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Lambert Conditional Operators on $L^2(\Sigma)$

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Abstract
In this paper, we discuss measure theoretic characterizations for Lambert conditional operators in some operator classes on $L^2(\Sigma)$ such as, p-hyponormal, centered, n-normal and binormal. In addition, it is showed when these operators are orthogonal projection and some correlations between these types of operators are established.

Keywords Conditional expectation · Aluthge transform · Moore–Penrose inverse · Multiplication operator · Normal operator

Mathematics Subject Classification Primary 47B20; Secondary 47B25

1 Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ denote the linear bounded operators on $\mathcal{H}$. We use $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, to denote the range and the null space of $T \in B(\mathcal{H})$. For an operator $T \in B(\mathcal{H})$, the adjoint of T denoted by T^*. T is self-adjoint if $T^* = T$ and T is normal if $T^*T = TT^*$. We write $T \geq 0$ if T is a positive operator, meaning $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. An operator T is quasinormal if $T(T^*T) = (T^*T)T$ and is binormal or weakly centered operator if T^*T and TT^* commute. An orthogonal projection is an operator $P \in B(\mathcal{H})$ such that $P^2 = P = P^*$. T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $0 < p \leq 1$. If $p = 1$, T is called hyponormal.

Every bounded operator T on a Hilbert space $\mathcal{H}$ can be written as $T = U|T|$, where $|T| = \sqrt{T^*T}$ and $UU^*U = U$. Moreover U and $|T|$ are unique if $\mathcal{N}(U) = \mathcal{N}(|T|)$. In

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this case, $U|T|$ is said to be the polar decomposition of T. Associated with $T \in B(\mathcal{H})$ and $0 < \lambda \leq 1$, there is a useful related operator $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$, called the λ-Aluthge transformation of T. $\Delta_{\frac{1}{2}}(T) = \widetilde{T}$ is the Aluthge transformation and $\Delta_1(T) = \widehat{T}$ is the Duggal transformation of T . Let $B_C(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For $T \in B_C(\mathcal{H})$, the Moore–Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in B_C(\mathcal{H})$ that satisfying the equations

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

We recall that $T^\dagger$ exists if and only if $T \in B_C(\mathcal{H})$. For other important properties of $\widetilde{T}$ and $T^\dagger$ see e.g. [1,2,6,21]. In [17] Morrel and Muhly introduced the concept of a centered operator. An operator $T \in B(\mathcal{H})$ with polar decomposition $U|T|$ is said to be centered if the doubly infinite sequence $\{T^n T^{*n}, T^{*m} T^m : n, m \geq 0\}$ consists of mutually commuting operators. Let $U_n|T^n|$ be the polar decomposition of T^n. It is shown in [17] that T is centered if and only if $U_n = U^n$, for each $n \in \mathbb{N}$. Note that every centered operator T is binormal. The relations between these classes are studied in [10]. Let $P = P^2 \in B(\mathcal{H})$ be an idempotent. Then we can represent element $T \in B(\mathcal{H})$ as

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

where $T_{11} = PTP, T_{12} = PT(I-P), T_{21} = (I-P)TP$ and $T_{22} = (I-P)T(I-P)$. Then, $\mathcal{R}(P)$ is an invariant subspace of T iff $T_{21} = 0$ and is reducing subspace of T iff $T_{12} = 0 = T_{21}$.

Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a sub-σ-finite algebra of Σ. Inequalities between measurable functions is interpreted in the almost everywhere sense, and equality between sets is interpreted up to a set of measure 0. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. We use the notation of [8] which is a basic reference for details. The support of a measurable function $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a real number r the power f^r is defined by $f^r(x) = sgn(f(x))|f(x)|^r$. The associated conditional expectation with respect to $\mathcal{A}$ is denoted by $E^{\mathcal{A}}_\mu$, or when no confusion will arise, simply E. Let $\mathcal{D}(E) = \{f \in L^0(\Sigma) : E(|f|) \in L^0(\mathcal{A})\}$ be the domain of E (see [7]). It is known that $\cup_{p\geq 1} L^p(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\} \subseteq \mathcal{D}(E)$. Conditional expectation operator E is an orthogonal projection onto $L^2(\mathcal{A})$. Let $u, w \in L^0(\Sigma)$ and $\sigma(u) \subseteq A \in \Sigma$. Then $\chi_A u = u = \chi_{\sigma(u)} u$. So, if $\sigma(u) \subseteq \sigma(w)$ then $\frac{u}{w}\chi_{\sigma(w)} = \frac{u}{w}$. Indeed, we adhere to the convention that $\frac{0}{0} = 0$. If u is an $\mathcal{A}$-measurable function, then $\sigma(u) \in \mathcal{A}$ but the converse is not hold. Now let $\{u_1, u_2, u_1 u_2\} \subset \mathcal{D}(E)$. Put $S_i = \sigma(E(|u_i|^2))$. Then by conditional Cauchy–Schwarz inequality, $\chi_{S_i} u_i = u_i$, $\chi_{S_i} E(u_i) = E(u_i)$ and $\chi_{S_i} E(u_1 u_2) = E(u_1 u_2)$. For details on the properties of E see e.g. [3,8,13,15,20].

Lambert conditional operators are closely related to the multiplication operators, integral and averaging operators and to the operators called conditional expectation-type which has been introduced in [3] and [14]. Let $1 < p < \infty$. Moy in [18] showed that if $T \in B(L^p(\Sigma))$, $T(L^\infty(\Sigma)) \subseteq L^\infty(\Sigma)$ and $T(fT(g)) = T(f)T(g)$ for all $f, g \in L^\infty(\Sigma)$, then $T = E^{\mathcal{A}}M_u$ for some $\mathcal{A} \subseteq \Sigma$. For $w, u \in \mathcal{D}(E)$, the mapping $T : L^2(\Sigma) \supseteq \mathcal{D}(T) \to L^2(\Sigma)$ given by $T(f) = wE(uf)$ for $f \in \mathcal{D}(T) = \{f \in L^2(\Sigma) : T(f) \in L^2(\Sigma)\}$, is well-defined and linear. Such an operator is called a Lambert conditional operator induced by the pair (w, u). Let q be the conjugate exponent to p. Lambert and Herron in [8,14] considered the conditional operator $T_1 = EM_u$ on $L^p(\Sigma)$ and they showed that $T_1 \in B(L^p(\Sigma))$ if and only if $E(|u|^q) \in L^\infty(\mathcal{A})$ and in this case $\|T_1\|^q = \|E(|u|^q)\|_\infty$. Let $T = M_w EM_u$ and set $\nu = u(E(|w|^p))^{1/p}$. Then $\|Tf\| = \|EM_\nu f\|$ for all $f \in L^p(\Sigma)$. It follows that $T \in B(L^p(\Sigma))$ if and only if $E(|u|^q)^{1/q}E(|w|^p)^{1/p} \in L^\infty(\mathcal{A})$. In this case, $\|T\| = \|E(|u|^q)^{1/q}E(|w|^p)^{1/p}\|_\infty$ (see [5]). Throughout this paper we consider the case $p = 2$ and assume that $T \in B(L^2(\Sigma))$. For further information on conditional type operators, see e.g. [5,9,14,16].

Let $\mathcal{K}$ denote the set of all bounded Lambert conditional operators on $L^2(\Sigma)$ and let $T \in \mathcal{K}$. In the next section, first we review some basic results on the elements of $\mathcal{K}$ and state some general assumptions and then we present a method for computing the null space and the range of T^n. Also, we give some necessary and sufficient conditions for T being normal, quasinormal, p-hyponormal, centered, binormal and n-normal. In fact, we show that all of these classes in $\mathcal{K}$ coincide. In addition, the reverse order law for the Moore–Penrose inverse is established and it is showed when these operators are partial isometry or orthogonal projection. Lastly, we show that the Duggal transformation and the λ-Aluthge transform of T coincide. To explain the results, some examples are then presented.

2 Characterizations

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Relative to the direct sum decomposition $\mathcal{M} \oplus \mathcal{M}^\perp$, an element $f \in \mathcal{H}$ can be written uniquely as $f = f_1 + f_2$ where $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^\perp$. Let $T \in B(\mathcal{H})$. Then $Tf = Tf_1 + Tf_2 = [(Tf_1)_1 + (Tf_1)_2] + [(Tf_2)_1 + (Tf_2)_2]$. Let $P \in B(\mathcal{H})$ be an orthogonal projection onto $\mathcal{M}$. For $1 \le i, j \le 2$, put $T_{i,j}(f_j) = (Tf_j)_i$. Then

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} PT|_{\mathcal{M}} & PT|_{\mathcal{M}^\perp} \\ (I-P)T|_{\mathcal{M}} & (I-P)T|_{\mathcal{M}^\perp} \end{bmatrix} \tag{2.1}$$

is the matrix representation of T. In particular, let $\mathcal{H} = L^2(\Sigma)$, $P = E^{\mathcal{A}} = E$, $\mathcal{M} = \mathcal{R}(E) = L^2(\mathcal{A})$ and $\mathcal{M}^\perp = \mathcal{N}_2(E) = \{f - Ef : f \in L^2(\Sigma)\}$. Note that $f_1 = Ef$ and $f_2 = f - f_1$, for all $f \in \mathcal{D}(E)$. Let $f_1 = 0$ and $f_2 \ge 0$. Since $\sigma(f) \subseteq \sigma(f_1)$, so $f = 0$. Consequently, $\{f \in \mathcal{D}(E) : f > 0\} \cap \mathcal{N}_2(E) = \emptyset$. In general, the structure of $\mathcal{N}_2(E)$ is very complicated (see [16]). Note that $E(|f|^2) = E((f_1 + f_2)(\bar{f_1} + \bar{f_2})) = |f_1|^2 + E(|f_2|^2)$. Using above argument, $E(|f_2|^2) = 0$ if and only if $f_2 = 0$. So, $|E(f)|^2 = E(|f|^2)$ if and only if $f \in L^0(\mathcal{A})$.

Notice that T_{11} and T_{22} are the compressions of T to the closed subspace $\mathcal{M}$ and $\mathcal{M}^\perp$, respectively. Let $\Lambda : B(\mathcal{M}) \to B(\mathcal{H})$ be a mapping defined as $\Lambda(T) = PTP$, where P be an orthogonal projection onto $\mathcal{M}$. Then Λ is linear and $\Lambda(T^*) = PT^*P =$

$(\Lambda(T))^*$, because $P = P^* = (P_{|\mathcal{M}})^*$. Let $T_1, T_2 \in B(\mathcal{M})$. Since $P^2 = P$ and $PT_2 = T_2$, then $\Lambda(T_1T_2) = (PT_1P)(PT_2P) = \Lambda(T_1)\Lambda(T_2)$. Moreover, $\Lambda(T_2) = 0$, implies that $PT_{2|\mathcal{M}} = PT_2P = 0$, and hence $T_2 = 0$. Consequently, Λ is a $*$-monomorphism and thus $B(\mathcal{M})$ can be considered a C^*-subalgebra of a C^*-algebra $B(\mathcal{H})$. It is a well-known fact that for any C^*-subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ and $b \in \mathfrak{B}$, $\text{Spec}_{\mathfrak{B}}(b) \cup \{0\} = \text{Spec}_{\mathfrak{A}}(b) \cup \{0\}$, where $\text{Spec}_{\mathfrak{B}}(b)$ is denote the spectrum of b as an element of $\mathfrak{B}$. Consequently, for $T \in B(\mathcal{H})$ with matrix representation (2.1) we have

$$\begin{aligned} \text{Spec}_{B(\mathcal{M})}(T_{11}) \cup \{0\} &= \text{Spec}_{B(\mathcal{H})}(T_{11}) \cup \{0\}; \\ \text{Spec}_{B(\mathcal{M}^\perp)}(T_{22}) \cup \{0\} &= \text{Spec}_{B(\mathcal{H})}(T_{22}) \cup \{0\}. \end{aligned}$$

Let $f, g \in L^2(\Sigma)$. Then $f_1, g_1 \in L^2(\mathcal{A})$, $f_2, g_2 \in \mathcal{N}_2(E)$ and $E(f_1g_2) = E(g_2f_1) = 0$. It follows that $(fg)_1 = E(fg) = f_1g_1 + E(f_2g_2)$ and $(fg)_2 = (I - E)(fg) = fg - f_1g_1 - E(f_2g_2)$. Let $K = E(|u|^2)E(|w|^2)$. We recall that $T = M_wEM_u$ is bounded in $L^2(\Sigma)$ if and only if $K \in L^\infty(\Sigma)$. In this case, $\|T\|^2 = \|K\|_\infty$ and $\mathcal{D}(T) = L^2(\Sigma)$ (see [5]). Now, let $T = M_wEM_u \in B(L^2(\Sigma))$ and let $f = f_1 + f_2 \in L^2(\Sigma)$. Then we have

$$\begin{aligned} Tf_1 &= (w_1 + w_2)E((u_1 + u_2)f_1) = w_1u_1f_1 + w_2u_1f_1 = (Tf_1)_1 + (Tf_1)_2; \\ Tf_2 &= (w_1 + w_2)E((u_1 + u_2)f_2) = E(w_1u_2f_2) + w_2E(u_2f_2) = (Tf_2)_1 + (Tf_2)_2. \end{aligned}$$

Thus,

$$\begin{aligned} T_{11}(f_1) &= M_{w_1u_1}f_1, \quad & T_{21}(f_1) &= M_{w_2u_1}f_1 \\ T_{12}(f_2) &= EM_{w_1u_2}f_2 \quad & T_{22}(f_2) &= M_{w_2}EM_{u_2}f_2. \end{aligned}$$

Consequently, the matrix representation of $T = M_wEM_u$ and T^* with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}_2(E)$ are

$$T = \begin{bmatrix} M_{w_1u_1} & EM_{w_1u_2} \\ M_{w_2u_1} & M_{w_2}EM_{u_2} \end{bmatrix} \text{ and } T^* = \begin{bmatrix} M_{\overline{w_1u_1}} & EM_{\overline{w_2u_1}} \\ M_{\overline{w_1u_2}} & M_{\bar{u}_2}EM_{\bar{w}_2} \end{bmatrix}. \tag{2.2}$$

In particular, if $w = 1$ then $w_1 = 1$ and $w_2 = 0$ and hence the matrix representation of $T_1 := EM_u$ and T_1^* are

$$T_1 = \begin{bmatrix} M_{u_1} & EM_{u_2} \\ 0 & 0 \end{bmatrix} \text{ and } T_1^* = \begin{bmatrix} M_{\bar{u}_1} & 0 \\ M_{\bar{u}_2} & 0 \end{bmatrix}. \tag{2.3}$$

Note that $\mathcal{N}(EM_u) = \{f \in L^2(\Sigma) : u_1f_1 + E(u_2f_2) = 0\}$. In the following we characterize the null space and the range of M_wEM_u.

Lemma 2.1 *Let $T_1 = EM_u \in B(L^2(\Sigma))$. Then $\mathcal{N}(T_1) = (\bar{u}L^2(\mathcal{A}))^\perp$.*

Proof Let $f \in L^2(\Sigma)$ and let $g \in L^2(\mathcal{A})$. Since $EM_u f \in L^2(\mathcal{A})$, then

$$\begin{aligned} f \in (\bar{u}L^2(\mathcal{A}))^{\perp} &\iff \langle g, EM_u f\rangle = \langle Eg, uf\rangle = \langle g, uf\rangle = \langle \bar{u}g, f\rangle = 0 \\ &\iff f \in \mathcal{N}(T_1). \end{aligned}$$

Thus, $\mathcal{N}(T_1) = \{f \in L^2(\Sigma) : f \perp \bar{u}L^2(\mathcal{A})\}$. □

Proposition 2.2 *Let $T = M_w EM_u \in B(L^2(\Sigma))$ and let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &(a)\ \mathcal{N}(T^n) = \{\bar{u}|E(uw)|^{n-1}\sqrt{E(|w|^2)}L^2(\mathcal{A}))\}^{\perp}; \\ &(b)\ \overline{\mathcal{R}(T^n)} = \{w|E(uw)|^{n-1}\sqrt{E(|u|^2)}L^2\overline{(\mathcal{A}))}\}. \end{aligned}$$

Moreover, if $|E(uw)| \geq k$ on $\sigma(E(uw))$ for some $k > 0$, then $\mathcal{N}(T^n) = \mathcal{N}(T^2)$ and $\mathcal{R}(T^2) = \mathcal{R}(T^n)$.

Proof Let $n = 1$ and $f \in L^2(\Sigma)$. Since

$$\|M_w EM_u f\| = \|EM_{u\sqrt{E(|w|^2)}} f\|, \tag{2.4}$$

then by Lemma 2.1 we have

$$\begin{aligned} \mathcal{N}(T) &= \mathcal{N}(EM_{u\sqrt{E(|w|^2)}}) = \{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\}^{\perp}; \\ \overline{\mathcal{R}(T)} &= \mathcal{N}(T^*)^{\perp} = \{w\sqrt{E(|u|^2)}L^2\overline{(\mathcal{A}))}\}. \end{aligned}$$

Now, (a) and (b) follows from equality $T^n = M_{E(uw)^{n-1}w} EM_u$.

Put $v_1 = \bar{u}\sqrt{E(|w|^2)}$ and $A = \sigma(E(uw))$. Then for each $g \in L^2(\mathcal{A})$ and $n > 2$ we have $\|\frac{g\chi_A}{|E(uw)|^{n-2}}\|_2 \leq \frac{1}{k^{n-2}}\|g\|_2$. Thus, $\frac{g\chi_A}{|E(uw)|^{n-2}} \in L^2(\mathcal{A})$ and so

$$v_1|E(uw)|g = v_1|E(uw)|^{n-1}\frac{g\chi_A}{|E(uw)|^{n-2}} \in v_1|E(uw)|^{n-1}L^2(\mathcal{A}).$$

Consequently, $v_1|E(uw)|L^2(\mathcal{A}) \subseteq v_1|E(uw)|^{n-1}L^2(\mathcal{A})$ and hence

$$\mathcal{N}(T^2) = (v_1|E(uw)|L^2(\mathcal{A}))^{\perp} \supseteq (v_1|E(uw)|^{n-1}L^2(\mathcal{A}))^{\perp} = \mathcal{N}(T^n).$$

Now, by a similar argument we have

$$\overline{\mathcal{R}(T^n)} = \mathcal{N}(T^{n*})^{\perp} = \{w|E(uw)|^{n-1}\sqrt{E(|u|^2)}L^2\overline{(\mathcal{A}))}\}.$$

Set $v_2 = w\sqrt{E(|u|^2)}$. By a similar argument as above,

$$v_2|E(uw)|L^2(\mathcal{A}) \subseteq v_2|E(uw)|^{n-1}L^2(\mathcal{A}),$$

and hence $\overline{\mathcal{R}(T^2)} \subseteq \overline{\mathcal{R}(T^n)}$. This completes the proof. □

It is worth nothing that if $|E(uw)|^2 = E(|u|^2)E(|w|^2) = K^2$, then for all $n \in \mathbb{N}$, $\|T^n\| = \||E(uw)|^{n-1}K\|_\infty = \|K\|_\infty^n = \|T\|^n$ and hence T, in this case, is normaloid.

Recall that the multiplication operator $M_u \in B(L^2(\Sigma))$ has closed range if and only if there exists $k > 0$ such that $\|M_u(f\chi_{\sigma(u)})\| \geq k\|f\chi_{\sigma(u)}\|$, for all $f \in L^2(\Sigma)$. Equivalently, $\|M_u f\| \geq k\|f\|$ for all $f \in \mathcal{N}(M_u)^\perp = \chi_{\sigma(u)}L^2(\Sigma)$. Using Lemma 2.1, $T_1 \in B(L^2(\Sigma))$ has closed range if and only if there exists $k > 0$ such that $\|T_1 f\| \geq k\|f\|$, for all $f \in \overline{\bar{u}L^2(\mathcal{A})}$. So we have the following proposition.

Proposition 2.3 *Let* $T_1 = EM_u \in B(L^2(\Sigma))$ *and* $S = \sigma(E(|u|^2))$. *Then the followings hold.*

(a) If T_1 *has closed range, then* $E(|u|^2) \geq k$ *on* S *for some* $k > 0$.
(b) If $\sigma(u) \in \mathcal{A}$ *and* $E(|u|^2) \geq \alpha$ *on* S *for some* $\alpha > 0$, *then* T_1 *has closed range.*

Proof (a) Suppose $E(|u|^2)$ is not bounded away from zero on its support. Then for fixed $\varepsilon > 0$, there exists $A \in \mathcal{A}$ with $A \subseteq S$ and $0 < \mu(A) < \infty$ such that $E(|u|^2)\chi_A < \varepsilon$. Put $f_0 = \bar{u}\chi_A$. Then f_0 is nonzero and $\|f_0\|^2 = \int_A |u|^2 d\mu = \int_A E(|u|^2)d\mu < \varepsilon\mu(A) < \infty$. Thus, $f_0 \in \overline{\bar{u}L^2(\mathcal{A})} \cap L^2(\Sigma)$ and satisfies

$$\begin{aligned}\|T_1 f_0\|^2 = \|E(uf_0)\|^2 = \|E(|u|^2)\chi_A\|^2 &= \int_A (E(|u|^2)\chi_A)E(|u|^2)d\mu \\ &\leq \varepsilon \int_A E(|u|^2)d\mu = \varepsilon \int_A |u|^2 d\mu = \varepsilon \int_X |\bar{u}\chi_A|^2 d\mu = \varepsilon\|f_0\|^2.\end{aligned}$$

But this is a contraction.

(b) Let $\sigma(u) \in \mathcal{A}$. Since $\sigma(u) = \sigma(|u|) \subseteq \sigma(E(|u|^2)) = S$, then for all $f \in L^2(\mathcal{A})$ we have

$$\begin{aligned}\|M_{\bar{u}}(f\chi_{\sigma(u)})\|^2 = \int_{\sigma(u)} |uf|^2 d\mu &= \int_{\sigma(u)} E(|u|^2)|f|^2 d\mu \\ &\geq \alpha \int_{\sigma(u)} \chi_S |f|^2 d\mu = \alpha\|f\chi_{\sigma(u)}\|^2.\end{aligned}$$

It follows that $M_{\bar{u}} : L^2(\mathcal{A}) \to L^2(\Sigma)$ have closed range. Hence $\overline{\bar{u}L^2(\mathcal{A})} = \bar{u}L^2(\mathcal{A})$. Let $\bar{u}g \in \bar{u}L^2(\mathcal{A})$ be arbitrary. Since $\chi_S E(|u|^2) = E(|u|^2)$, then we have

$$\begin{aligned}\|T_1(\bar{u}g)\|^2 = \|EM_u(\bar{u}g)\|^2 = \|E(|u|^2)g\|^2 &= \int_X (E(|u|^2))(E(|u|^2)|g|^2)d\mu \\ &\geq \alpha \int_X E(|u|^2)|g|^2 d\mu = \alpha \int_X |\bar{u}g|^2 d\mu = \alpha\|\bar{u}g\|^2.\end{aligned}$$

This completes the proof. □

Using (2.4), we have the following corollary.

Corollary 2.4 *Let $T = M_w EM_u \in B(L^2(\Sigma))$ has closed range. Then $K = E(|u|^2)E(|w|^2)$ is bounded away from zero on $\sigma(K)$. Also, if $\sigma(u\sqrt{E(|w|^2)}) \in \mathcal{A}$ and K is bounded away from zero on $\sigma(K)$, then T has closed range.*

Remark 2.5 Recall that asc(T) and des(T), the ascent and descent of $T \in B(\mathcal{H})$, is the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$, respectively. If $T = M_w EM_u \in B(L^2(\Sigma))$, $\sigma(u\sqrt{E(|w|^2)}) \in \mathcal{A}$ and $E(|u|^2)E(|w|^2)$ is bounded away from zero on its support, then by Proposition 2.2 and Corollary 2.4, asc$(T) =$ des$(T) = 2$.

Let $EM_u \in B(L^2(\Sigma))$ and $EM_u = 0$. Then by (2.3), $M_{u_1} = 0$ on $L^2(\mathcal{A})$ and $EM_{u_2} = 0$ on $\mathcal{N}_2(E)$. Let $\sigma(u_1) = \cup A_n$ with $A_n \subseteq A_{n+1}$, $0 < \mu(A_n) < \infty$. Then $\chi_{A_n} \nearrow \chi_{\sigma(u_1)}$. Set $f_n = \chi_{A_n}\bar{u}_1$. Then for each $n \in \mathbb{N}$,

$$\int_X |\chi_{A_n}\bar{u}_1|^2 d\mu = \int_{A_n} E(|u_1|^2)d\mu \leq \int_{A_n} E(|u|^2)d\mu \leq \|E(|u|^2)\|_\infty \mu(A_n) < \infty.$$

Thus, $f_n \in L^2(\mathcal{A})$ and $0 = M_{u_1}(f_n) = \chi_{A_n}|u_1|^2 \to |u_1|^2$. It follows that $u_1 = 0$. Moreover, $EM_{u_2} = 0$ on $\mathcal{N}_2(E)$ if and only if $u_2\mathcal{N}_2(E) \subseteq \mathcal{N}_2(E)$. However, if $u \geq 0$, then $u = 0$. Thus, the mapping $u \mapsto EM_u$ is not one-to-one, in general. However, this occurs under certain conditions.

Proposition 2.6 *Let $\mathcal{K}_n = \{u \in L^n(\Sigma) : E(|u|^2) \in L^\infty(\mathcal{A})\}$.*

(a) If $\mu(X) < \infty$, then the mapping $\Lambda_1 : \mathcal{K}_0 \to B(L^2(\Sigma))$ defined by $\Lambda_1(u) = EM_u$ is one-to-one.
(b) The mapping $\Lambda_2 : \mathcal{K}_2 \to B(L^2(\Sigma))$ defined by $\Lambda_2(u) = EM_u$ is one-to-one.

Proof Let $EM_u = 0$. Using (2.3), $M_{u_1} = 0$ and $EM_{u_2} = 0$ on $L^2(\mathcal{A})$ and $\mathcal{N}_2(E)$, respectively. Then by the above discussion, $u_1 = 0$. Also, since $E(|u|^2) = |u_1|^2 + E(|u_2|^2)$ then we have

$$\int_X |\bar{u}_2|^2 d\mu \leq \int_X E(|u|^2)d\mu \leq \begin{cases} \|E(|u|^2)\|_\infty \mu(X) & \text{if } \mu(X) < \infty, \\ \|u\|_2^2 & \text{if } u \in L^2(\Sigma). \end{cases}$$

Thus, $\bar{u}_2 \in \mathcal{N}(E) \cap L^2(\Sigma) = \mathcal{N}_2(E)$. It follows that $E(|u_1|^2) = EM_{u_2}(\bar{u}_2) = 0$, and hence $u = u_1 + u_2 = 0$. □

Corollary 2.7 *Let $M_w EM_u \in B(L^2(\Sigma))$. If $\mu(X) < \infty$ or $u\sqrt{E(|w|^2)} \in L^2(\Sigma)$, then $M_w EM_u = 0$ implies that $u = 0$ on $\sigma(E(|w|)$.*

Proof Since $\|M_w EM_u\| = \|EM_{u\sqrt{E(|w|^2)}}\|$, then the desired conclusion follows from Proposition 2.6. □

Let $T = M_w EM_u \in B(L^2(\Sigma))$. If $w = g\bar{u}$, for some $0 \leq g \in L^0(\mathcal{A})$, then

$$\langle Tf, f\rangle = \langle M_{g\bar{u}} EM_u f, f\rangle = \int_X g\bar{u}E(uf)\bar{f}d\mu$$

$$= \int_X gE(uf)\overline{E(uf)}d\mu = \int_X g|E(uf)|^2 d\mu \geq 0$$

for all $f \in L^2(\Sigma)$, and so $T \geq 0$. In this case, $(M_{g\bar{u}}EM_u)^p(f) = g^p\bar{u} E(|u|^2)^{p-1}E(uf)$ for $p \in (0, \infty)$ and $f \in L^2(\Sigma)$ [11, Lemma 2.1]. Now, suppose $T \geq 0$. Then $T_{11} = ET|_{L^2(\mathcal{A})} \geq 0$ and $T_{22} = (I - E)T|_{\mathcal{N}_2(E)} \geq 0$. Indeed, for $f \in L^2(\mathcal{A})$ and $g \in \mathcal{N}_2(E)$ we have

$$\langle T_{11}f, f\rangle = \langle ETf, f\rangle = \langle Tf, Ef\rangle = \langle Tf, f\rangle \geq 0;$$
$$\langle T_{22}f, f\rangle = \langle (I - E)Tg, g\rangle = \langle Tg, g\rangle - \langle Tg, Eg\rangle = \langle Tg, g\rangle \geq 0.$$

Theorem 2.8 *Let $T = M_w EM_u \in B(L^2(\Sigma))$. If $T \geq 0$, then $E(uw) \geq 0$, $\bar{u}E(uw) = wE(|u|^2)$ and $\sqrt{T_{22}} = M_{\nu\bar{u}_2}EM_{u_2}$ where $\nu = \frac{\sqrt{E(u_2w_2)}}{E(|u_2|^2)}$.*

Proof Let $T \geq 0$. Using above argument $T_{11} = M_{w_1u_1}$ and $T_{22} = M_{w_2}EM_{u_2}$ are positive operators on $L^2(\mathcal{A})$ and $\mathcal{N}_2(E)$, respectively. It follows that $w_1u_1 \geq 0$ and

$$0 \leq \int_X w_2E(u_2f)\bar{f}d\mu = \int_X E(u_2f)E(w_2\bar{f})d\mu, \quad \forall f \in \mathcal{N}_2(E). \tag{2.5}$$

Since $\mathcal{A}$ is σ-finite, there exists $\{A_n\}_n \subseteq \mathcal{A}$ such that $X = \cup_n A_n$, $A_n \subseteq A_{n+1}$ with $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. In this case $\chi_{A_n} \nearrow \chi_X$. Put $f_n = \bar{u}_2\sqrt{E(|w_2|^2)}\ \chi_{A_n}$. Then $\|f_n\|^2 = \int_{A_n} |u_2|^2E(|w_2|^2)d\mu \leq \|T\|^2\mu(A_n) < \infty$, and hence $f_n \in \mathcal{N}_2(E)$. Replacing f in (2.5) by f_n, we obtain

$$\int_{A_n} E(|u_2|^2)E(|w_2|^2)E(u_2w_2)d\mu \geq 0, \quad \forall n \in \mathbb{N}.$$

Thus, $E(u_2w_2) \geq 0$ on each A_n, and so $E(u_2w_2) \geq 0$ on X. From this and $w_1u_1 \geq 0$, we conclude that $E(uw) = E(u_1w_1 + u_2w_2) = u_1w_1 + E(u_2w_2) \geq 0$ on X. Now, from the equalities $T_{22}f_0 = T_{22}^*f_0$ and $T_{12}f_0 = T_{21}^*f_0$ and that $E(u_2w_2) \geq 0$ and $w_i\chi_{\sigma(E|w_i|)} = w_i$ we deduce that

$$\begin{aligned} w_2E(|u_2|^2) &= \bar{u}_2E(u_2w_2); \\ w_1E(|u_2|^2) &= \bar{u}_1E(u_2w_2). \end{aligned} \tag{2.6}$$

Thus, $(w_1 + w_2)E(|u_2|^2) = (\bar{u}_1 + \bar{u}_2)E(u_2w_2)$ and hence

$$wE(|u_2|^2) = \bar{u}E(u_2w_2). \tag{2.7}$$

On the other hand, since $w_1|u_1|^2 = \bar{u}_1(u_1w_1)$, $T_{21} = T_{12}^*$ and $w_1u_1 \geq 0$, then $w_2|u_1|^2 = (u_2w_1)\bar{u}_1 = \bar{u}_2(u_1w_1)$. Consequently,

$$w|u_1|^2 = \bar{u}(u_1w_1). \tag{2.8}$$

Now, from (2.7) and (2.8) we have

$$w(|u_1|^2 + E(|u_2|^2)) = \bar{u}(u_1 w_1 + E(u_2 w_2))$$
$$\Longrightarrow \bar{w}E(|u|^2) = uE(uw).$$

Using (2.6), $\frac{w_2 \chi_{\sigma(u_2)}}{\bar{u}_2} = \frac{E(u_2 w_2)}{E(|u_2|^2)}$. Then by [11, Lemma 2.1] we get that

$$\sqrt{T_{22}} = \sqrt{M_{\frac{w_2\chi_{\sigma(u_2)}}{\bar{u}_2}\bar{u}_2} EM_{u_2}} = \sqrt{M_{\frac{E(u_2w_2)}{E(|u_2|^2)}\bar{u}_2} EM_{u_2}}$$
$$= M_{\frac{\sqrt{E(u_2w_2)}}{E(|u_2|^2)}\bar{u}_2} EM_{u_2}.$$

This completes the proof. □

Corollary 2.9 [12] *Let $T = M_w EM_u \in B(L^2(\Sigma))$. Then T is positive if and only if $w = g\bar{u}$ for some $0 \le g \in L^0(\mathcal{A})$.*

Corollary 2.10 *Let $T = M_w EM_u \in B(L^2(\Sigma))$. If $T \ge 0$ and $E(uw) \le 1$, then T is an contraction on $L^2(\Sigma)$.*

Proof Let $E(uw) \le 1$ and put $g = \frac{E(uw)}{E(|u|^2)}$. Then $0 \le g \in L^0(\mathcal{A})$ and by Corollary 2.9, $w = g\bar{u}$. It follows that $|E(uw)|^2 = g^2(E(|u|^2))^2 = E(|u|^2)E(|w|^2)$. Consequently, $\|T\|^2 = \|E(|u|^2)E(|w|^2)\|_\infty = \|E(uw)\|_\infty^2 \le 1$, and hence T is a contraction. □

Bounded self-adjoint, normal and quasinormal Lambert conditional operators have recently been characterized in [12].

Proposition 2.11 [12] *Let $T = M_w EM_u \in B(L^2(\Sigma))$. Then the followings hold.*

(a) T is self-adjoint if and only if $T = M_{g\bar{u}} EM_u$ for some $\bar{g} = g \in L^0(\mathcal{A})$.
(b) T is normal if and only if $T = M_{g\bar{u}} EM_u$ for some $g \in L^0(\mathcal{A})$.
(c) T is quasinormal if and only if T is normal.

Corollary 2.12 *Let $T = M_w EM_u \in B(L^2(\Sigma))$. Then T is normal if and only if so is T^n, for $n \in \mathbb{N}$.*

Proof Suppose $n \in \mathbb{N}$ and $T^n = M_{wE(uw)^{n-1}} EM_u$ is normal. Using Proposition 2.11(b), $wE(uw)^{n-1} = g_1\bar{u}$ for some $g_1 \in L^0(\mathcal{A})$. It follows that $w = g\bar{u}$ where $g = \frac{g_1}{E(uw)^{n-1}} \chi_{\sigma(E(uw))} \in L^0(\mathcal{A})$. □

Corollary 2.13 *Let $T = M_{w_1} EM_u$, $S = M_{w_2} EM_v$ be two bounded self-adjoint operators on $L^2(\Sigma)$. If TS is normal, then it is self-adjoint.*

Proof Using Proposition 2.11(a), $T = M_{g_1\bar{u}1} EM_u$ and $S = M_{g_2\bar{v}} EM_v$ for some $g_i = \bar{g}_i \in L^2(\mathcal{A})$. Since $TS = M_{g_1 g_2 \bar{u} E(u\bar{v})} EM_v$ is normal, then $g_1 g_2 \bar{u} E(u\bar{v}) = g_3\bar{v}$ for some $g_3 \in L^2(\mathcal{A})$. After multiplying both sides by v and then taking E we obtain $g_1 g_2 |E(u\bar{v})|^2 = g_3 E(|v|^2)$. Thus, $g_3 = g_1 g_2 \frac{|E(u\bar{v})|^2}{E(|v|^2)}$, and so $\bar{g}_3 = g_3$. It follows that TS is self-adjoint. □

In the following we study measure-theoretic characterizations for p-hyponorma, centered and binormal Lambert conditional operators in $L^2(\Sigma)$ space.

Theorem 2.14 *Let $T = M_w E M_u \in B(L^2(\Sigma))$. Then the followings hold.*

(a) T is binormal if and only if T is normal. In this case $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.
(b) T is centered if and only if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.
(c) T is p-hyponormal if and only if T is normal.

Proof (a) It is easy to check that

$$\begin{aligned}(TT^*)(T^*T) &= (M_{wE(|u|^2)}EM_{\bar{w}})(M_{\bar{u}E(|w|^2)}EM_u)\\ &= M_{wE(|u|^2)E(\overline{uw})E(|w|^2)}EM_u;\\ (T^*T)(TT^*) &= (M_{\bar{u}E(|w|^2)}EM_u)(M_{wE(|u|^2)}EM_{\bar{w}})\\ &= M_{\bar{u}E(|w|^2)E(uw)E(|u|^2)}EM_{\bar{w}}.\end{aligned}$$

Thus, T is binormal if and only if

$$wE(|u|^2)E(\overline{uw})E(|w|^2)E(uf) = \bar{u}E(|w|^2)E(uw)E(|u|^2)E(\bar{w}f), \tag{2.9}$$

for all $f \in L^2(\Sigma)$. Put $f_n = \bar{u}\sqrt{E(|w|^2}\chi_{A_n}$. After substituting f in (2.9) and taking limit on n, we obtain

$$\begin{aligned}&wE(|u|^2)E(\overline{uw})E(|w|^2)E(|u|^2)\sqrt{E(|w|^2}\\ &= \bar{u}E(|w|^2)E(uw)E(|u|^2)E(\overline{uw})\sqrt{E(|w|^2},\end{aligned}$$

and hence

$$wE(|u|^2)\chi_{\sigma(E(uw))} = \bar{u}E(uw). \tag{2.10}$$

It follows that $w = g\bar{u}$ where $g = \frac{E(uw)}{E(|u|^2)} \in L^0(\mathcal{A})$. Now, multiplying both sides of (2.10) by $\bar{w}$ and then taking E we obtain $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.

(b) Let $U|T|$ be the polar decomposition of T. Then by [5, Theorem 3.6],

$$U = M_{\frac{w}{\sqrt{E(|w|^2)E(|u|^2)}}}EM_u; \quad |T| = M_{\bar{u}\sqrt{\frac{E(|w|^2)}{E(|u|^2)}}}EM_u.$$

Now let $U_n|T^n|$ be the polar decomposition of $T^n = M_{E(uw)^{n-1}w}EM_u$. Put $K = \sqrt{E(|u|^2)E(|w|^2)}$. Then we have

$$U^n = M_{\frac{E(uw)^{n-1}w}{K^{n-1}K}}EM_u, \quad \text{and} \quad U_n = M_{\frac{E(uw)^{n-1}w}{|E(uw)|^{n-1}K}}EM_u,$$

for all $n \in \mathbb{N}$. Consequently, if $|E(uw)| = K$ then $U_n = U^n$, and so T is centered. The converse follows from part (a).

(c) Let T is p-hyponormal. Then by [5, Theorem 3.4(a)], T is hyponormal. So, for each $f \in L^2(\Sigma)$ we have

$$\int_X \left\{\bar{u}E(|w|^2)E(uf) - wE(|u|^2)E(\bar{w}f)\right\} \bar{f}d\mu \geq 0. \tag{2.11}$$

Put $f_n = \bar{u}\sqrt{E(|w|^2}\chi_{A_n}$. After substituting f in (2.11), we obtain

$$\int_{A_n} E(|u|^2)E(|w|^2)\left\{|u|^2E(|w|^2) - uw\overline{E(uw)}\right\}d\mu$$
$$= \int_{A_n} E(|u|^2)E(|w|^2)\left\{|w|^2E(|u|^2) - uw\overline{E(uw)}\right\}d\mu \geq 0.$$

It follows that $Im\left\{|w|^2E(|u|^2) - uw\overline{E(uw)}\right\} = 0$ on each A_n, and hence $uw\overline{E(uw)}$ is real-valued on X. From this fact and the Cauchy–Schwarz inequality we have

$$uw\overline{E(uw)} \leq |w|^2E(|u|^2). \tag{2.12}$$

For $A_n \in \mathcal{A}$ with $0 < \mu(A_n) < \infty$, put $f_n = w\sqrt{E(|u|^2}\chi_{A_n}$. On substituting f in the inequality (2.11) and using $\overline{uw}E(uw) = uw\overline{E(uw)}$ we obtain

$$\int_{A_n} E(|u|^2)E(|w|^2)\left\{uw\overline{E(uw)} - |w|^2E(|u|^2)\right\}d\mu \geq 0.$$

Using (2.12) we obtain that $uw\overline{E(uw)} = |w|^2E(|u|^2)$ on each A_n, and so $uw\overline{E(uw)} = |w|^2E(|u|^2)$ on X. Consequently, $|E(uw)|^2 = E(|u|^2)E(|w|^2)$. Now, the desired conclusion follows from (b). □

Corollary 2.15 *Let $T = M_wEM_u \in B(L^2(\Sigma))$ and $n \in \mathbb{N}$. Then T^n is normal (quasinormal, p-hyponormal, binormal) if and only if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.*

Remark 2.16 Set $\mathcal{K} = \{M_wEM_u : u, w, uw \in \mathcal{D}(E),\ E(|u|^2)E(|w|^2) \in L^\infty(\Sigma)\}$. Then $\mathcal{K}$ is closed under multiplication by scaler, multiplication and positive root elements. Indeed, for $T_i = M_{w_i}EM_{u_i}$ and $T_j = M_{w_j}EM_{u_j}$ in $\mathcal{K}$, we have $T_iT_j = M_{w_iE(u_iw_j)}EM_{u_j} \in \mathcal{K}$. Moreover, if $0 \leq T = M_wEM_u \in \mathcal{K}$ and $n \in \mathbb{N}$ then by Corollary 2.9 and [11, Lemma 2.1] we have

$$T^{\frac{1}{n}} = M_{g^{\frac{1}{n}}\bar{u}E(|u|^2)^{\frac{1-n}{n}}}EM_u \in \mathcal{K}.$$

Also,

$$\begin{aligned}\|T_iT_j\| &= \|E(|w_i|^2)^{\frac{1}{2}}|E(u_iw_j)|E(|u_j|^2)^{\frac{1}{2}}\|_\infty \\ &\leq \|E(|w_i|^2)^{\frac{1}{2}}E(|u_i|^2)^{\frac{1}{2}}E(|w_j|^2)^{\frac{1}{2}}E(|u_j|^2)^{\frac{1}{2}}\|_\infty \\ &\leq \|T_i\|\ \|T_j\| < \infty.\end{aligned}$$

However, $\mathcal{K}$ is not closed under the sum. Put

$$\mathfrak{L} = \left\{ \sum_{i=1}^{n} T_{w_i,u_i} \,:\, n \in \mathbb{N},\ T_{w_i,u_i} = M_{w_i} E M_{u_i} \in \mathcal{K} \right\}.$$

Then $\mathfrak{L}$ is an algebra. Moreover, $(\Sigma_{i=1}^{n} T_{w_i,u_i})^* = \Sigma_{i=1}^{n} T_{\bar{u}_i,\bar{w}_i}$ and $(\Sigma_{i=1}^{n} T_{w_i,u_i}) M_\nu = M_\nu(\Sigma_{i=1}^{n} T_{w_i,u_i})$, for all $\Sigma_{i=1}^{n} T_{w_i,u_i} \in \mathfrak{L}$ and $\nu \in L^\infty(\mathcal{A})$. Thus, $\mathfrak{L}$ is in fact a $*$-algebra and $\mathfrak{L}' \supseteq \{M_\nu : \nu \in L^\infty(\mathcal{A})\}$, where $\mathfrak{L}' = \{A \in B(L^2(\Sigma)) : AT = TA,\ \forall T \in \mathfrak{L}\}$ is the commutant of $\mathfrak{L}$. Put $\mathfrak{R} = \{EM_u : E(|u|^2) \in L^\infty(\mathcal{A})\}$. Then by [8, Theorem 3.1.2], $\mathfrak{R}' = \{M_\nu : \nu \in L^\infty(\mathcal{A})\}$ whenever $\mu(X) = 1$. Since $\mathfrak{R} \subseteq \mathfrak{L}$, then $\mathfrak{L}' \subseteq \mathfrak{R}'$. Thus, $A \in \mathfrak{L}'$ if and only if $A = M_\nu$ with $\nu \in L^\infty(\mathcal{A})$. Consequently, if (X, Σ, μ) be a probability measure space, then $\mathfrak{L}'$, the commutant of $\mathfrak{L}$, is $\{M_\nu : \nu \in L^\infty(\mathcal{A})\}$.

Lemma 2.17 [5,11] *Let $T \in \mathcal{K}$. Then the followings hold.*

(a) The Aluthge transformation of T is $\widetilde{T} = M_{\frac{E(uw)\bar{u}}{E(|u|^2)}} EM_u$.

(b) If T has closed range, then the Moore–Penrose inverse of T is

$$T^\dagger = M_{\frac{\bar{u}}{E(|u|^2)E(|w|^2)}} EM_{\bar{w}}.$$

Corollary 2.18 *The following assertions hold.*

(a) $\{\widetilde{T} : T \in \mathcal{K}\} \subseteq \mathcal{K}$.
(b) $(\mathcal{K} \cap B_C(L^2(\Sigma)))^\dagger \subseteq \mathcal{K} \cap B_C(L^2(\Sigma))$.
(c) For every $T \in \mathcal{K}$, $\widetilde{T}$ is always normal.
(d) U is normal if and only if so is $T = U|T| \in \mathcal{K}$.
(e) $T^\dagger$ is normal if and only if so is $T \in B_C(L^2(\Sigma))$.

Put $g_1 = E(|w|^2)^{1/2} E(|u|^2)^{-1/2}$, $g_2 = E(|w|^2)^{-1/2} E(|u|^2)^{-1/2}$ and $a = E(|u|^2)$. Then by [11, Lemma 2.1] and Lemma 2.17, we have

$$|T|^\lambda = (M_{g_1\bar{u}} EM_u)^\lambda = M_{(g_1^\lambda a^{\lambda-1})\bar{u}} EM_u;$$
$$|T|^{1-\lambda} = M_{(g_1^{1-\lambda} a^{-\lambda})\bar{u}} EM_u$$

and $U = M_{g_2 w} EM_u$. It follows that $|T|^\lambda U |T|^{1-\lambda} = M_{(g_1 g_2 E(uw))\bar{u}} EM_u$ and $\Delta_\lambda(T) = M_{b\bar{u}} EM_u$, where $b = g_1 g_2 E(uw) \in L^0(\mathcal{A})$. Then, by Proposition 2.11(b), $\Delta_\lambda(T)$ is always normal. Put $\mathcal{N} = \{M_w EM_u \in \mathcal{K} : M_w EM_u \text{ is normal}\}$. Then $\Delta_\lambda(\mathcal{K}) \subseteq \mathcal{N}$, and so $\mathcal{N}$ is invariant under Δ_λ. On the other hand, if $T = M_w EM_u \in \mathcal{N}$, then by Proposition 2.11(b), $w = g\bar{u}$ where $g = \frac{E(|w|^2)}{E(uw)} \chi_{\sigma(E(uw))}$. It follows that

$$b = g_1 g_2 E(uw) = \frac{gE(|u|^2)}{E(|u|^2)} = g\chi_{E(|u|^2)} = g.$$

Consequently, $\Delta_\lambda(\mathcal{N}) = \mathcal{N}$ and all points of $\mathcal{N}$ are fixed points for the λ-Aluthge transform Δ_λ. Also, it is easy to check that $|T|U = \widetilde{T}$ and by the conditional

Cauchy–Schwarz inequality, $\|\Delta_\lambda(T)\| = \|E(uw)\|_\infty \leq \|T\|$. So, if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then $\|\Delta_\lambda(T)\| = \|T\|$. Now, let $U^2 = U^*$. Then we have

$$M_{g_2^2 wE(uw)}EM_u = M_{g_2\bar{u}}EM_{\bar{w}}. \tag{2.13}$$

Put $f_n = \bar{u}\sqrt{E(|w|^2}\chi_{A_n}$. After substituting f_n in (2.13) and using the similar argument in Proposition 2.11(b), we obtain $g_2 wE(uw)E(|u|^2) = \bar{u}\overline{E(uw)}$. Then $w = g\bar{u}$, where $w = \frac{\overline{E(uw)}}{g_2 E(uw)E(|u|^2)}$.

We recall that $L^2(\mathcal{A})$ is a reducing subspace of T if and only if $w\chi_{\sigma(E(u))}$ and $u\chi_{\sigma(E(w))}$ are $\mathcal{A}$-measurable functions [12]. Now, let $T \in \mathcal{N}$. Then by Proposition 2.11(b), $T = M_{g\bar{u}}EM_u$ where $g = \frac{E(|w|^2)}{E(uw)}\chi_{\sigma(E(uw))}$. In view of matrix representation of T and T^*, $L^2(\mathcal{A})$ is invariant under T and T^* if and only if $(g\bar{u}_2)u_1 = 0$ and $(g\bar{u}_1)u_2 = 0$. Thus, $L^2(\mathcal{A})$ is a reducing subspace for T if and only if $u\chi_{\sigma(E(u))\cap\sigma(E(uw))} \in L^0(\mathcal{A})$. Note that $T|_{L^2(\mathcal{A})} = M_{wE(u)}$ is always normal and for $M_w EM_u \in \mathcal{N}$, $\sigma(E(uw)) = \sigma(E(|u|^2)) \cap \sigma(E(|w|^2))$.

Now, let $T \in B_C(L^2(\Sigma))$. Put $z = \frac{\chi_{\sigma(K)}}{K}$, where $K = E(|u|^2)E(|w|^2)$. Then, by Lemma 2.17(b), $T^\dagger = M_z T^*$. Thus, the matrix representation of $T^\dagger$ with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}_2(E)$ is

$$T^\dagger = \begin{bmatrix} M_z & 0 \\ 0 & M_z \end{bmatrix} \begin{bmatrix} M_{\overline{w_1 u_1}} & EM_{\overline{w_2 u_1}} \\ M_{\overline{w_1 u_2}} & M_{\bar{u}_2}EM_{\bar{w}_2} \end{bmatrix}.$$

It follows that

$$\begin{aligned} T^\dagger(L^2(\mathcal{A})) \subseteq L^2(\mathcal{A}) &\Longleftrightarrow zw_1u_2 = 0 \\ &\Longleftrightarrow zu\chi_{\sigma(E(w))} \in L^0(\mathcal{A}) \\ &\Longleftrightarrow u\chi_{\sigma(K)\cap\sigma(E(w))} \in L^0(\mathcal{A}) \\ &\Longleftrightarrow u\chi_{\sigma(E(|u|^2))\cap\sigma(E(w))} \in L^0(\mathcal{A}), \text{ since } \sigma(E(w)) \subseteq E(|w|^2). \end{aligned}$$

These observations establish the following result.

Theorem 2.19 *The following assertions hold.*

(a) $\Delta_\lambda(\mathcal{K}) \subseteq \mathcal{N}$ *and* $\Delta_\lambda(\mathcal{N}) = \mathcal{N}$.
(b) *For each* $0 < \lambda \leq 1$, $n \in \mathbb{N}$ *and* $T \in \mathcal{K}$, $\Delta_\lambda(T) = \widetilde{T} = \widehat{T} = \Delta_\lambda^n(T)$, *where* $\widehat{T} = |T|U$ *is the Duggal transformation of* T *and* $\Delta_\lambda^n(T)$ *is the n-times iterated* λ*–Aluthge transform of* T.
(c) *If* $U^2 = U^*$, *then* T *is normal.*
(d) $L^2(\mathcal{A})$ *is a reducing subspace of* $T \in \mathcal{N}$ *if and only if* $u\chi_B \in L^0(\mathcal{A})$, *where* $B = \sigma(E(|w|^2)) \cap \sigma(E(u))$.
(e) *Let* $T \in B_C(L^2(\Sigma))$. *Then* $L^2(\mathcal{A})$ *is a invariant subspace of* $T^\dagger$ *if and only if* $u\chi_C \in L^0(\mathcal{A})$, *where* $C = \sigma(E(|u|^2)) \cap \sigma(E(w))$.

Let T and S are in $B(\mathcal{H})$. The equality $(TS)^\dagger = S^\dagger T^\dagger$ is called the reverse order low for the Moore-Penrose inverse whenever both sides of equality be well defined.

Proposition 2.20 *Let $T = M_w E M_u$, $S = M_{w_1} E M_{u_1}$ and $\{T, S, TS\} \subseteq B_C(L^2(\Sigma))$. Then $(TS)^\dagger = S^\dagger T^\dagger$ if and only if $|E(uw_1)|^2 = E(|u|^2)E(|w_1|^2)$.*

Proof Put $K = E(|u|^2)E(|w|^2)$ and $K_1 = E(|u_1|^2)E(|w_1|^2)$. Direct computations show that $T^\dagger = M_{\frac{\bar{u}}{K}} E M_{\bar{w}}$, $S^\dagger = M_{\frac{\bar{u}_1}{K_1}} E M_{\bar{w}_1}$ and for all $f \in L^2(\Sigma)$ we have

$$(S^\dagger T^\dagger)(f) = \frac{\overline{E(uw_1)u_1}}{E(|u|^2)E(|w|^2)E(|u_1|^2)E(|w_1|^2)} E(\bar{w} f); \tag{2.14}$$

$$(TS)^\dagger(f) = \frac{\overline{E(uw_1)u_1}}{E(|u_1|^2)E(|w|^2)|E(uw_1)|^2} E(\bar{w} f). \tag{2.15}$$

Let $|E(uw_1)|^2 = E(|u|^2)E(|w_1|^2)$. Then by (2.14) and (2.15), $(TS)^\dagger = S^\dagger T^\dagger$. Conversely, let $(TS)^\dagger = S^\dagger T^\dagger$. Put

$$g = \overline{E(uw_1)} \left\{ \frac{1}{KK_1} - \frac{1}{E(|u_1|^2)E(|w|^2)|E(uw_1)|^2} \right\}.$$

Then $M_g T^* = M_{\bar{u}}((TS)^\dagger - S^\dagger T^\dagger) = 0$, and so

$$\|M_g T^*\| = \| |g| E(|u|^2)^{\frac{1}{2}} E(|w|^2)^{\frac{1}{2}} \|_\infty = 0.$$

Consequently, $g = 0$ and hence $|E(uw_1)|^2 = E(|u|^2)E(|w_1|^2)$. □

In view of matrix representation of T and T^* (2.2), we have

$$T^*T = \begin{bmatrix} M_{|u_1|^2 E(|w|^2)} & E M_{\bar{u}_1 u_2 E(|w|^2)} \\ M_{\bar{u}_2 u_1 E(|w|^2)} & M_{\bar{u}_2 E(|w|^2)} E M_{u_2} \end{bmatrix}.$$

Thus, for $\mathcal{A} \neq \Sigma$, T^*T, under any other conditions, can not be an isometry. However, it can be partial isometry. A bounded operator $T \in B(\mathcal{H})$ is a partial isometry if $\|Tx\| = \|x\|$ for all $x \in \mathcal{N}(T)^\perp$ or equivalently, $TT^*T = T$. Powers and roots of a partial isometry are not necessarily partial isometries.

Proposition 2.21 *Let $T \in \mathcal{K}$ and $K = E(|w|^2)E(|u|^2)$. Then the followings hold.*

(a) T is a partial isometry if and only if $K = 1$ on $\sigma(K)$.
(b) If T^{n_0} is a partial isometry for some $n_0 \geq 2$, then for all $n \in \mathbb{N}$, T^n is a partial isometry whenever $\|T\| \leq 1$.

Proof (a) Direct computations show that $TT^*T = M_K T$. Thus, T is a partial isometry iff $M_{K-1}T = M_{(K-1)w} E M_u = 0$ iff $\| |K-1| \sqrt{K} \|_\infty = \|M_{K-1}T\| = 0$ iff $K = \chi_{\sigma(K)}$.

(b) Put $S = \chi_{\sigma(E(uw))}$. Using (a), $|E(uw)|^{2(n_0-1)} K = \chi_S$. Since, by the conditional Cauchy–Schwarz inequality, $|E(uw)|^2 \leq K$ then $K^{n_0} \geq \chi_S$. On the other hand, because $K \leq \|T\|^2$ then $K \leq \chi_S$ whenever $\|T\| \leq 1$. Consequently, $K = \chi_S = |E(uw)|$, and hence $|E(uw)|^{2(n-1)} K = \chi_S$ for all $n \in \mathbb{N}$.

□

Douglas in [4] showed that $P \in B(L^1(\Sigma))$ is a contractive projection if and only if $P = M_{\bar{u}}EM_u$ for some $u \in L^0(\Sigma)$ with $|u| = 1$. The following theorem gives a necessary and sufficient condition for the Lambert conditional operator M_wEM_u on $L^2(\Sigma)$ to be an orthogonal projection.

Theorem 2.22 *Let $T \in \mathcal{K}$ be a nonzero operator. Then T is an orthogonal projection if and only if $w = \frac{\bar{u}}{E(|u|^2)}$.*

Proof Let $w = \frac{\bar{u}}{E(|u|^2)}$. Then it is easy to check that $\|T\| = 1$ and $T^2 = T = T^*$. Conversely, let T be an orthogonal projection. Using (2.2), the equality $T = T^*$ and $T^2 = T$ becomes:

$$\begin{bmatrix} M_{w_1u_1} & EM_{w_1u_2} \\ M_{w_2u_1} & M_{w_2}EM_{u_2} \end{bmatrix} = \begin{bmatrix} M_{\overline{w_1u_1}} & EM_{\overline{w_2u_1}} \\ M_{\overline{w_1u_2}} & M_{\bar{u}_2}EM_{\bar{w}_2} \end{bmatrix}. \tag{2.16}$$

and

$$\begin{cases} M_{w_1^2u_1^2} + EM_{w_1u_2w_2u_1} = M_{w_1u_1} & (1); \\ M_{w_1u_1}EM_{w_1u_2} + EM_{w_1u_2w_2}EM_{u_2} = EM_{w_1u_2} & (2); \\ M_{w_1u_1w_2u_1} + M_{w_2}EM_{w_2u_1u_2} = M_{w_2u_1} & (3); \\ M_{w_2u_1}EM_{w_1u_2} + M_{w_2}EM_{u_2w_2}EM_{u_2} = M_{w_2}EM_{u_2} & (4). \end{cases}$$

We consider the following cases;

$$\begin{cases} \text{(I) one of } u_i\text{'s is zero;} \\ \text{(I) one of } w_i\text{'s is zero;} \\ \text{(III) } u_2w_2 \in \mathcal{N}(E) \text{ and } w_1u_1 = 1; \\ \text{(IV) } w_2u_2 \in L^2(\mathcal{A}) \text{ and } w_1u_1 + w_2u_2 = 1. \end{cases}$$

(I) If $u_1 = 0$ or $u_2 = 0$, we have

$$u_1 = 0 \overset{(2.16)}{\Longrightarrow} w_1u_2 = 0 \overset{T\neq 0}{\Longrightarrow} w_1 = 0 \overset{(4)}{\Longrightarrow} E(u_2w_2) = 1 \Longrightarrow E(uw) = 1.$$

$$u_2 = 0 \overset{(2.16)}{\Longrightarrow} w_2u_1 = 0 \overset{T\neq 0}{\Longrightarrow} w_2 = 0 \overset{(1)}{\Longrightarrow} u_1w_1 = 1 \Longrightarrow E(uw) = 1.$$

(II) Suppose that one of w_i's is zero. Then we get that

$$w_1 = 0 \overset{(2.16)}{\Longrightarrow} w_2u_1 = 0 \overset{T\neq 0}{\Longrightarrow} u_1 = 0 \overset{(4)}{\Longrightarrow} E(u_2w_2) = 1 \Longrightarrow E(uw) = 1.$$

$$w_2 = 0 \overset{(2.16)}{\Longrightarrow} w_1u_2 = 0 \overset{T\neq 0}{\Longrightarrow} u_2 = 0 \overset{(1)}{\Longrightarrow} u_1w_1 = 1 \Longrightarrow E(uw) = 1.$$

In both cases (III) and (IV), we have $E(uw) = E(u_1w_1 + u_2w_2) = 1$. On the other hand, since T is self-adjoint, then by Proposition 2.11(a), $T = M_{g\bar{u}}EM_u$ for some

$g = \bar{g} \in L^0(\mathcal{A})$. It follows that $1 = E(uw) = gE(|u|^2)$, and so $\sigma(g) = X$. Thus, $g = \frac{1}{E(|u|^2)}$, and hence $T = M_{\frac{\bar{u}}{E(|u|^2)}} EM_u$. □

Let $T \in B(\mathcal{H})$. T is said to be generalized projection [19], if $T^2 = T^*$. In this case, $TT^* = T^*T = T^3$, and so T is normal. So, when $T \in \mathcal{K}$ be a nonzero generalized projection, then by Proposition 2.11(b), $T = M_{g\bar{u}} EM_u$ for some $g \in L^0(\mathcal{A})$.

Proposition 2.23 *Let $g \in L^0(\mathcal{A})$ and let $T = M_{g\bar{u}} EM_u \in \mathcal{K}$ be a nonzero generalized projection. Then $g = \frac{1}{E(|u|^2)} \chi_{\sigma(E(|u|^2))}$.*

Proof The equality $T^2 = T^*$ implies that $M_{g^2 E(|u|^2)\bar{u}} EM_u = M_{\overline{ug}} EM_u$ and hence $M_{(g^2 E(|u|^2) - \bar{g})} M_{\bar{u}} EM_u = 0$. Then by multiplying both sides of this by M_g we obtain $M_{(g^2 E(|u|^2) - \bar{g})} T = 0$. Thus, $\|(g^2 E(|u|^2) - \bar{g}) E(|\bar{u}g|^2) E(|u|^2)\|_\infty = 0$. Because $T \neq 0$, $g^2 E(|u|^2) = \bar{g}$. So if we multiply both sides of this by g, we conclude that $g \geq 0$ and $g = \frac{1}{E(|u|^2)} \chi_{\sigma(E(|u|^2))}$. This completes the proof. □

Corollary 2.24 *The following assertions hold.*

(a) $EM_u \in B(L^2(\Sigma))$ *is an orthogonal projection if and only if* $u = 1$ *on* X.
(b) *Let* $0 \neq T \in \mathcal{K}$ *with* $\sigma(E(|u|^2)) = X$. *Then* T *is an orthogonal projection if and only if* T *is a generalized projection on* $L^2(\Sigma)$.

Proposition 2.25 *Let $T \in \mathcal{K}$. Then T is quasinilpotent if and only if $E(uw) = 0$ on X.*

Proof Let $n \in \mathbb{N}$. By induction we have $T^n = M_{(E(uw))^{n-1} w} EM_u$, and hence $\|T^n\| = \|(E(uw))^{n-1} \sqrt{E(|u|^2) E(|w|^2)}\|_\infty$. It follows that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \|E(uw)\|_\infty = 0$, whenever $E(uw) = 0$ on X.

Conversely, suppose T is quasinilpotent. We show that $E(uw) = 0$. Suppose, on the contrary that there exists $\lambda \neq 0$ and $B \in \mathcal{A}$ with $0 < \mu(B) < \infty$ such that

$$E(uw)\chi_B = \lambda \chi_B. \tag{2.17}$$

Multiplying both sides of (2.17) by $\lambda^{n-1} w \sqrt{E(|u|^2} \chi_B$ we obtain

$$E(uw))^{n-1} w E(uw \sqrt{E(|u|^2} \chi_B) = \lambda^n w \sqrt{E(|u|^2} \chi_B. \tag{2.18}$$

Put $f = w\sqrt{E(|u|^2} \chi_B$. Evidently, f is in $L^2(\Sigma)$. Also f is nonzero because if $f = 0$, then $|E(uw)|^2 \chi_B \leq E(|u|^2) E(|w|^2) \chi_B = E(|f|^2) = 0$. So $E(uw)\chi_B = 0$, and hence $\lambda = 0$. But this is a contradiction. Now, from (2.18) we obtain that

$$0 = \lim_{n\to\infty} \|T^n f\|^{\frac{1}{n}} = \lim_{n\to\infty} |\lambda| \|f\|^{\frac{1}{n}} = |\lambda|.$$

But again, this is a contradiction. Thus, $E(uw) = 0$ on X. □

Corollary 2.26 *Let $T \in \mathcal{K}$ and $n_0 \geq 2$. If $T^{n_0} = 0$, then $E(uw) = 0$.*

Example 2.27 Let $X = \{1, 2, 3, 4\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/4$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{\{1, 3\}, \{2, 4\}\}$. Then $L^2(\Sigma) \cong \mathbb{C}^4$ and

$$\begin{aligned} E(f) &= \left(\frac{1}{\mu(A_1)} \int_{A_1} f d\mu\right) \chi_{A_1} + \left(\frac{1}{\mu(A_2)} \int_{A_2} f d\mu\right) \chi_{A_2} \\ &= \frac{f_1 + f_3}{2} \chi_{A_1} + \frac{f_2 + f_4}{2} \chi_{A_2}, \end{aligned}$$

where $A_1 = \{1, 3\}$ and $A_2 = \{2, 4\}$. Then matrix representation of E with respect to the standard orthonormal basis is

$$E = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

It can be easily checked that $E^2 = E = E^*$, $\mathcal{N}_2(E) = \langle (a, a, -a, -b) : a, b \in \mathbb{C} \rangle$, $\mathcal{R}(E) = \langle (a, b, a, b) : a, b \in \mathbb{C} \rangle$ and $\mathcal{R}(E) \perp \mathcal{N}_2(E)$. For $w = (w_1, w_2, w_3, w_4)$ and $u = (u_1, u_2, u_3, u_4)$ in $\mathbb{C}^4$ we have

$$\begin{aligned} T = M_w E M_u &= \begin{bmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & w_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & u_4 \end{bmatrix}. \\ &= \begin{bmatrix} \frac{w_1 u_1}{2} & 0 & \frac{w_1 u_3}{2} & 0 \\ 0 & \frac{w_2 u_2}{2} & 0 & \frac{w_2 u_4}{2} \\ \frac{w_3 u_1}{2} & 0 & \frac{w_3 u_3}{2} & 0 \\ 0 & \frac{w_4 u_2}{2} & 0 & \frac{w_4 u_4}{2} \end{bmatrix}. \end{aligned} \tag{2.19}$$

Let $u_i \neq 0$ and $w_i \neq 0$. Put $G = (E(|w|^2))^{1/2}(E(|u|^2))^{-1/2}$. Then

$$G(1) = G(3) = \sqrt{\frac{|w_1|^2 + |w_3|^2}{|u_1|^2 + |u_3|^2}};$$

$$G(2) = G(4) = \sqrt{\frac{|w_2|^2 + |w_4|^2}{|u_2|^2 + |u_4|^2}}.$$

For $1 \leq i \leq 4$, take $W_i = G(i)\bar{u}_i$. Then we have

$$|T| = M_W E M_u = \begin{bmatrix} \frac{W_1 u_1}{2} & 0 & \frac{W_1 u_3}{2} & 0 \\ 0 & \frac{W_2 u_2}{2} & 0 & \frac{W_2 u_4}{2} \\ \frac{W_3 u_1}{2} & 0 & \frac{W_3 u_3}{2} & 0 \\ 0 & \frac{W_4 u_2}{2} & 0 & \frac{W_4 u_4}{2} \end{bmatrix}. \tag{2.20}$$

Now, put $K^2 = E(|w|^2)E(|u|^2)$. Then we have

$$K(1) = K(3) = \frac{1}{\sqrt{(|w_1|^2 + |w_3|^2)(|u_1|^2 + |u_3|^2)}};$$
$$K(2) = K(4) = \frac{1}{\sqrt{(|w_2|^2 + |w_4|^2)(|u_2|^2 + |u_4|^2)}}.$$

Set $V_i = K(i)w_i$. Then

$$U = M_V E M_u = \begin{bmatrix} \frac{V_1u_1}{2} & 0 & \frac{V_1u_3}{2} & 0 \\ 0 & \frac{V_2u_2}{2} & 0 & \frac{V_2u_4}{2} \\ \frac{V_3u_1}{2} & 0 & \frac{V_3u_3}{2} & 0 \\ 0 & \frac{V_4u_2}{2} & 0 & \frac{V_4u_4}{2} \end{bmatrix}. \tag{2.21}$$

Put $b_i = K^2(i)$. Then we get that

$$T^\dagger = M_{K^2}T^* = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix} \begin{bmatrix} \frac{\bar{w}_1\bar{u}_1}{2} & 0 & \frac{\bar{w}_3\bar{u}_1}{2} & 0 \\ 0 & \frac{\bar{w}_2\bar{u}_2}{2} & 0 & \frac{\bar{w}_4\bar{u}_2}{2} \\ \frac{\bar{w}_1\bar{u}_3}{2} & 0 & \frac{\bar{w}_3\bar{u}_3}{2} & 0 \\ 0 & \frac{\bar{w}_2\bar{u}_4}{2} & 0 & \frac{\bar{w}_4\bar{u}_4}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{b_1\bar{w}_1\bar{u}_1}{2} & 0 & \frac{b_1\bar{w}_3\bar{u}_1}{2} & 0 \\ 0 & \frac{b_2\bar{w}_2\bar{u}_2}{2} & 0 & \frac{b_2\bar{w}_4\bar{u}_2}{2} \\ \frac{b_1\bar{w}_1\bar{u}_3}{2} & 0 & \frac{b_1\bar{w}_3\bar{u}_3}{2} & 0 \\ 0 & \frac{b_2\bar{w}_2\bar{u}_4}{2} & 0 & \frac{b_2\bar{w}_4\bar{u}_4}{2} \end{bmatrix}. \tag{2.22}$$

Also, since

$$E(|u|^2) = \left(\frac{|u_1|^2 + |u_3|^2}{2}, \frac{|u_2|^2 + |u_4|^2}{2}, \frac{|u_1|^2 + |u_3|^2}{2}, \frac{|u_2|^2 + |u_4|^2}{2}\right);$$
$$E(uw) = \left(\frac{u_1w_1 + u_3w_3}{2}, \frac{u_2w_2 + u_4w_4}{2}, \frac{u_1w_1 + u_3w_3}{2}, \frac{u_2w_2 + u_4w_4}{2}\right),$$

then

$$\frac{E(uw)}{E(|u|^2)} = (a_1, a_2, a_1, a_2), \text{ where } a_1 = \frac{u_1w_1 + u_3w_3}{|u_1|^2 + |u_3|^2} \text{ and } a_2 = \frac{u_2w_2 + u_4w_4}{|u_2|^2 + |u_4|^2}.$$

Consequently,

$$\widetilde{T} = M_{\frac{E(uw)}{E(|u|^2)}}(M_{\bar{u}}EM_u) = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} \frac{|u_1|^2}{2} & 0 & \frac{\bar{u}_1u_3}{2} & 0 \\ 0 & \frac{|u_2|^2}{2} & 0 & \frac{\bar{u}_2u_4}{2} \\ \frac{\bar{u}_3u_1}{2} & 0 & \frac{|u_3|^2}{2} & 0 \\ 0 & \frac{\bar{u}_4u_2}{2} & 0 & \frac{|u_4|^2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_1|u_1|^2}{2} & 0 & \frac{a_1\bar{u}_1u_3}{2} & 0 \\ 0 & \frac{a_2|u_2|^2}{2} & 0 & \frac{a_2\bar{u}_2u_4}{2} \\ \frac{a_1\bar{u}_3u_1}{2} & 0 & \frac{a_1|u_3|^2}{2} & 0 \\ 0 & \frac{a_2\bar{u}_4u_2}{2} & 0 & \frac{a_2|u_4|^2}{2} \end{bmatrix}. \tag{2.23}$$

Now, set $u = (1, i, 2, 2i)$ and $w = (i, 1, 2i, 2)$. It is easy to check that $G = (1, 1, 1, 1)$, $K = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$, $W = \bar{u}$ and $V = \frac{2}{5}w$. Then by (2.19), (2.20) and (2.21) we have

$$T = \begin{bmatrix} \frac{i}{2} & 0 & i & 0 \\ 0 & \frac{i}{2} & 0 & i \\ i & 0 & 2i & 0 \\ 0 & i & 0 & 2i \end{bmatrix};$$

$$|T| = \begin{bmatrix} \frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} \frac{i}{5} & 0 & \frac{2i}{5} & 0 \\ 0 & \frac{i}{5} & 0 & \frac{2i}{5} \\ \frac{2i}{5} & 0 & \frac{4i}{5} & 0 \\ 0 & \frac{2i}{5} & 0 & \frac{4i}{5} \end{bmatrix}.$$

It follows that $U|T| = |T|U$, $T^* \neq T$, but $TT^* = T^*T$, and so $\widetilde{T} = T$ and $|E(uw)|^2 = E(|u|^2)E(|w|^2)$. Also, by (2.22) and (2.23) we have

$$T^\dagger = \frac{4}{25} \begin{bmatrix} \frac{-i}{2} & 0 & -i & 0 \\ 0 & \frac{-i}{2} & 0 & -i \\ -i & 0 & -2i & 0 \\ 0 & -i & 0 & -2i \end{bmatrix}.$$

Finally, take $u = (i, -1, 2i, 3)$ and $w = (\frac{-2}{5}i, \frac{-1}{5}, \frac{-4}{5}i, \frac{3}{5})$. Direct computations show that

$$T = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{1}{10} & 0 & \frac{-3}{10} \\ \frac{2}{5} & 0 & \frac{4}{5} & 0 \\ 0 & \frac{-3}{10} & 0 & \frac{9}{10} \end{bmatrix} = T^2 = T^*.$$

Indeed, $E(|u|^2) = (\frac{5}{2}, 5, \frac{5}{2}, 5)$ and $\frac{\bar{u}}{E(|u|^2)} = w$. It follows by Theorem 2.22 that T is a orthogonal projection. Moreover, $\mathcal{N}(T) = \langle (-2a, 3b, a, b) : a, b \in \mathbb{C} \rangle$, $\mathcal{R}(T) = \langle (a, b, 2a, -3b) : a, b \in \mathbb{C} \rangle$.

Let (X, Σ, μ) be a complete probability space We recall that an $\mathcal{A}$-atom of the measure μ is an element $C \in \mathcal{A}$ with $\mu(C) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq C$ then either $\mu(F) = 0$ or $\mu(F) = \mu(C)$. Let $\mathcal{A} = \{\emptyset, X\}$. Then X is an $\mathcal{A}$-atom. Since every $L^2(\mathcal{A})$-function is constant on any $\mathcal{A}$-atom, then we have

$$E(f) = E(f)\mu(X) = \int_X E(f)d\mu = \int_X f dA, \quad f \in L^2(\Sigma).$$

Since E is a contraction, $E(f) \in L^\infty(\Sigma)$ whenever so is f. But the converse, in general, is not true. For this, let $f \in L^2(\Sigma) \setminus L^\infty(\Sigma)$. Since $|E(f)| \le E(|f^2|)^{1/2} = (\int_X |f|^2 d\mu)^{1/2} = \|f\|_2 < \infty$, then $E(f) \in L^\infty(\Sigma)$ whence f is not in $L^\infty(\Sigma)$.

Now, let $\mathcal{A} = \langle X_i \rangle_{i \in I}$ be the σ-algebra generated by the countable collection of the non-null disjoint measurable subsets of X. As the same way, $E(f)$ is constant on any $\mathcal{A}$-atom X_i. Then for all $f \in L^2(\Sigma)$ and $i \in I$, $\mu(X_i)E(f)\chi_{X_i} = (\int_{X_i} f dA)\chi_{X_i}$. It follows that

$$E(f) = \sum_{i=1}^{\infty} \frac{1}{\mu(X_i)} \left(\int_{X_i} f d\mu \right) \chi_{X_i}.$$

Example 2.28 Let $X = [0, 1]$, $d\mu = dx$, Σ be the Lebesgue measurable sets and let $\mathcal{A} = \{\emptyset, X\}$. Then $Tf(x) = w(x)E(uf)(x) = w(x) \int_0^1 u(x) f(x)dx$ and $T^* f(x) = u(x) \int_0^1 w(x) f(x)dx$ for all $f \in L^2(\Sigma)$. In this case f is $\mathcal{A}$-measurable if and only if f is a constat function. So, $L^2(\mathcal{A}) \cong \mathbb{R}$ and $\mathcal{N}_2(E) = \{f \in L^2(\Sigma) : \int_0^1 f(x)dx = 0\}$. Using the previous results, $AT = TA$ if and only if $A = \alpha I$ for some $\alpha \in \mathbb{R}$.

Put $u(x) = -2x^2 - x + 1$, $w(x) = \frac{1}{x+1}$. Then $E(uw)(x) = \int_0^1 (-2x + 1)dx = 0$, $E(|u|^2) = \frac{4}{3}$ and $E(|w|^2) = \frac{1}{2}$. Thus, $Tf(x) = \frac{1}{x+1} \int_0^1 (-2x^2 - x + 1) f(x)dx$ with $\|T\|^2 = \frac{2}{3}$ is quasinilpotent and so $\text{Spec}(T) = \{0\}$. Also, $TT^* \neq T^*T$ but T has closed range, $T^\dagger f(x) = \frac{3}{2}(-2x^2 - x + 1) \int_0^1 \frac{f(x)}{x+1} dx$ and $\widetilde{T} = 0$.

Now, if we take $u(x) = 2x+1$ and $w(x) = e^x$, then we have $u_1(x) = E(u)(x) = 3$, $w_1(x) = E(w)(x) = e - 1$, $u_2(x) = 2x - 2$ and $w_2(x) = e^x - e + 1$. Thus the matrix representation of T with respect to the decomposition $L^2(\Sigma) = \mathbb{R} \oplus \mathcal{N}_2(E)$ is

$$T \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 3(e-1) f_1 & 2(e-1) \int_0^1 (x-1) f_2(x)dx \\ 3(e^x - e + 1) f_1 & 2(e^x - e + 1) \int_0^1 (x-1) f_2(x)dx \end{bmatrix},$$

where $f_1 = \int_0^1 f(x)dx$ and $f_2 = f - \int_0^1 f(x)dx$.

Finally, if $u = \sqrt{2}\chi_{[0,\frac{1}{2}]}$ and $w = \sqrt{3}\chi_{[0,\frac{1}{3}]}$ then $E(|u|^2) = 1 = E(|w|^2)$ and $|E(uw)| = \frac{\sqrt{6}}{3}$. Thus, T is a partial isometry but neither T^2 nor T^n is a partial isometry.

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