $D_u(f) \in L^2(\Sigma)$ for all $f \in L^2(\Sigma)$, is called a conditional multiplier on $L^2(\Sigma)$. These multipliers on $L^2(\Sigma)$ were initially introduced in [21] and then extended on C*-algebras in [9]. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each $L^2(\Sigma)$ convergent sequence assures us that $u \in \mathcal{D}(E)$ is a conditional multiplier if and only if $D_u = M_{E(u)} - M_u E \in B\left(L^2(\Sigma)\right)$. In [21] the relationship between a probability space (X, Σ, μ) and a sub-sigma algebra $\mathcal{B}$ of Σ is studied by using the algebra $\mathfrak{D} = \left\{D_u : u \in L^2(\Sigma), E\left(|u|^2\right) \in L^\infty(\Sigma)\right\}$ of bounded operators on $L^2(\Sigma)$. In [15–17], another type of Lambert multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of the conditional expectation operator.

In the next section we prove the existence of a set $B_0^\varphi \in \mathcal{B}$ which is maximum set with respect to $\left\{S \in \Sigma : S \cap \varphi^{-1}(\Sigma) \subseteq \mathcal{B}\right\}$. A collection of composition conditional type multipliers is defined by $\mathfrak{L}_\varphi = \left\{u \in L^2(\Sigma) : D_u^\varphi \in B\left(L^2(\Sigma)\right)\right\}$, where $D_u^\varphi = D_u C_\varphi$. We show that $\mathfrak{D}_\varphi = \left\{D_u^\varphi : u \in \mathfrak{L}_\varphi\right\}$, the algebra of all composition conditional type operators on $L^2(\Sigma)$, restricted to $L^2\left(B_0^\varphi\right)$ is the zero operator algebra. When $\varphi = id$, the identity map on X, set $B_0^{id} = B_0$. It was shown in [21] that there exists a function of full support in the kernel of E whenever $\mu\left(B_0\right) = 0$. We extend this result to case where $\mu\left(B_0^\varphi\right) = 0$. We define $(\varphi_0)_0$-type and $(\varphi_1)_0$-type sub-sigma algebras of Σ which are extentions of type-0 and type-1 subalgebras in [21]. These concepts turn out to play a fundamental role in the theory to be developed. We prove that $\mathfrak{D}_\varphi$ is closed in the weak operator topology and then we establish criteria for normality for elements of $\mathfrak{D}_\varphi$. In addition, we obtain the commutant of $\mathfrak{D}_\varphi$ and then we discuss an open problem stated in [21]. Finally, we define a suitable norm and a Lie product on $\mathfrak{L}_\varphi$ and show that, under these structures, $\mathfrak{L}_\varphi$ becomes a Banach–Lie algebra.

2 | CHARACTERIZATIONS OF $\mathfrak{D}_\varphi$

Let $\mathcal{B} \subseteq \Sigma$ be a sub-sigma algebra and let C_φ be a non-singular measurable transformation for which the composition operator $C_\varphi \in B\left(L^2(\Sigma)\right)\}$. For each $u \in L^2(\Sigma)$, we define the composition conditional type operator D_u^φ on $L^2(\Sigma)$ by $D_u^\varphi(f) = E(u)f \circ \varphi - uE(f \circ \varphi)$. Note that $\mathcal{R}\left(C_\varphi\right) \subseteq \mathcal{D}(E)$ and $D_u^\varphi = D_u C_\varphi$, where D_u is a conditional type operator and C_φ is a composition operator defined by $D_u(f) = E(u)f - uE(f)$ and $C_\varphi f = f \circ \varphi$, respectively. For example, let $X = [0,1] \times [0,1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{B} = \{B \times [0,1] : B$ is a Lebesgue set in $[0,1]\}$. Then, for each $f \in L^2(\Sigma)$, $(Ef)(x,y) = \int_0^1 f(x,t)\, dt$, which is independent of the second coordinate. In this case we have

$$D_u^\varphi(f) = f(\varphi(x,y)) \int_0^1 u(x,t)\, dt - u(x,y) \int_0^1 f(\varphi(x,t))\, dt.$$

In general, the structure of $\mathcal{N}(E)$, the null space of E, is very complicated. For example, there are no strictly positive elements in $\mathcal{N}(E)$ and there is not a proper sub-sigma algebra $\mathcal{A}$ in Σ for which $\mathcal{N}(E) = L^2(\mathcal{A}) \ominus \mathbb{C}1$ (see [20]). Since for each

$D_u^\varphi \in \mathfrak{D}_\varphi$, $\mathcal{R}(D_u^\varphi) \subseteq \mathcal{N}(E)$, so the study of $\mathfrak{D}_\varphi$ opens a new window through which one may observe $\mathcal{N}(E)$. But in description of $\mathfrak{D}_\varphi$, we will observe that (Theorem 2.6) the representation $u \mapsto D_u^\varphi$ is faithful if and only if the maximum set B_0^φ has measure zero. So the $(\varphi_0)_0$-type subalgebras of Σ (Definition 2.9) will play a major role in our work. On the other hand, products of conditional expectation, multiplication and composition operators and their differences appears more often in the service of the study of other operators such as operators generated by random measures and averaging operators on order ideals in Banach lattices (e.g., [10,12]).

One of the interesting features of composition conditional type operator is that the conditional type operator alone may not define a bounded operator (see Example 2.8). Let $\mathfrak{L}_\varphi := \{u \in L^2(\Sigma) : D_u^\varphi \in B(L^2(\Sigma))\}$ and $\mathfrak{D}_\varphi := \{D_u^\varphi : u \in \mathfrak{L}_\varphi\}$. Then $\mathfrak{L}_\varphi$ and $\mathfrak{D}_\varphi$ are vector spaces and $E(\mathfrak{D}_\varphi) = \{0\}$. If $u \in L^\infty(\Sigma)$, then $\|D_u^\varphi\| \le 2\|u\|_\infty \sqrt{\|h\|_\infty}$. In this case, $L^\infty(\Sigma) \subseteq \mathfrak{L}_\varphi$ and $\mathfrak{D}_\varphi \subseteq B(L^2(\Sigma))$. Let $\mathcal{M}^\varphi := \{S \in \Sigma : S \cap \varphi^{-1}(\Sigma) \subseteq \mathcal{B}\}$, in which $S \cap \varphi^{-1}(\Sigma) = \{S \cap \varphi^{-1}(A) : A \in \Sigma\}$. Then $\mathcal{M}^\varphi \subseteq \mathcal{B}$. In the following lemma we show that $\mathcal{M}^\varphi$ contains a maximum set in the sense of measure theory.

Lemma 2.1. *For each sub-sigma algebra $\mathcal{B} \subseteq \Sigma$, the collection $\mathcal{M}^\varphi$ contains a maximum set B_0^φ; that is, there exists $B_0^\varphi \in \mathcal{M}^\varphi$ such that for each non-null set $S \in \Sigma$ with $S \nsubseteq B_0^\varphi$, we can find $A \in \Sigma$ such that $S \cap \varphi^{-1}(A) \notin \mathcal{B}$.*

Proof. Let $r = \sup\{\mu(S) : S \in \mathcal{M}^\varphi\}$. If $r = 0$, then each element of $\mathcal{M}^\varphi$ is a maximum set. Let $r > 0$ and $\{S_n\}_{n=1}^\infty \subseteq \mathcal{M}^\varphi$ be a sequence of sets with $\mu(S_n) > 0$ such that $\mu(S_n) \uparrow r$. Put $B_0^\varphi = \bigcup_n S_n$. Then $B_0^\varphi \in \mathcal{M}^\varphi$ and $\mu(S_n) \le \mu(B_0^\varphi) \le r$, for all $n \in \mathbb{N}$. It follows that $\mu(B_0^\varphi) = r$. Now, we show that B_0^φ is a maximum set. Let $S \in \Sigma$ be a non-null set and $S \nsubseteq B_0^\varphi$. If $S \in \mathcal{M}^\varphi$, then $S \cup B_0^\varphi \in \mathcal{M}^\varphi$ and so

$$\mu(S \cup B_0^\varphi) = \mu(B_0^\varphi) + \mu(S - B_0^\varphi) = r + \mu(S - B_0^\varphi) > r.$$

But this is a contradiction. Thus $S \notin \mathcal{M}^\varphi$ and hence $S \cap \varphi^{-1}(A) \notin \mathcal{B}$, for some $A \in \Sigma$. □

Corollary 2.2. *If $\mu(B_0^\varphi) = 0$, then $S \notin \mathcal{M}^\varphi$ for each $S \in \Sigma$ with $\mu(S) > 0$.*

Proposition 2.3. *Let $\mathcal{B} \subseteq \Sigma$ be a complete sub-σ-finite algebra and $A \notin \mathcal{B}$ with $\mu(A) > 0$. Then there exists $C \subseteq A$ with $\mu(C) > 0$ such that C does not contain any $\mathcal{B}$-measurable set of positive measure.*

Proof. Let $M = \{S \in \mathcal{B} : S \subseteq A\}$. Then M is nonempty because $\emptyset \in M$. Set $r = \sup\{\mu(C) : C \in M\}$. Let $\{B_n\} \subseteq M$ and $\mu(B_n) \to r$. Then $B := \bigcup_n B_n \in M$ and $\mu(B_n) \le \mu(B) \le r$. Take $C = A - B$. We claim that C is the desired set. Indeed, if $\mu(C) = 0$, then $A \overset{a.e.}{=} B \in \mathcal{B}$. Since $\mathcal{B}$ is a complete subalgebra, $A \in \mathcal{B}$. Moreover, if there is a $\mathcal{B}$-measurable set B_1 of positive measure such that $B_1 \subseteq C$, then $B_1 \cup B \in M$ and so $\mu(B_1 \cup B) > r$. But this is a contradiction. □

Corollary 2.4. *Let $\mu(B_0^\varphi) = 0$. Then for each $A \in \Sigma$ with $\mu(A) > 0$, there exists $C \subseteq A$ with $\mu(C) > 0$ such that C does not contain any $\mathcal{B}$-measurable set of positive measure.*

Proof. If $A \notin \mathcal{B}$, then by Proposition 2.3 there is not any thing to prove. So we assume that $A \in \mathcal{B}$. Then by Corollary 2.2, $A \notin \mathcal{M}^\varphi$. So $A_1 := A \cap \varphi^{-1}(D) \notin \mathcal{B}$ for some $D \in \Sigma$. Noting that $\mathcal{B}$ is a complete subalgebra, we have $\mu(A_1) > 0$ and $A_1 \notin \mathcal{B}$. Again, the desired result follows from Proposition 2.3 with A_1 replaced by A. □

In the following we show that the representation $u \mapsto D_u^\varphi$ is faithful if and only if the maximum set B_0^φ has measure zero.

Lemma 2.5. *$D_u^\varphi = 0$ if and only if $u \in L^0(\mathcal{B})$ and $S_u \in \mathcal{M}^\varphi$.*

Proof. Let $D_u^\varphi = 0$. Then for each $f \in L^2(\Sigma)$, $E(u) f \circ \varphi = uE(f \circ \varphi)$. Put $f = 1$. Then $E(u) = u$, and so u is a $\mathcal{B}$-measurable function. Now, for $A \in \Sigma$ take $f = \chi_A$. Then $u\chi_{\varphi^{-1}(A)} = uE(\chi_{\varphi^{-1}(A)})$, and thus $E(\chi_{\varphi^{-1}(A)})\chi_{S_u} = \chi_{\varphi^{-1}(A)}\chi_{S_u}$. Hence $E(\chi_{\varphi^{-1}(A)}\chi_{S_u}) = \chi_{\varphi^{-1}(A)}\chi_{S_u}$, since $S_u \in \mathcal{B}$. Thus, for all $A \in \Sigma$, $S_u \cap \varphi^{-1}(A) \in \mathcal{B}$. Conversely, suppose u is a $\mathcal{B}$-measurable function, $S_u \in \mathcal{M}^\varphi$ and $f \in L^2(\Sigma)$. Then we have

$$D_u^\varphi(f) = 0 \iff E(f \circ \varphi)u = (f \circ \varphi)u \iff E(f \circ \varphi)\chi_{S_u} = (f \circ \varphi)\chi_{S_u} \iff E(f \circ \varphi\chi_{S_u}) = (f \circ \varphi)\chi_{S_u},$$

and so, $D_u^\varphi(f) = 0$ if and only if $(f \circ \varphi)\chi_{S_u} \in L^0(\mathcal{B})$. Let G be a Lebesgue measurable set in $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If $0 \notin G$, then $((f \circ \varphi)\chi_{S_u})^{-1}(G) = \varphi^{-1}(f^{-1}(G)) \cap S_u \in \mathcal{B}$. Now, suppose that $0 \in G$. Then

$$((f \circ \varphi)\chi_{S_u})^{-1}(G - \{0\}) \in \mathcal{B};$$

$$((f \circ \varphi)\chi_{S_u})^{-1}(\{0\}) = (\varphi^{-1}(f^{-1}(\{0\}^c)) \cap S_u)^c \in \mathcal{B}.$$

Thus, $\left((f\circ\varphi)\chi_{S_u}\right)^{-1}(G)\in\mathcal{B}$. This completes the proof. □

Theorem 2.6. *The linear map $u\mapsto D_u^\varphi$ is injective if and only if $\mu\left(B_0^\varphi\right)=0$.*

Proof. Let $D_u^\varphi=0$. Then by Lemma 2.5, $S_u\in\mathcal{M}^\varphi$. Thus if $\mu\left(B_0^\varphi\right)=0$, then we have $\mu(S_u)=0$ and hence $u=0$. Conversely, suppose that $\mu\left(B_0^\varphi\right)\neq 0$. Put $u=\chi_{B_0^\varphi}$. Then $S_u=B_0^\varphi\in\mathcal{M}^\varphi\subseteq\mathcal{B}$. Consequently, by Lemma 2.5 we have again $D_u^\varphi=0$. □

Motivated by the above fact, in the following definition we shall be concerned principally with sub-sigma algebras which are of $(\varphi_0)_0$-type and $(\varphi_1)_0$-type.

Definition 2.7. Let $\mathcal{B}\subseteq\Sigma$, $E_\varphi=E^{\varphi^{-1}(\Sigma)}$, $h=\frac{\mu\circ\varphi^{-1}}{d\mu}$ and let C_φ be a composition operator on $L^2(\Sigma)$. Put

$$\mathfrak{M}\left(\mathcal{N}\left(EC_\varphi\right)\right)=\left\{u\in L^2(\Sigma):u\left(\mathcal{N}\left(EC_\varphi\right)\right)\subseteq L^2(\Sigma)\right\}.$$

(a) $\mathcal{B}$ is a $(\varphi_0)_0$-type sub-sigma algebra of Σ if $\mu\left(B_0^\varphi\right)=0$ and $\mathcal{N}(E)\subseteq\mathcal{N}\left(EC_\varphi\right)$.

(b) $\mathcal{B}$ is a $(\varphi_1)_0$-type sub-sigma algebra of Σ if for each $u\in L^0(\mathcal{B})$, $hE_\varphi\left(|u|^2\right)\circ\varphi^{-1}\in L^\infty(\mathcal{B})$, whenever $hE_\varphi\left(|u|^2\right)\circ\varphi^{-1}\in\mathfrak{M}\left(\mathcal{N}\left(EC_\varphi\right)\right)$.

In particular, when φ is the identity map, then the $(\varphi_0)_0$-type and $(\varphi_1)_0$-type subalgebras are called type-0 and type-1 subalgebras, respectively. The terminology follows that of Lambert and Weinstock [21]. Note that if $\mu\left(B_0^\varphi\right)=0$, then $\mu(S)=0$ for every $S\in\mathcal{M}^\varphi$. But B_0, the maximum set of $\mathcal{M}=\{S\in\Sigma:S\cap\Sigma\subseteq\mathcal{B}\}$, is in $\mathcal{M}^\varphi$. Therefore, if $\mathcal{B}$ is $(\varphi_0)_0$-type then $\mathcal{B}$ is type-0 sub-sigma algebra of Σ. However, the converse of this fact is not true in general.

Example 2.8. (i) Consider the space $\ell^2=L^2\left(\mathbb{N},2^{\mathbb{N}},\mu\right)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\})=\left\{\frac{n^{-4}}{K}\right\}$, where $K=\sum_{n=1}^\infty\frac{1}{n^4}$. Suppose that $\mathcal{B}$ is generated by $\{B_1,B_2,\ldots\}$, where $B_n=\{2n-1,2n\}$ for all $n\in\mathbb{N}$. For each n, B_n is a $\mathcal{B}$-atom and $\{n\}\notin\mathcal{B}$. Hence, $\mathcal{M}=\{\emptyset\}$ and so $\mathcal{B}_0=\emptyset$. Let $\varphi:\mathbb{N}\to\mathbb{N}$ be any self-map with $\varphi(n)=1$. Since Σ contains no nonempty null-set, so the self-map φ is non-singular. for each $f=(f_1,f_2,\ldots)\in\ell^2$, we have

$$\|C_\varphi(f)\|_2^2=|f_1|^2\le|f_1|^2+\sum_{n=1}^\infty\frac{|f_n|^2}{n^4}=K\|f\|_2^2,$$

that is, C_φ is bounded on ℓ^2. Also $\varphi^{-1}(\Sigma)=\{\emptyset,\mathbb{N}\}$ and hence $B_0^\varphi=\mathbb{N}$. Consequently, $\mu\left(B_0^\varphi\right)\neq 0$ but $\mu\left(B_0\right)=0$. Therefore $\mathcal{B}$ is $(\varphi_0)_0$-type but not type-0. Now, we show that D_u^φ is bounded on ℓ^2 but D_u is not. For this, let $f=(f_1,f_2,\ldots)\in\ell^2$. Then

$$E(f)=\sum_{n=1}^\infty\left(\frac{1}{\mu(B_n)}\int_{B_n}f\,d\mu\right)\chi_{B_n}=\frac{1}{K}\sum_{n=1}^\infty\left(\frac{16n^4(2n-1)^4}{16n^4+(2n-1)^4}\left(\frac{f_{2n-1}}{(2n-1)^4}+\frac{f_{2n}}{16n^4}\right)\right)\chi_{\{2n-1,2n\}}.$$

Define $u:\mathbb{N}\to\mathbb{R}$ by $u(n)=n$. Then $u\in\ell^2$ and

$$E\left(|u|^2\right)=\frac{1}{K}\sum_{n=1}^\infty\left(\frac{16n^4(2n-1)^2+4n^2(2n-1)^4}{16n^4+(2n-1)^4}\right)\chi_{\{2n-1,2n\}}.$$

Clearly, $E\left(|u|^2\right)\notin L^2(\mathbb{N},\mathcal{B},\mu)$. Hence D_u is not bounded on ℓ^2, see [11, Lemma 5] and also Theorem 2.10 below. On the other hand, by hypothesis, $C_\varphi(f)=(f_1,f_1,\ldots)$. Then, since E is contraction, we have

$$\|D_u^\varphi(f)\|_2\le\|E(u)-u\|_2|f_1|\le 2\|u\|_2\left\{|f_1|^2+\sum_{n=2}^\infty\frac{|f_n|^2}{n^4}\right\}^{\frac{1}{2}}=2\sqrt{K}\|u\|_2\|f\|_2.$$

Hence, D_u^φ is bounded on ℓ^2 but D_u is not.

(ii) Let $X=\{1,2,3,4\}$, $\Sigma=2^X$, $\mu(\{n\})=1/4$ and let $\mathcal{B}$ be the σ-algebra generated by the partition $\{\{1,2\},\{3,4\}\}$. Then $\mathcal{B}$ is a type-0 subalgebra of Σ, $L^2(\Sigma)=\mathbb{C}^4$ and

$$E(f)=\left(\frac{f_1+f_2}{2}\right)\chi_{\{1,2\}}+\left(\frac{f_3+f_4}{2}\right)\chi_{\{3,4\}},$$

where $f = (f_1, f_2, f_3, f_4) \in \mathbb{C}^4$. With respect to the standard orthonormal basis,

$$E = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \text{where} \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

For $u = (u_1, u_2, u_3, u_4) \in \mathbb{C}^4$, $M_u = \operatorname{diag}\{u_1, u_2, u_3, u_4\}$ and and

$$M_{E(u)} = \operatorname{diag}\left\{ \frac{u_1 + u_2}{2}, \frac{u_1 + u_2}{2}, \frac{u_3 + u_4}{2}, \frac{u_3 + u_4}{2} \right\}.$$

It follows that

$$D_u = M_{E(u)} - M_u E = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} \frac{u_2}{2} & -\frac{u_1}{2} \\ -\frac{u_2}{2} & \frac{u_1}{2} \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} \frac{u_4}{2} & -\frac{u_3}{2} \\ -\frac{u_4}{2} & \frac{u_3}{2} \end{bmatrix}.$$

Define $\varphi : X \to X$ as $\varphi = \chi_{\{1,3\}} + 2\chi_{\{2,4\}}$. Then $\varphi^{-1}(\Sigma)$ is generated by the partition $\{\{1,3\}, \{2,4\}\}$, $\varphi^{-1}(\mathcal{B}) = \{\emptyset, X\}$,

$$E_\varphi = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad C_\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then we obtain

$$D_u^\varphi = D_u C_\varphi = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}.$$

Note that $h(1) = h(2) = 2$, $h(3) = h(4) = 0$ and $h \circ \varphi = 2$. Also,

$$\mathcal{N}(E) = \langle (a, -a, b, -b) : a, b \in \mathbb{C} \rangle,$$

$$\mathcal{N}\left(E_\varphi\right) = \langle (a, b, -a, -b) : a, b \in \mathbb{C} \rangle.$$

Moreover, since $\mathcal{M}^\varphi = \emptyset$, then $\mathcal{B}$ is a $(\varphi_0)_0$-type subalgebra of Σ and $\mathfrak{L}_\varphi = L^2(\Sigma)$.

Proposition 2.9. *If $\mathcal{B}$ is $(\varphi_0)_0$-type, then $\mathcal{B}$ is $(\varphi_1)_0$-type.*

Proof. Let $u \in L^2(\mathcal{B})$ and $f := hE_\varphi\left(|u|^2\right) \circ \varphi^{-1} \in \mathfrak{M}\left(\mathcal{N}\left(EC_\varphi\right)\right)$. Define $L_f : \mathcal{N}\left(EC_\varphi\right) \to L^2(\Sigma)$ by $L_f(g) = fg$. Then by the closed graph theorem L_f is bounded. Let $M = \|L_f\|$. We show that $\|f\|_\infty \leq M$. For fixed $\varepsilon > 0$, put $C = \{|f| \geq M + \varepsilon\}$. If $\mu(C) > 0$, then by Lemma 2.5, $D_{\chi_c}^\varphi \neq 0$. So $G := D_{\chi_c}^\varphi(g) \neq 0$ for some $g \in L^2(\Sigma)$. Since $E(G) = 0$ and $\mathcal{N}(E) \subseteq \mathcal{N}\left(EC_\varphi\right)$, then $G \in \mathcal{N}\left(EC_\varphi\right) \subseteq L^2(\Sigma)$. In addition, $\mathcal{B}$-measurability of C implies that $G = \chi_C(g \circ \varphi - E(g \circ \varphi))$ and hence $S_G \subseteq C$. Therefore,

$$(M + \varepsilon)^2 \|G\|^2 \leq \int_{S_G} |f|^2 |G|^2 d\mu = \|L_f(G)\|^2 \leq M^2 \|G\|^2,$$

which is impossible. Hence C has measure zero. □

Theorem 2.10. *Let $\mathcal{B}$ be a $(\varphi_0)_0$-type sub-sigma algebra of Σ, $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$ and $u \in L^2(\Sigma)$. Then $D_u^\varphi \in B\big(L^2(\Sigma)\big)$ if and only if $J_1 := hE_\varphi\big(E\big(|u|^2\big)\big) \circ \varphi^{-1} \in L^\infty(\Sigma)$.*

Proof. Let $J_1 \in L^\infty(\Sigma)$. Since $|E(u)|^2 \leq E\big(|u|^2\big)$, it follows that $hE_\varphi\big(|E(u)|^2\big) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Then by [11, Theorem 2.1(i)], the operators $M_u EC_\varphi$ and $M_{E(u)}C_\varphi$ are bounded. But $D_u^\varphi = M_{E(u)}C_\varphi - M_u EC_\varphi$, and so D_u^φ is bounded.

Conversely, suppose D_u^φ is bounded on $L^2(\Sigma)$. If $f \in \mathcal{N}\big(EC_\varphi\big)$, then $D_u^\varphi(f) = M_{E(u)}C_\varphi(f) \in L^2(\Sigma)$. It follows that

$$\int_X hE_\varphi\big(|E(u)|^2\big) \circ \varphi^{-1} |f|^2 d\mu = \int_X |E(u)|^2 |f|^2 \circ \varphi \, d\mu < \infty,$$

and so $hE_\varphi\big(|E(u)|^2\big) \circ \varphi^{-1} \in \mathfrak{M}\big(\mathcal{N}\big(EC_\varphi\big)\big)$. As $\mathcal{B}$ is $(\varphi_0)_0$-type we have $hE_\varphi\big(|E(u)|^2\big) \circ \varphi^{-1} \in L^\infty(\mathcal{B})$ by Proposition 2.9. Now, we claim that $J_2 := hE_\varphi\big(E\big(|E(u) - u|^2\big)\big) \circ \varphi^{-1}$ is also in $L^\infty(\mathcal{B})$. To do this, let $f \in L^2(\mathcal{B})$. Since $\{J_2, f \circ \varphi\} \subset L^0(\mathcal{B})$, which follows from hypotheses, then $D_u^\varphi(f) = (E(u) - u) f \circ \varphi \in L^2(\Sigma)$ and so we have

$$\int_X J_2 |f|^2 \, d\mu = \int_X E\big(|E(u) - u|^2\big)\big(|f|^2 \circ \varphi\big) \, d\mu = \int_X |E(u) - u|^2 \big(|f|^2 \circ \varphi\big) \, d\mu < \infty.$$

Consequently, $\sqrt{J_2} \in \mathfrak{M}\big(L^2(\mathcal{B})\big)$ and hence $J_2 \in L^\infty(\mathcal{B})$. On the other hand, by the conditional variance formula $E\big(|E(u) - u|^2\big) = E\big(|u|^2\big) - |E(u)|^2$, it is easy to check that $J_2 = J_1 - hE_\varphi\big(|E(u)|^2\big) \circ \varphi^{-1}$. We then obtain that $J_1 \in L^\infty(\Sigma)$. □

In the first part of the proof of Theorem 2.10, we did not require that $\mathcal{B}$ be a $(\varphi_0)_0$-type sub-sigma algebra of Σ.

Recall that a closed subspace $M \subseteq \mathcal{H}$ is said to be invariant for an operator $T \in B(\mathcal{H})$ whenever $T(M) \subseteq M$. If M and its orthogonal complement $M^\perp$ are both invariant for T, then we say that M reduces T. The proof of the next lemma is left to the reader

Lemma 2.11. *Let $C_\varphi \in B\big(L^2(\Sigma)\big)$. Then the following are equivalent.*

(a) $L^2(\mathcal{B})$ *reduces* C_φ, *i.e.,* $EC_\varphi = C_\varphi E$.

(b) $\mathcal{N}(E) \subseteq \mathcal{N}\big(EC_\varphi\big)$ *and* $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$.

(c) $h \in L^2(\mathcal{B})$ *and* $EE_\varphi = E_\varphi E = E^{\varphi^{-1}(\mathcal{B})}$.

Proof. The equivalence of (a) and (c) can be found in [4, Theorem 5(b)]. For $(a) \Leftrightarrow (b)$, we notice that $C_\varphi(\mathcal{N}(E)) \subseteq \mathcal{N}(E)$ if and only if $\mathcal{N}(E) \subseteq \mathcal{N}\big(EC_\varphi\big)$, and $C_\varphi\big(L^2(\mathcal{B})\big) \subseteq L^2(\mathcal{B})$ if and only if $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. □

Corollary 2.12. *Let $L^2(\mathcal{B})$ be a reducing subspace of $C_\varphi \in B\big(L^2(\Sigma)\big)$. Then $\mathcal{B}$ is a $(\varphi_0)_0$-type sub-sigma algebra of Σ if and only if $\mu\big(B_0^\varphi\big) = 0$.*

From now on, we will make the following assumptions on $\mathcal{B}$ and φ.

- $\mathcal{B}$ is a $(\varphi_0)_0$-type sub-sigma algebra of Σ.
- $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$.

In this setting, $\mathcal{N}(E)$ and so $\mathcal{R}(E) = L^2(\mathcal{B})$ are reducing subspaces of C_φ. Relative to the direct sum decomposition $L^2(\Sigma) = L^2(\mathcal{B}) \oplus \mathcal{N}(E)$, the matrix form of each D_u^φ in $\mathfrak{D}_\varphi$ is

$$\begin{bmatrix} ED_u^\varphi E & ED_u^\varphi(I - E) \\ (I - E)D_u^\varphi E & (I - E)D_u^\varphi(I - E) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ M_{E(u)-u}C_\varphi & M_{E(u)}C_\varphi \end{bmatrix}.$$

Lemma 2.13. *There exists $\psi \in \mathcal{N}(E)$ with $0 < |\psi \circ \varphi| \leq 1$.*

Proof. Since every $(\varphi_0)_0$-type sub-sigma algebra of Σ is type-0, then the desired result follows from the Lambert–Weinstock lemma (see [21]). □

Example 2.14. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$, Σ be the Lebesgue sets and let $\mathcal{B}$ the sub-sigma algebra of Σ generated by the sets symmetric about the origin. Then for each $f \in L^2(\Sigma)$ we have $(Ef)(x) = (f(x) + f(-x))/2$. This example is due to Alan Lambert. Let $\varphi : X \to X$ defined by $\varphi(x) = \sqrt[3]{x}$. Then $h(x) = 3x^2 \in L^\infty(\Sigma)$, $\varphi^{-1}(\Sigma) = \Sigma$ (i.e., $E_\varphi = I$) and

$$EC_\varphi(f)(x) = \frac{f\left(\sqrt[3]{x}\right) + f\left(-\sqrt[3]{x}\right)}{2} = C_\varphi E(f)(x), \quad f \in L^2(\Sigma).$$

Thus, $L^2(\mathcal{B})$ is a reducing subspace for C_φ. Note that $\mathcal{M}^\varphi = \mathcal{M} = \{\emptyset\}$ and so $\mu\left(B_0^\varphi\right) = \mu\left(B_0\right) = 0$. It follows that $\mathcal{B}$ is a $(\varphi_0)_0$-type sub-sigma algebra of Σ. Let $u(x) = 1/\sqrt[4]{|x|}$. Then $\int_{-1}^{1} u^2(x)\,dx = 2$. Hence, $u \in L^2(\Sigma)$ but $E\left(u^2\right) = u^2 \notin L^\infty(\Sigma)$. It follows that $D_u \notin B\left(L^2(\Sigma)\right)$. However, it is easy to check that

$$h(x)E_\varphi\left(E\left(|u|^2\right)\right) \circ \varphi^{-1}(x) = \frac{3x^2}{\sqrt{|x|^3}} \in L^\infty(\Sigma),$$

and hence $D_u^\varphi \in B\left(L^2(\Sigma)\right)$.

Theorem 2.15. *$\mathfrak{D}_\varphi = \left\{D_u^\varphi : u \in \mathfrak{L}_\varphi\right\}$ is closed in the weak operator topology.*

Proof. Let $\{u_\alpha\} \subseteq \mathfrak{L}_\varphi$ such that $D_{u_\alpha}^\varphi \xrightarrow{w} D$ for some $D \in B\left(L^2(\Sigma)\right)$. We show that $D = D_{u_0}^\varphi$ for some $u_0 \in \mathfrak{L}_\varphi$. Recall that since $h \in L^\infty(\Sigma)$, $L^\infty(\Sigma) \subseteq \mathfrak{L}_\varphi \subseteq L^2(\Sigma) \subseteq L^1(\Sigma)$. If $f \in \mathcal{N}(EC_\varphi)$, then $E(u_\alpha)f \circ \varphi \xrightarrow{w} D(f)$. Choose ψ as in Lemma 2.13. Since $\psi \circ \varphi$ is essentially bounded, we get that $(\psi \circ \varphi)E(u_\alpha)(f \circ \varphi) \xrightarrow{w} (\psi \circ \varphi)D(f)$; hence $(f \circ \varphi)E(u_\alpha)(\psi \circ \varphi) \xrightarrow{w} (f \circ \varphi)D(\psi)$. Thus, $(f \circ \varphi)D(\psi) = (\psi \circ \varphi)D(f)$. Consequently, for each $f \in \mathcal{N}(E) \subseteq \mathcal{N}\left(EC_\varphi\right)$, $D(f) = u(f \circ \varphi)$, where $u = \frac{D(\psi)}{\psi \circ \varphi}$. Since $\mathcal{R}(D) \subseteq \mathcal{N}(E)$, then $D(-1) \in \mathcal{N}(E)$ and $u\mathcal{N}(E) \subseteq \mathcal{N}(E)$. Then, by [21, Theorem 2], $u \in L^\infty(\Sigma)$. We claim that u is $\mathcal{B}$-measurable. To prove this, we take $a = Re(a)$ and $b = Im(u)$. Since $u\,\mathcal{N}(E) \subseteq \mathcal{N}(E)$, then

$$a\mathcal{N}(E) = \frac{u + \bar{u}}{2}\mathcal{N}(E) \subseteq \mathcal{N}(E). \tag{2.1}$$

Fix any $\varepsilon > 0$. Set $A = \{x \in X : E(a)(x) \geq a(x) + \varepsilon\}$. Since $\chi_A - E(\chi_A) \in \mathcal{N}(E)$, then by (2.1), $a\left(\chi_A - E(\chi_A)\right) \in \mathcal{N}(E)$ and hence

$$E(a\chi_A) = E(a)E(\chi_A). \tag{2.2}$$

But $a\chi_A \leq (E(a) - \varepsilon)\chi_A$. After taking E and using (2.2) we obtain $E(\chi_A) \leq 0$. It follows that $E(\chi_A) = 0$ and so $\chi_A = 0$. Thus, $\mu(A) = 0$. Since ε was chosen arbitrarily, $E(a) \leq a$. Now, put $B = \{x \in X : E(a)(x) \leq a(x) + \varepsilon\}$. A similar argument shows that $\mu(B) = 0$, and so $E(a) \geq a$. Consequently, $E(a) = a$. The same reasoning applies to b, so that $E(u) = u$. Now let $u_0 = u + f_1$, where $f_1 = D(-1)$. We claim that $D = D_{u_0}^\varphi$. Indeed, suppose $T : \mathcal{D}_T = \left\{f \in L^2(\Sigma) : D_{u_0}^\varphi(f) \in L^2(\Sigma)\right\} \to L^2(\Sigma)$ is a linear transformation defined by $T(f) = D_{u_0}^\varphi(f)$. Since $E(u) = u$ and $E(f_1) = 0$, then

$$T(f) = u(f \circ \varphi - E(f \circ \varphi)) - E(f \circ \varphi)f_1, \quad f \in \mathcal{D}_T. \tag{2.3}$$

Note that $T(L^\infty(\Sigma)) \subseteq L^2(\Sigma)$. We show then T is closed. Let $f_n \to f$ and $T(f_n) \to K$ in L^2-norm. Since C_φ is bounded and E is a contraction, then $f_n \circ \varphi \to f \circ \varphi$ and $E(f_n \circ \varphi) \to E(f \circ \varphi)$ in L^2-norm. So we can choose a subsequence $\left\{f_{n_k}\right\} \subseteq \{f_n\}$ such that $f_{n_k} \circ \varphi \xrightarrow{a.e.} f \circ \varphi$, $E\left(f_{n_k} \circ \varphi\right) \xrightarrow{a.e.} E\left(f \circ \varphi\right)$ and $T\left(f_{n_k}\right) \xrightarrow{a.e.} K$. By the first two convergence we obtain $T\left(f_{n_k}\right) \xrightarrow{a.e.} T(f)$, and consequently $T(f) = K$.

Now, we show that T agrees with D on the dense set $L^\infty(\Sigma)$, and consequently $T = D$. Since $L^2(\Sigma) = \mathcal{N}(E) \oplus \mathcal{R}(E)$, it suffices to show that $T = D$ on $L^\infty(\Sigma) \cap \mathcal{N}(E)$ and $L^\infty(\Sigma) \cap \mathcal{R}(E)$, respectively. To get these, we first let $f \in L^\infty(\Sigma) \cap \mathcal{N}(E)$. Then by (2.3) we have $T(f) = u(f \circ \varphi) = D(f)$. Now let $f \in L^\infty(\Sigma) \cap \mathcal{R}(E)$. Since $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$, then $E(f \circ \varphi) = f \circ \varphi$, and

$$\begin{aligned} T(f) &= -E(f \circ \varphi)f_1 = -(f \circ \varphi)D(-1) = -\left(\lim_w D_{u_\alpha}^\varphi(-1)\right)(f \circ \varphi) \\ &= \lim_w (E(u_\alpha) - u_\alpha)(f \circ \varphi) = \lim_w D_{u_\alpha}^\varphi(f) = D(f). \end{aligned}$$

This completes the proof. □

Take $\varphi = id$, be the identity map. For $u \in \mathcal{D}(E)$, set $u = u_1 + u_2$, where $u_1 = E(u)$ and $u_2 = u - E(u)$. If $u \in \mathfrak{L}$, we have the following matrix representation of D_u and D_u^* with respect to decomposition $L^2(\Sigma) = L^2(\mathcal{B}) \oplus \mathcal{N}(E)$:

$$D_u = \begin{bmatrix} 0 & 0 \\ -M_{u_2} & M_{u_1} \end{bmatrix} \quad \text{and} \quad D_u^* = \begin{bmatrix} 0 & -M_{\overline{u_2}} \\ 0 & M_{\overline{u_1}} \end{bmatrix}.$$

Then, $D_u = D_u^*$ if and only if $\overline{u_1} = u_1$ and $u_2 = 0$. Thus, D_u is self-adjoint if and only if u is a real-valued member of $L^\infty(\mathcal{B})$. Also, since

$$D_u D_u^* = \begin{bmatrix} 0 & 0 \\ 0 & M_{|u_1|^2+|u_2|^2} \end{bmatrix} \quad \text{and} \quad D_u^* D_u = \begin{bmatrix} M_{|u_2|^2} & M_{-u_1\overline{u_2}} \\ M_{-\overline{u_1}u_2} & M_{|u_1|^2} \end{bmatrix},$$

so $D_u D_u^* = D_u^* D_u$ if and only if $u_2 = 0$. Thus, D_u is normal if and only if $u \in L^\infty(\mathcal{B})$. Moreover, $D_u D_u^* = D_u$ if and only if $u_2 = 0$ and $0 \le u_1 = u_1^2$. It follows that D_u is an orthogonal projection if and only if $u = \chi_B$ for some $B \in \mathcal{B}$. These results were obtained in [21] by another way.

In the following, Theorem 2.20, we give and prove a criterion for the normality of D_u^φ on $L^2(\Sigma)$. Here the matrix representation method seems to be a bit complicated. To prove this result, we need the following proposition.

Proposition 2.16. *Let $D_u^\varphi \in B(L^2(\Sigma))$ and $f \in L^2(\Sigma)$. Then the following assertions hold.*

(a) $(D_u^\varphi)^*(f) = hE_\varphi(\overline{E(u)}f) \circ \varphi^{-1} - hE_\varphi(E(\bar{u}f)) \circ \varphi^{-1}$.
(b) $(D_u^\varphi)^*(L^2(\mathcal{B})) = \{0\}$.
(c) $u \in L^2(\mathcal{B})$ *if and only if* $D_u^\varphi(1) = 0$.
(d) *If* D_u^φ *is normal, then* $u \in L^2(\mathcal{B})$.
(e) $D_u^\varphi(D_u^\varphi)^*(f) = E(u)(h \circ \varphi)E_\varphi(\overline{E(u)}f - E(\bar{u}f)) - uE(h \circ \varphi E_\varphi(\overline{E(u)}f - E(\bar{u}f)))$.
(f) $(D_u^\varphi)^* D_u^\varphi(f) = hE_\varphi(|E(u)|^2(f \circ \varphi) - uE(\bar{u})E(f \circ \varphi) - E(\bar{u}f \circ \varphi)E(u) + E(|u|^2)E(f \circ \varphi)) \circ \varphi^{-1}$.

Proof. We recall that $C_\varphi^*(f) = hE_\varphi(f) \circ \varphi^{-1}$, for all $f \in L^2(\Sigma)$. To prove (d), let $(D_u^\varphi)^* D_u^\varphi = D_u^\varphi(D_u^\varphi)^*$. Since $1 \in \mathcal{N}((D_u^\varphi)^*)$, then $1 \in \mathcal{N}((D_u^\varphi)^* D_u^\varphi)$ and so $1 \in \mathcal{N}(D_u^\varphi)$. Consequently, by (c), $u \in L^2(\mathcal{B})$. The remainder of the proof is left to the reader. □

Corollary 2.17. *Let $u \in L^2(\mathcal{B})$. Then for each $f \in L^2(\Sigma)$ we have*

(a) $D_u^\varphi(D_u^\varphi)^*(f) = u(h \circ \varphi)E_\varphi(\bar{u}f - \bar{u}E(f))$.
(b) $(D_u^\varphi)^* D_u^\varphi(f) = hfE_\varphi(|u|^2) \circ \varphi^{-1} - hE_\varphi(|u|^2 E(f \circ \varphi)) \circ \varphi^{-1}$.
(c) *If* $f \in L^2(\mathcal{B})$, *then* $D_u^\varphi(D_u^\varphi)^*(f) = (D_u^\varphi)^* D_u^\varphi(f) = 0$.

Theorem 2.18. *Let $\mathcal{B}$ be a $(\varphi_0)_0$-type sub-sigma algebra of Σ. Then D_u^φ is normal on $L^2(\Sigma)$ if and only if the following three conditions hold.*

(i) $u \in L^2(\mathcal{B})$,
(ii) $hE_\varphi(|u|^2) \circ \varphi^{-1} = u(h \circ \varphi)E_\varphi(\bar{u}) \in L^\infty(\mathcal{B})$,
(iii) $M_{uE_\varphi(\bar{u})} = M_u E_\varphi M_{\bar{u}}$ *on* $\mathcal{N}(E)$.

Proof. Let D_u^φ be normal. Then by Proposition 2.16(d), $u \in L^2(\mathcal{B})$. Also by Corollary 2.17(c), $D_u^\varphi(D_u^\varphi)^*$ agree with $(D_u^\varphi)^* D_u^\varphi$ on $L^2(\mathcal{B})$. But for every $f \in \mathcal{N}(E)$ we have

$$(D_u^\varphi)^* D_u^\varphi(f) = hE_\varphi(|u|^2) \circ \varphi^{-1} f = u(h \circ \varphi)E_\varphi(\bar{u}f) = D_u^\varphi(D_u^\varphi)^*(f). \tag{2.4}$$

Invoking Lemma 2.13, we choose the function $\psi \in \mathcal{N}(E)$ with $0 < |\psi \circ \varphi| \le 1$. Then we have

$$h(\psi \circ \varphi)E_\varphi(|u|^2) \circ \varphi^{-1} = u(h \circ \varphi)E_\varphi(\bar{u}(\psi \circ \varphi)),$$

and so, $hE_\varphi(|u|^2)\circ\varphi^{-1} = u(h\circ\varphi)E_\varphi(\bar{u})$. Note that $hE_\varphi(|u|^2)\circ\varphi^{-1}\in L^\infty(\mathcal{B})$. So condition (ii) holds. Now, by condition (ii) and (2.4) we obtain $u(h\circ\varphi)fE_\varphi(\bar{u}) = u(h\circ\varphi)E_\varphi(\bar{u}f)$ for all $f\in\mathcal{N}(E)$. Since $S_{h\circ\varphi} = X$, then $ufE_\varphi(\bar{u}) = uE_\varphi(\bar{u}f)$ and therefore condition (iii) is also holds.

Conversely, let the conditions (i), (ii) and (iii) hold. Then by (2.4) we have

$$\left(D_u^\varphi\right)^* D_u^\varphi(f) = hE_\varphi\left(|u|^2\right)\circ\varphi^{-1} f \overset{(ii)}{=} u(h\circ\varphi)E_\varphi(\bar{u})f \overset{(iii)}{=} u(h\circ\varphi)E_\varphi(\bar{u}f) = D_u^\varphi\left(D_u^\varphi\right)^*(f)$$

for all $f\in\mathcal{N}(E)$. Thus $\left(D_u^\varphi\right)^* D_u^\varphi$ agree with $D_u^\varphi\left(D_u^\varphi\right)^*$ on $\mathcal{N}(E)$. Now, the desired conclusion follows from Corollary 2.17(c). □

Put $\mathcal{K}_2 = \left\{u\in L^2(\Sigma) : uL^2(\mathcal{B})\subseteq L^2(\Sigma)\right\}$. Then $\mathcal{K}_2$ is a vector space which contains $L^\infty(\Sigma)$. Indeed, if $\mathcal{B}=\Sigma$, then $\mathcal{K}_2 = L^\infty(\Sigma)$. For $u\in\mathcal{K}_2$, let $M_u : L^2(\mathcal{B})\to L^2(\Sigma)$ be the corresponding multiplication operator. Lembert [19] proved that EM_u is bounded if and only if $E\left(|u|^2\right)\in L^\infty(\mathcal{B})$ and in this case $\|M_u\| = \sqrt{\|E(|u|^2)\|_\infty}$. In general, for $u\in L^0(\Sigma)$, M_u is closed and densely defined on its natural domain $\left\{f\in L^2(\Sigma) : uf\in L^2(\Sigma)\right\}$ (see [1]). We now examine the commutant $\mathfrak{D}'_\varphi = \left\{T\in B\left(L^2(\Sigma)\right) : D_u^\varphi T = TD_u^\varphi \text{ for all } u\in\mathfrak{L}_\varphi\right\}$ of $\mathfrak{D}_\varphi$.

Lemma 2.19. *Let $f\in L^\infty(\Sigma)$. Then*

(i) $M_f\in\{C_\varphi\}'$ if and only if $f\circ\varphi = f$.
(ii) $M_f\in\mathfrak{D}'_\varphi$ if and only if $f\circ\varphi = f$ and $f\in L^\infty(\mathcal{B})$.

Proof. Part (i) is obvious. To prove (ii), let $M_f\in\mathfrak{D}'_\varphi$. Then for each $g\in L^2(\Sigma)$, $D_u^\varphi M_f(g) = M_f D_u^\varphi(g)$, so that

$$E(u)(f\circ\varphi)(g\circ\varphi) - uE((f\circ\varphi)(g\circ\varphi)) = f(E(u)g\circ\varphi - E((g\circ\varphi)u). \tag{2.5}$$

Letting $u=1$ in (2.5), we have

$$(f\circ\varphi)(g\circ\varphi) - E((f\circ\varphi)(g\circ\varphi)) = f(g\circ\varphi - E(g\circ\varphi)). \tag{2.6}$$

Take $g=1$ in (2.6). Then $E(f\circ\varphi) = f\circ\varphi$ and hence $f\circ\varphi$ is a $\mathcal{B}$-measurable function. So, from (2.6) we obtain

$$(f\circ\varphi)(g\circ\varphi - E(g\circ\varphi) = f(g\circ\varphi - E(g\circ\varphi)). \tag{2.7}$$

By Lemma 2.13, Choose $\psi\in\mathcal{N}(E)$ with $0<|\psi\circ\varphi|\le 1$. By hypothesis, $EC_\varphi = C_\varphi E$. So $\psi\circ\varphi\in\mathcal{N}(E)$. Replacing g by ψ, we can rewrite (2.7) as $f\circ\varphi = f$. Thus $f\in L^\infty(\mathcal{B})$. The converse is obvious. □

Lemma 2.20. *Let $T\in B\left(L^2(\Sigma)\right)$ and let for each $u\in\mathfrak{L}_\varphi$, $D_u^\varphi T = TD_u^\varphi$. Then the following assertions hold.*

(i) There exists $f\in L^2(\mathcal{B})$ such that $T = M_f$ on $\mathcal{N}(E)\cap\mathfrak{L}_\varphi$.
(ii) If $h>0$ then for each $g\in L^\infty(\mathcal{B})$, $T(g) = M_f(g)$.

Proof. (i) Let $T\in\mathfrak{D}'_\varphi$. Then for each $g\in L^2(\Sigma)$,

$$T(E(u)g\circ\varphi - uE(g\circ\varphi)) = E(u)(T(g)\circ\varphi) - uE(T(g)\circ\varphi). \tag{2.8}$$

Letting $u=1$ in (2.8), we have

$$T(g\circ\varphi - E(g\circ\varphi)) = T(g)\circ\varphi - E(T(g)\circ\varphi).$$

This shows that $T(g)\circ\varphi\in L^2(\mathcal{B})$, for all $g\in L^2(\mathcal{B})$. Let $f = T(1)\circ\varphi$. Then $f\in L^2(\mathcal{B})$. We show that $T = M_f$ on $\mathcal{N}(E)\cap\mathfrak{L}_\varphi$. Take $g=1$ in (2.8). Then for each $u\in\mathfrak{L}_\varphi$, $T(E(u)-u) = M_f(E(u)-u)$. Consequently, $T(v) = M_f(v)$ for each $v\in\mathcal{N}(E)\cap\mathfrak{L}_\varphi$.

(ii) Choose $\psi\in\mathcal{N}(E)$ with $0<|\psi\circ\varphi|\le 1$ and let $g\in L^\infty(\mathcal{B})$. Then $D_{\psi\circ\varphi}^\varphi(g) = -(g\circ\varphi)(\psi\circ\varphi) = D_{g\circ\varphi}^\varphi(\psi)$. From $T\in\mathfrak{D}'_\varphi$ and (i), we have $T(\psi) = f\psi$. Then

$$D_{\psi\circ\varphi}^\varphi(T(g)) = T\left(D_{\psi\circ\varphi}^\varphi(g)\right) = -T\left(D_{g\circ\varphi}^\varphi(\psi)\right) = -D_{g\circ\varphi}^\varphi(T(\psi)) = -D_{g\circ\varphi}^\varphi(f\psi).$$

On the other hand, since $E(\psi\circ\varphi)=0$, $T(g)\circ\varphi\in L^2(\mathcal{B})$ and $E(g\circ\varphi)=g\circ\varphi$, we have

$$D^{\varphi}_{\psi\circ\varphi}(T(g))=-E(T(g)\circ\varphi)(\psi\circ\varphi)=-(T(g)\circ\varphi)(\psi\circ\varphi)$$

and $-D^{\varphi}_{g\circ\varphi}(f\psi)=-(f\circ\varphi)(g\circ\varphi)(\psi\circ\varphi)$. Since $\psi\circ\varphi\neq 0$, then we obtain

$$T(g)\circ\varphi=(f\circ\varphi)(g\circ\varphi),\quad g\in L^\infty(\mathcal{B}). \tag{2.9}$$

Set $g=1$ in (2.7). Then $f=T(1)\circ\varphi=f\circ\varphi$. So we can rewrite 2.9 as $C_\varphi T(g)=T(g)\circ\varphi=f(g\circ\varphi)=C_\varphi M_f(g)$, for all $g\in L^\infty(\mathcal{B})$. Now, if C_φ is one-to-one; equivalently $h>0$, then $T(g)=M_f(g)$ for all $g\in L^\infty(\mathcal{B})$. But $L^2(\Sigma)=L^2(\mathcal{B})\oplus\mathcal{N}(E)$. Thus, by part (i), $T(g)=M_f(g)$ for all $g\in L^\infty(\Sigma)$. Since $L^\infty(\Sigma)\subseteq\mathfrak{L}_\varphi$, $L^\infty(\Sigma)$ is dense in $L^2(\Sigma)$ and M_f is closed, then $T=M_f$. □

By Lemma 2.19 and 2.20 we have the following theorem.

Theorem 2.21. *Let $h>0$. Then $\mathfrak{D}'_\varphi=\{M_u : u\circ\varphi=u,\ u\in L^\infty(\mathcal{B})\}$.*

In [21], Lambert and Weinstock considered the set $\mathcal{K}_2=\{u\in L^2(\Sigma) : E(|u|^2)\in L^\infty(\Sigma)\}$ and they asked the following question:

For which subalgebras $\mathcal{B}$ does $\mathcal{K}_2=L^\infty(\Sigma)$?

In [19], Lambert prove that M_uE is bounded on $L^2(\Sigma)$ if and only if $E(|u|^2)\in L^\infty(\mathcal{B})$. Let $u\in\mathcal{K}_2$. Since $|E(u)|^2\le E(|u|^2)$, the boundedness of M_uE implies that $E(u)\in L^\infty(\mathcal{B})$, and so $M_{E(u)}$ is bounded. Consequently, in this case, $D_u=M_{E(u)}-M_uE$ is bounded. On the other hand, if $\mathcal{B}$ is a type-0 sub-σ-finite algebra of Σ and D_u is bounded on $L^2(\Sigma)$, then $E(|u|^2)\in L^\infty(\mathcal{B})$. Thus, $\mathcal{K}_2=\{u\in L^2(\Sigma) : D_u\in B(L^2(\Sigma))\}$. Hence, by [19, Proposition 2.2] we obtain the following proposition.

Proposition 2.22. *Let $\mathcal{B}$ is a type-0 sub-σ algebra of Σ and let $\mathfrak{L}=\{u\in L^2(\Sigma) : D_u\in B(L^2(\Sigma))\}$. Then the following assertions hold.*

(a) $\mathfrak{L}=\mathcal{K}_2$.
(b) $\mathfrak{L}=L^2(\Sigma)$ *if and only if $\mathcal{B}$ is generated by a finite partition of X.*
(c) $\mathfrak{L}=L^\infty(\Sigma)$ *if and only if there is a constant C so that every $f\in L^1(\Sigma)$, $|f|\le CE^{\mathcal{B}}(|f|)$.*

Lemma 2.23. *Let $\mathcal{B}$ be a $(\varphi_0)_0$-type sub-sigma algebra of Σ. Then the following assertions hold.*

(a) If $h\circ\varphi$ is bounded away from zero then for each $f\in L^2(\Sigma)$, $h(E^{\varphi^{-1}(\mathcal{B})}(f))\circ\varphi^{-1}\in L^\infty(\Sigma)$ if and only if $E^{\varphi^{-1}(\mathcal{B})}(f)\in L^\infty(\Sigma)$.
(b) If $h\circ\varphi$ is bounded away from zero, then $\mathfrak{L}_\varphi=\{u\in L^2(\Sigma) : E^{\varphi^{-1}(\mathcal{B})}(|u|^2)\in L^\infty(\Sigma)\}$.

Proof.

(a) Let $f\in L^2(\Sigma)$ and let $g:=h(E^{\varphi^{-1}(\mathcal{B})}(f))\circ\varphi^{-1}\in L^\infty(\Sigma)$. Since $L^\infty(\Sigma)$ reduces C_φ, then $E^{\varphi^{-1}(\mathcal{B})}(f)=\frac{C_\varphi(g)}{h\circ\varphi}\in L^\infty(\Sigma)$. Conversely, let $E^{\varphi^{-1}(\mathcal{B})}(f)\in L^\infty(\Sigma)$. Then $g=C^*_\varphi(E^{\varphi^{-1}(\mathcal{B})}(f))\in L^\infty(\Sigma)$.
(b) By Theorem 2.10, $\mathfrak{L}_\varphi=\{u\in L^2(\Sigma) : hE_\varphi(E(|u|^2))\circ\varphi^{-1}\in L^\infty(\Sigma)\}$. Now, the desired conclusion follows from (a) and Lemma 2.11. □

Corollary 2.24. *Let $\mathcal{B}$ be a $(\varphi_0)_0$-type sub-sigma algebra of Σ. Then the following assertions hold.*

(a) $\mathfrak{L}_\varphi=L^2(\Sigma)$ *if and only if $\varphi^{-1}(\mathcal{B})$ is generated by a finite partition of X.*
(b) $\mathfrak{L}_\varphi=L^\infty(\Sigma)$ *if and only if there is a constant C so that every $f\in L^1(\Sigma)$, $|f|\le CE^{\varphi^{-1}(\mathcal{B})}(|f|)$.*

Let $f,g\in L^2(\Sigma)$. Then by conditional Cauchy–Schwarz inequality we have

$$E(|f+g|^2)\le E(|f|^2)+E(|g|^2)+2(E(|f|^2))^{\frac12}(E(|g|^2))^{\frac12}=\left(\sqrt{E(|f|^2)}+\sqrt{E(|g|^2)}\right)^2.$$

Thus, we have the following corollary.

Corollary 2.25. *Let $f, g \in L^2(\Sigma)$. Then*

$$\sqrt{hE_\varphi\big(E\big(|f+g|^2\big)\big)\circ\varphi^{-1}} \le \sqrt{hE_\varphi\big(E\big(|f|^2\big)\big)\circ\varphi^{-1}} + \sqrt{hE_\varphi\big(E\big(|g|^2\big)\big)\circ\varphi^{-1}}.$$

We recall that $\mathfrak{L}_\varphi = \big\{u \in L^2(\Sigma) : hE_\varphi\big(E\big(|u|^2\big)\big)\circ\varphi^{-1} \in L^\infty(\Sigma)\big\}$ is a closed subspace of $L^2(\Sigma)$. For $u \in \mathfrak{L}_\varphi$, define

$$\|u\|_{\mathfrak{L}_\varphi} = \sqrt{\left\|hE_\varphi\big(E\big(|u|^2\big)\big)\circ\varphi^{-1}\right\|_\infty} = \sqrt{\left\|C_\varphi^*\big(E\big(|u|^2\big)\big)\right\|_\infty}.$$

Let $\|u\|_{\mathfrak{L}_\varphi} = 0$. Then $C_\varphi^*\big(E\big(|u|^2\big)\big) = 0$, and so $(h\circ\varphi)E_\varphi\big(E\big(|u|^2\big)\big) = C_\varphi C_\varphi^*\big(E\big(|u|^2\big)\big) = 0$. Since $h\circ\varphi > 0$ and $E_\varphi E = E^{\varphi^{-1}(\mathcal{B})}$, then $E^{\varphi^{-1}(\mathcal{B})}\big(|u|^2\big) = 0$ and thus $u = 0$, because $\mathcal{N}\big(E^{\varphi^{-1}(\Sigma)}\big)$ contains no positive element. Triangle inequality follows from Corollary 2.25.

Hence $\big(\mathfrak{L}_\varphi, \|\cdot\|_{\mathfrak{L}_\varphi}\big)$ is a normed space.

Lemma 2.26. *Let $u \in \mathfrak{L}_\varphi$. Then $\|u\|_2 \le \sqrt{\mu(X)}\|u\|_{\mathfrak{L}_\varphi}$.*

Proof.

$$\|u\|_2^2 = \int_X |u|^2 d\mu = \int_X hE_\varphi\big(E\big(|u|^2\big)\big)\circ\varphi^{-1} d\mu \le \int_X \left\|hE_\varphi\big(E\big(|u|^2\big)\big)\circ\varphi^{-1}\right\|_\infty d\mu = \mu(X)\|u\|_{\mathfrak{L}_\varphi}^2.$$

□

Proposition 2.27. $\big(\mathfrak{L}_\varphi, \|\cdot\|_{\mathfrak{L}_\varphi}\big)$ *is a Banach space.*

Proof. Let $\{u_n\} \subseteq \mathfrak{L}_\varphi$ be a Cauchy sequence. Then by Lemma 2.26 there is an element $u \in L^2(\Sigma)$ such that $\|u_n - u\|_2 \to 0$. Also, for $\varepsilon > 0$ fixed, we can find an $n_0 \in \mathbb{N}$ such that $hE_\varphi\big(E\big(|u_{n_0} - u_n|^2\big)\big)\circ\varphi^{-1} < \varepsilon$, for all $n \ge n_0$. Let

$$S = \Big\{x \in X : h(x)E_\varphi\Big(E\Big(|u_{n_0} - u|^2\Big)\Big)\circ\varphi^{-1}(x) \ge 2\varepsilon\Big\}.$$

Since $L^2(\mathcal{B})$ reduces C_φ^*, then $S \in \mathcal{B}$ and so $\varphi^{-1}(S) \in \varphi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. It follows that

$$\begin{aligned}
\varepsilon\mu(S) &\ge \int_S hE_\varphi\Big(E\Big(|u_{n_0} - u_n|^2\Big)\Big)\circ\varphi^{-1} d\mu \\
&= \int_{\varphi^{-1}(S)} E\Big(|u_{n_0} - u_n|^2\Big)\, d\mu = \int_{\varphi^{-1}(S)} |u_{n_0} - u_n|^2 d\mu \\
&= \Big\|\Big(u_{n_0} - u_n\Big)\chi_{\varphi^{-1}(S)}\Big\|_2^2 \to \Big\|\Big(u_{n_0} - u\Big)\chi_{\varphi^{-1}(S)}\Big\|_2^2 \quad (\text{as } n \to \infty) \\
&= \int_{\varphi^{-1}(S)} |u_{n_0} - u|^2 d\mu = \int_{\varphi^{-1}(S)} E\Big(|u_{n_0} - u|^2\Big)\, d\mu \\
&= \int_S hE_\varphi\Big(E\Big(|u_{n_0} - u|^2\Big)\Big)\circ\varphi^{-1} d\mu \ge 2\varepsilon\mu(S),
\end{aligned}$$

which is impossible. Thus, $\mu(S) = 0$ and so $hE_\varphi\big(E\big(|u_{n_0} - u|^2\big)\big)\circ\varphi^{-1} \in L^\infty(\mathcal{B})$. So $u_{n_0} - u \in \mathfrak{L}_\varphi$ and thus $u \in \mathfrak{L}_\varphi$. Moreover, we obtain $hE_\varphi\big(E\big(|u_{n_0} - u|^2\big)\big)\circ\varphi^{-1} < \varepsilon$ for some $u_{n_0} \in \mathfrak{L}_\varphi$ and consequently, $\|u_{n_0} - u\|_{\mathfrak{L}_\varphi} \le \sqrt{\varepsilon}$. This completes the proof. □

A Lie algebra over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a vector space $\mathfrak{L}$ over $\mathbb{F}$ endowed with a Lie product $\mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}$, $(f, g) \to [f, g]$ which is bilinear, $[f, g] = -[g, f]$ and satisfies the Jacobi identity, i.e.

$$[f, [g, k]] + [g, [k, f]] + [k, [f, g]] = 0,$$

for every $f, g, k \in \mathfrak{L}$. A Banach–Lie algebra is a normed algebra $\big(\mathfrak{L}, \|\cdot\|_\mathfrak{L}\big)$ that satisfies $\|[f, g]\|_\mathfrak{L} \le C\|f\|_\mathfrak{L}\|g\|_\mathfrak{L}$ for a positive $C > 0$ and all f, g in $\mathfrak{L}$.

Now, we define [,] on $\mathfrak{L}_\varphi \times \mathfrak{L}_\varphi$ by

$$[f,g] = D^{\varphi}_{E_\varphi(f)}\left(\sqrt{h}E_\varphi(g)\circ\varphi^{-1}\right) = E(E_\varphi(f))\left(\sqrt{h\circ\varphi}E_\varphi(g)\right) - E_\varphi(f)E\left(\sqrt{h\circ\varphi}E_\varphi(g)\right).$$

For simplicity, set $E_\varphi(f) = f_\varphi$. Since $L^2(\mathcal{B})$ reduces C_φ, then h is $\mathcal{B}$-measurable ([4, Theorem 5(b)]) and so $h\circ\varphi \in L^2\left(\varphi^{-1}(\mathcal{B})\right) \subseteq L^2(\mathcal{B})$. Thus

$$[f,g] = \sqrt{h\circ\varphi}\,\left(E\left(f_\varphi\right)g_\varphi - E\left(g_\varphi\right)f_\varphi\right) = \sqrt{h\circ\varphi}\,D_{f_\varphi}\left(g_\varphi\right).$$

Note that $\left(f_\varphi\right)_\varphi = f_\varphi$ and $[f,g]_\varphi = [f,g]$, because $[f,g]$ is $\varphi^{-1}(\Sigma)$-measurable (Corollary 2.11(c)). Moreover,

$$[f,g] = -\sqrt{h\circ\varphi}\,D_{g_\varphi}\left(f_\varphi\right) = -[g,f]$$

and $E([f,g]) = 0$, because $E\left(D^{\varphi}_u\right) = 0$. It follows that

$$\begin{aligned}[f,[g,k]] &= \sqrt{h\circ\varphi}\,\left(E\left(f_\varphi\right)[g,k]_\varphi - E([g,k]_\varphi)f_\varphi\right)\\ &= \sqrt{h\circ\varphi}\,\left(E\left(f_\varphi\right)\left(\sqrt{h\circ\varphi}\left(E\left(g_\varphi\right)k_\varphi - E\left(k_\varphi\right)g_\varphi\right)\right)\right)\\ &= h\circ\varphi\,\left(E\left(f_\varphi\right)E\left(g_\varphi\right)k_\varphi - E\left(f_\varphi\right)E\left(k_\varphi\right)g_\varphi\right),\end{aligned}$$

for every $f,g,k \in \mathfrak{L}_\varphi$. Hence the Jacobi identity holds on $\mathfrak{L}_\varphi$, and so $\left(\mathfrak{L}_\varphi, \|\cdot\|_{\mathfrak{L}_\varphi}\right)$ is a complete Lie algebra.

Lemma 2.28. *Let $f,g \in \mathfrak{L}_\varphi$. Then the following assertions hold.*

(a) $f_\varphi \in \mathfrak{L}_\varphi$.
(b) $E\left(|[f,g]|^2\right) \le 4(h\circ\varphi)E_\varphi\left(E\left(|f|^2\right)\right)E_\varphi\left(E\left(|g|^2\right)\right)$.
(c) $\|[f,g]\|_{\mathfrak{L}_\varphi} \le 2\|f\|_{\mathfrak{L}_\varphi}\|g\|_{\mathfrak{L}_\varphi}$.

Proof. (a) Let $f \in \mathfrak{L}_\varphi$. Since $EE_\varphi = E_\varphi E$, then $|f_\varphi|^2 = |E_\varphi(f)|^2 \le E_\varphi\left(|f|^2\right)$ and so $E\left(|f_\varphi|^2\right) \le EE_\varphi\left(|f|^2\right) = E_\varphi E\left(|f|^2\right)$. It follows that

$$hE_\varphi\left(E\left(|f_\varphi|^2\right)\right)\circ\varphi^{-1} \le hE_\varphi\left(E\left(|f|^2\right)\right)\circ\varphi^{-1} \in L^\infty(\mathcal{B}).$$

(b) For $f,g \in \mathfrak{L}_\varphi$ we have $|[f,g]|^2 = \left|\sqrt{h\circ\varphi}\,\left(E\left(f_\varphi\right)g_\varphi - E\left(g_\varphi\right)f_\varphi\right)\right|^2$. Then

$$|[f,g]|^2 \le (h\circ\varphi)\left(|E(f_\varphi)g_\varphi|^2 + |E(g_\varphi)f_\varphi|^2 + 2|E(f_\varphi)E(g_\varphi)f_\varphi g_\varphi|\right).$$

Using $EE_\varphi = E_\varphi E$, conditional Cauchy–Schwarz inequality and taking conditional expectation E of both sides the above equation, gives

$$E\left(|[f,g]|^2\right) \le 4(h\circ\varphi)E\left(|f_\varphi|^2\right)E\left(|g_\varphi|^2\right) \le 4(h\circ\varphi)E_\varphi\left(E\left(|f|^2\right)\right)E_\varphi\left(E\left(|g|^2\right)\right).$$

(c) Using (b) and $\mathcal{B}$-measurability of h, we obtain

$$\begin{aligned}hE_\varphi\left(E\left(|[f,g]|^2\right)\right)\circ\varphi^{-1} &\le 4h\left(E_\varphi\left(h\circ\varphi E_\varphi\left(E\left(|f|^2\right)\right)E_\varphi\left(E\left(|g|^2\right)\right)\right)\right)\circ\varphi^{-1}\\ &= 4h^2\left(E_\varphi\left(E\left(|f|^2\right)\right)\circ\varphi^{-1}\right)\left(E_\varphi\left(E\left(|g|^2\right)\right)\circ\varphi^{-1}\right)\\ &\le 4\|f\|^2_{\mathfrak{L}_\varphi}\|g\|^2_{\mathfrak{L}_\varphi}.\end{aligned}$$

This completes the proof. □

Corollary 2.29. $\left(\mathfrak{L}_\varphi, \|\cdot\|_{\mathfrak{L}_\varphi}, [\ ,\ ]\right)$ *is a Banach–Lie algebra.*

Example 2.30. Let $g = (g_1,g_2,g_3,g_4) \in \mathbb{C}^4$. Under the hypotheses of Example 2.8(ii) we have $\|h\|_\infty = 2$ and

$$E_\varphi(g) = \frac{1}{2}(g_1+g_3, g_2+g_4, g_1+g_3, g_2+g_4);$$

$$\sqrt{h}E_\varphi(g)\circ\varphi^{-1} = \frac{\sqrt{2}}{2}(g_1+g_3, g_2+g_4, , 0, 0);$$

$$[f,g] = D^{\varphi}_{E_\varphi(f)}\left(\sqrt{h}E_\varphi(g)\circ\varphi^{-1}\right) = \frac{\sqrt{2}}{8}(a,-a,a,-a),$$

where $f=(f_1,f_2,f_3,f_4)\in\mathbb{C}^4$ and

$$a = \det\begin{bmatrix} g_1+g_3 & g_2+g_4 \\ f_1+f_3 & f_2+f_4 \end{bmatrix}.$$

Also, a direct computation shows that

$$E\left(|f|^2\right) = \frac{1}{2}\left(|f_1|^2+|f_2|^2, |f_1|^2+|f_2|^2, |f_3|^2+|f_4|^2, |f_3|^2+|f_4|^2\right);$$

$$hE_\varphi\left(E\left(|f|^2\right)\right)\circ\varphi^{-1} = \frac{1}{2}\left(|f_1|^2+|f_2|^2+|f_3|^2+|f_4|^2, |f_1|^2+|f_2|^2+|f_3|^2+|f_4|^2, 0, 0\right);$$

$$\|f\|_{\mathfrak{L}_\varphi} = \sqrt{\|hE_\varphi(E(|f|^2))\circ\varphi^{-1}\|_\infty} = \sqrt{\frac{1}{2}\left(|f_1|^2+|f_2|^2+|f_3|^2+|f_4|^2\right)}.$$

Similarly, we have $|[f,g]|^2 = \frac{1}{32}\left(a^2,a^2,a^2,a^2\right)$ and

$$\sqrt{hE_\varphi\left(E\left(|[f,g]|^2\right)\right)\circ\varphi^{-1}} = \sqrt{\frac{1}{16}(a^2,a^2,0,0)} = \frac{1}{4}(|a|,|a|,0,0).$$

Thus $\|[f,g]\|_{\mathfrak{L}_\varphi} = \frac{|a|}{4}$. Then by Cauchy–Schwarz inequality we have

$$\begin{aligned}\|[f,g]\|_{\mathfrak{L}_\varphi} &\le \frac{1}{4}\left\{|f_2g_1+f_4g_3-f_1g_2-f_3g_4| + |f_4g_1+f_2g_3-f_3g_2-f_1g_4|\right\} \\ &\le \frac{1}{2}\left(\sqrt{|f_1|^2+|f_2|^2+|f_3|^2+|f_4|^2}\right)\left(\sqrt{|g_1|^2+|g_2|^2+|g_3|^2+|g_4|^2}\right) \\ &= \|f\|_{\mathfrak{L}_\varphi}\|g\|_{\mathfrak{L}_\varphi}.\end{aligned}$$

Thus, the inequality in Lemma 2.30(c) is not sharp.

ACKNOWLEDGEMENT

We would like to thank the referees for their careful reading and constructive comments.

REFERENCES

[1] W. Arveson, *A density theorem for operator algebras*, Duke Math. J. **34** (1967), 635–647.

[2] P. Budzynski, Z. J. Jablonski, I. Jung, and J. Stochel, *On unbounded composition operators in L^2-spaces*, Ann. Mat. Pura Appl. (4) **193** (2014), 663–688.

[3] P. Budzyński, Z. Jabloński, Il Bong Jung, and J. Stochel, *Unbounded weighted composition operators in L^2-spaces*, Lecture Notes in Math., vol. 2209, Springer, Cham, 2018.

[4] C. Burnap and A. Lambert, *Reducibility of composition operators on L^2*, J. Math. Anal. Appl. **178** (1993), 87–101.

[5] C. Burnap, I. Jung, and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory **53** (2005), 381–397.

[6] J. T. Campbell and W. Hornor, *Seminormal composition operators*, J. Operator Theory **29** (1993), 323–343.

[7] J. T. Campbell and J. Jamison, *On some classes of weighted composition operators*, Glasg. Math. J. **32** (1990), 87–94.

[8] J. T. Campbell, A. Lambert, and B. M. Weinstock, *Localizing sets and the structure of sigma-algebras*, Indiana Uiv. Math. J. **47** (1998), 913–938.

[9] J. Daughtry, A. Lambert, and B. M. Weinstock, *Operators on C^*-algebras induced by conditional expectations*, Rocky Mountain Math. J. **25** (1996), 1243–1275.

[10] P. Dodds, C. Huijsmans, and B. de Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.

[11] Y. Estaremi and M. R. Jabbarzadeh, *Weighted composition Lambert-type operators on L^p spaces*, Mediterr. J. Math. **11** (2014), 955–964.

[12] J. J. Grobler and B. de Pagter, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory **48** (2002), 15–40.
[13] J. Herron, *Weighted conditional expectation operators*, Oper. Matrices **5** (2011), 107–118.
[14] T. Hoover, A. Lambert, and J. Quinn, *The Markov process determined by a weighted composition operator*, Studia Math. **LXXII** (1982), 225–235.
[15] M. R. Jabbarzadeh, *A conditional expectation type operator on L^p spaces*, Oper. Matrices **4** (2010), 445–453.
[16] M. R. Jabbarzadeh and S. Khalil Sarbaz, *Lambert multipliers between L^p spaces*, Czechoslovak Math. J. **60** (2010), no. 1, 31–43.
[17] M. R. Jabbarzadeh and S. Khalil Sarbaz, *Lambert multipliers of the range of composition operators*, Mat. Vesnik **63** (2011), 191–199.
[18] A. Lambert, *Localising sets for sigma-algebras and related point transformations*, Proc. Royal Soc. Edinburgh Sect. A **118** (1991), no. 1–2, 111–118.
[19] A. Lambert, *L^p multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.
[20] A. Lambert, *The sigma algebra generated by the null space of a conditional expectation*, Houston J. Math. **28** (2002), 899–905.
[21] A. Lambert and B. M. Weinstock, *A class of operator algebras induced by probabilistic conditional expectations*, Michigan Math. J. **40** (1993), 359–376.
[22] P. K. Lin, *Extreme points of Banach lattices related to conditional expectations*, J. Math. Anal. Appl. **312** (2005), 138–147.
[23] M. M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.
[24] R. K. Singh and J. S. Manhas, *Composition operators on function spaces*, North–Holland Math. Stud. vol. 179, Amsterdam, 1993.

How to cite this article: Jabbarzadeh MR, Sharifi MH. Operators whose ranges are contained in the null space of conditional expectations. *Mathematische Nachrichten*. 2019;292:2427–2440. https://doi.org/10.1002/mana.201800066