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Generalized Inverses of Conditional Type Operators

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Abstract

In this paper, some characterizations of the Drazin and the Moore–Penrose inverses of the conditional type operators on $L^2(\Sigma)$ are established.

Keywords Conditional expectation · Drazin inverse · Moore–penrose inverse · Finite-rank

Mathematics Subject Classification Primary 47B20; Secondary 47B25

1 Introduction and Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces with inner product $\langle\, ,\, \rangle$. Let $B(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and let $B_C(\mathcal{H}, \mathcal{K})$ be the subspace of all $T \in B(\mathcal{H}, \mathcal{K})$ such that the range of T is closed in $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we write $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{K})$ and $B_C(\mathcal{H}) = B_C(\mathcal{H}, \mathcal{H})$. For $T \in B(\mathcal{H}, \mathcal{K})$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the kernel and the range of T, respectively. The Moore–Penrose inverse of $T \in B(\mathcal{H}, \mathcal{K})$ is the operator $S \in B(\mathcal{K}, \mathcal{H})$ which satisfies the Penrose equations

$$(1)\ TST = T, \quad (2)\ STS = S, \quad (3)\ (TS)^* = TS, \quad (4)\ \ (ST)^* = ST. \tag{1.1}$$

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The Moore–Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed in $\mathcal{K}$. If the Moore–Penrose inverse of T exists, then it is unique, and it is denoted by $T^{\dagger}$. Let $T\{i, \ldots, j\}$ denote the set of all operators S which satisfy the equations $(1) \leq (i), \ldots, (j) \leq (4)$. In this case $S \in T\{i, \ldots, j\}$ is a $\{i, \ldots, j\}$-inverse of T and is denoted by $T^{(i,\ldots,j)}$. Noth that $T^{(1,2,3,4)} = T^{\dagger}$. An element $T \in B(\mathcal{H})$ is said to have a Drazin inverse, or T is Drazin invertible if there exists $S \in B(\mathcal{H})$ such that $STS = S$, $TS = ST$ and $T^{k+1}S = T^k$ for some $k \in \mathbb{N}$. The minimal such k is called the Drazin index of T, and will be denoted by $ind(T)$. If T has Drazin inverse, then it is unique and denoted by T^D. When $k = 1$, the Drazin inverse reduced to the group inverse and it is denoted by $T^{\#}$. Recall that $\mathrm{asc}(T)$ and $\mathrm{des}(T)$, the ascent and descent of $T \in B(\mathcal{H})$, is the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$, respectively. It is well known that $\mathrm{asc}(T) = \mathrm{des}(T)$ if $\mathrm{asc}(T)$ and $\mathrm{des}(T)$ are finite (see [16]). For $T \in B(\mathcal{H})$, T^D exists if and only if T has finite ascent and descent. In this case, $\mathrm{ind}(T) = \mathrm{asc}(T) = \mathrm{des}(T) = n$. For other important properties of $T^{\dagger}$ and T^D, see e.g. [1,3]. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $T \in B(\mathcal{H})$ and let $Pj : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto $\mathcal{H}_j$ for $j = 1, 2$. Then $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ is the operator given by $T_{ij} = P_i T P_j \mid_{\mathcal{H}_j}$. In particular, $T(\mathcal{H}_1) \subseteq \mathcal{H}_1$ if and only if $T_{21} = 0$. Also, $\mathcal{H}_1$ reduces T if and only if $T_{12} = 0 = T_{21}$.

Let (X, Σ, μ) be a sigma-finite measure space and let $\mathcal{A}$ be a sigma-finite subalgebra of Σ. The space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$ and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all finite-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of a measurable function $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. An $\mathcal{A}$-atom of the measure μ is an element $B \in \mathcal{A}$ with $\mu(B) > 0$ such that for each $A \in \mathcal{A}$, if $A \subseteq B$ then either $\mu(A) = 0$ or $\mu(A) = \mu(B)$. As is well known, a sigma-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is decomposed into two disjoint sets Y and Z, where Y does not possess any $\mathcal{A}$-atoms and Z is a countable union of $\mathcal{A}$-atoms of finite measure. Moreover, we can easily check that $f_{|B}$, the restriction of $f \in L^0(\mathcal{A})$ to an $\mathcal{A}$-atom B, is constant (see [18]). Let $f \in L^2(\Sigma)$. Then $E(f)$, the conditional expectation of f, is the unique $\mathcal{A}$-measurable function such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu, \quad \forall A \in \mathcal{A}. \tag{1.2}$$

Put $\mathcal{D}(E) = \{f \in L^0(\Sigma) : E(|f|) \in L^0(\mathcal{A})\}$. Then $\mathcal{D}(E)$, the domain of E, contains $\{L^p(\Sigma) : 1 \leq p \leq \infty\} \cup \{f \in L^0(\Sigma) : f \geq 0\}$ (see [7]). As an operator on $L^2(\Sigma)$, $E := E^{\mathcal{A}}$ is an orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. Conditional expectation operator E will play a major role in our work. A detailed discussion and verification of most of properties of E may be found in [2,4,8,11,13,14,17]. Those properties of E used in our discussion are summarized below. In all cases we assume that $f, fg \in \mathcal{D}(E)$.

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- $\sigma(E(|f|))$ is the smallest $\mathcal{A}$-measurable set containing $\sigma(f)$.

∘ (Conditional Cauchy-Schwarz) $|E(fg)|^2 \leq E(|f|^2)E(|g|^2)$.

From now on we assume that $\{u, w, uw\} \subseteq \mathcal{D}(E)$. Operators of the form $M_w EM_u(f) = wE(uf)$ acting in $L^2(\Sigma)$ with $\mathcal{D}(M_w EM_u) = \{f \in L^2(\Sigma) : wE(uf) \in L^2(\Sigma)\}$ are called weighted conditional type operators. Several aspects of this operator were studied in [6,8–10,13]. Put $K = E(|u|^2)E(|w|^2)$. Estaremi [5] proved that $M_w EM_u : \mathcal{D}(T) \to L^2(\Sigma)$ is densely defined if and only if $K - 1$ is finite valued. Moreover, $T := M_w EM_u \in B(L^2(\Sigma))$ if and only if $\mathcal{D}(T) = L^2(\Sigma)$. In this case $T^* = M_{\bar{u}} EM_{\bar{w}}$ and $\|T\|^2 = \|K\|_\infty$.

Conditional operators and the various types of generalized inverse have been widely used in practise. In the next section we prove some basic results on the Drazin and Moore–Penrose inverse of conditional type operators on $L^2(\Sigma)$. Moreover, we provide a necessary and sufficient condition for special type products of general operators so that the reverse order low for the Moore–Penrose inverse is satisfied. Finally, one example is provided to illustrate the obtained results.

2 Characterizations

Put $K = E(|u|^2)E(|w|^2)$ and set $\mathcal{L} = \{M_w EM_u : u, w, uw \in \mathcal{D}(E), K \in L^\infty(\Sigma)\}$. For$\{M_w EM_u, M_r EM_s\} \subseteq \mathcal{L}$, $(M_w EM_u)(M_r EM_s) = M_{wE(ur)} EM_s \in \mathcal{L}$ and hence $\mathcal{L}$ is closed under multiplication. So, for $T \in B_C(L^2(\Sigma))$, one might guess that $\{T^\dagger, T^D\} \subseteq \mathcal{L}$. Recall that if $\mathcal{R}(T)$ is closed, then T is bounded below on $\mathcal{N}(T)^\perp$, i.e., there is $c > 0$ such that$\|Tf\| \geq c\|f\|$ for all $f \in \mathcal{N}(T)^\perp$. Now we shall prove the following lemma.

Lemma 2.1 *Let $T = M_w EM_u \in B_C(L^2(\Sigma))$. Then K is bounded away from zero on $\sigma(K)$.*

Proof Suppose K is not bounded away from zero on $\sigma(K)$. Then for fixed $\varepsilon > 0$, there exists $A \in \mathcal{A}$ with $A \subseteq S$ and $0 < \mu(A) < \infty$ such that $K\chi_A < \varepsilon$. Put $f_0 = \bar{u}\sqrt{E(|w|^2)}\chi_A$. Then for each $g \in \mathcal{N}(T)$ we have

$$\begin{aligned}
|\langle g, f_0\rangle|^2 &= |\int_A ug\sqrt{E(|w|^2)}d\mu|^2 = |\int_A E(u\sqrt{E(|w|^2)}g)d\mu|^2 \\
&\leq \int_X E(|w|^2)|E(ug)|^2 d\mu = \int_X |wE(ug)|^2 d\mu = \int_X |Tg|^2 d\mu = 0.
\end{aligned}$$

It follows that $f_0 \in L^2(\Sigma) \cap \mathcal{N}(T)^\perp$ and satisfies

$$\begin{aligned}
\|Tf_0\|^2 &= \|wE(|u|^2)\sqrt{E(|w|^2)}\chi_A\|^2 = \int_X K^2\chi_A d\mu \\
&\leq \varepsilon \int_X K\chi_A d\mu = \varepsilon \int_X |\bar{u}\sqrt{E(|w|^2)}\chi_A|^2 d\mu = \varepsilon\|f_0\|^2.
\end{aligned}$$

But this is a contradiction. □

Now, for $T \in B_C(L^2(\Sigma))$, set $S = M_{\frac{\chi_{\sigma(K)}}{K}} T^*$. Then by Lemma 2.1, $S \in B(L^2(\Sigma))$ and $TST = M_{\frac{\chi_{\sigma(K)}}{K}}(TT^*T) = M_{\frac{\chi_{\sigma(K)}}{K}}(M_{Kw}EM_u) = M_{w\chi_{\sigma(K)}}EM_u = T$. Also, we have $STS = M_{\frac{\chi_{\sigma(K)}}{K^2}}(T^*TT^*) = M_{\frac{\chi_{\sigma(K)}}{K^2}}(M_{K\bar{u}}EM_{\bar{w}}) = M_{\frac{\chi_{\sigma(K)}}{K}}T^* = S$, $(TS)^* = (M_{\frac{\chi_{\sigma(K)}}{K}}TT^*)^* = TS$ and $(ST)^* = (M_{\frac{\chi_{\sigma(K)}}{K}}T^*T)^* = ST$. Thus, $S = T^\dagger$. Since $T^* = M_{\bar{u}}EM_{\bar{w}}$, $\sigma(\bar{u}) \subseteq \sigma(E(|u|^2))$, $\chi_{\sigma(K)} = \chi_{\sigma(E(|u|^2))}\chi_{\sigma(E(|w|^2))}$ and that $T^\dagger$ has closed range, then $\bar{u}\chi_{\sigma(E(|u|^2))} = \bar{u}$ and

$$T^\dagger = M_{\frac{\bar{u}\chi_{\sigma(E(|w|^2))}}{E(|u|^2)E(|w|^2)}} EM_{\bar{w}} \in B_C(L^2(\Sigma)). \tag{2.1}$$

In particular, if $u = \bar{w}$ then it is easy to check that T and $T^\dagger$ are positive operators.

Proposition 2.2 *Let $T = M_wEM_u \in B_C(L^2(\Sigma))$ and let $S = M_\alpha EM_u \in \mathcal{L}$. If $E(uw)E(u\alpha) = 1$ and $wE(u\alpha) = \alpha E(uw)$, then $T^D = S \in B_C(L^2(\Sigma))$.*

Proof By hypothesis we have

$$\begin{aligned} STS &= (M_\alpha EM_u)(M_wEM_u)(M_\alpha EM_u) \\ &= M_{\alpha E(uw)E(u\alpha)}EM_u \\ &= M_\alpha EM_u = S; \end{aligned}$$

$$\begin{aligned} TS &= (M_wEM_u)(M_\alpha EM_u) \\ &= M_{wE(u\alpha)}EM_u \\ &= M_{\alpha E(uw)}EM_u \\ &= M_\alpha EM_uM_wEM_u = ST. \end{aligned}$$

Since for each $k \in \mathbb{N}$, $T^k = M_{w(E(uw))^{k-1}}EM_u$, we obtain

$$\begin{aligned} T^{k+1}S &= (M_{w(E(uw))^k}EM_u)(M_\alpha EM_u) \\ &= M_{w(E(uw))^kE(u\alpha)}EM_u \\ &= M_{w(E(uw))^{k-1}}EM_u = T^k. \end{aligned}$$

Note that the equality $E(uw)E(u\alpha) = 1$ and the conditional Cauchy-Schwarz inequality implies that $\sigma(E(uw)) = X$ and $E(|\alpha|^2)E(|u|^2) \geq |E(\alpha u)|^2 = |E(uw)|^{-2} \geq K^{-2} \geq \|T\|^{-4}$. Moreover, since $ind(T) = 1$, then $T^D = T^\# = S \in B_C(L^2(\Sigma))$. □

Set $\mathfrak{R} = \{EM_u : u \in \mathcal{D}(E)$ and $E(|u|^2) \in L^\infty(\mathcal{A})\}$. Then by [8, Theorem 3.1.2], $\mathfrak{R}' = \{M_\nu : \nu \in L^\infty(\mathcal{A})\}$, where $\mathfrak{R}' = \{A \in B(L^2(\Sigma)) : AT = TA,\ \forall T \in \mathfrak{R}\}$ is the commutant of $\mathfrak{R}$. It follows that $\mathcal{L}' = \mathfrak{R}'$. Let $T \in \mathcal{L}$ be Drazin invertible. Since $TT^D = T^DT$, then it seems that T^D has a factorization of the form $M_\nu T$ for some $\nu \in L^\infty(\mathcal{A})$.

Relative to the direct sum decomposition $L^2(\Sigma) = \mathcal{R}(E) \oplus \mathcal{N}(E)$, any element f of $L^2(\Sigma)$ can be written uniquely as $f = f_1 + f_2$ where $f_1 = E(f) \in L^2(\mathcal{A})$

and $f_2 = f - E(f) \in \mathcal{N}(E)$. Since $E(|f|^2) = E((f_1 + f_2)(\bar{f_1} + \bar{f_2})) = |f_1|^2 + E(|f_2|^2)$, then $\max\{|f_1|^2, E(|f_2|^2)\} \le E(|f|^2)$. In the following we calculate matrix representation of the Drazin inverse of $T \in \mathcal{L}$ with respect to the decomposition $L^2(\Sigma) = R(E) \oplus N(E)$.

Theorem 2.3 *Let $T = M_w E M_u \in \mathcal{L}$ and let $C = \sigma(E(uw))$. If $E(uw)$ is bounded away from zero on C, then T is Drazin invertible and*

$$T^D = \begin{pmatrix} M_{\frac{w_1 u_1 \chi_C}{E(uw)^2}} & E M_{\frac{w_1 u_2 \chi_C}{E(uw)^2}} \\ M_{\frac{w_2 u_1 \chi_C}{E(uw)^2}} & M_{\frac{w_2 \chi_C}{E(uw)^2}} E M_{u_2} \end{pmatrix} = M_{\frac{\chi_C}{(E(uw))^2}} T. \tag{2.2}$$

In particular, $E^D = E$.

Proof First, we recall that (see [10]) the matrix representation of $T = M_w E M_u \in \mathcal{L}$ with respect to the direct sum decomposition $L^2(\Sigma) = R(E) \oplus N(E)$ is

$$T = \begin{pmatrix} M_{w_1 u_1} & E M_{w_1 u_2} \\ M_{w_2 u_1} & M_{w_2} E M_{u_2} \end{pmatrix}. \tag{2.3}$$

For $\nu \in L^\infty(\mathcal{A})$, set

$$S = \begin{pmatrix} M_{w_1 u_1 \nu} & E M_{w_1 u_2 \nu} \\ M_{w_2 u_1 \nu} & M_{w_2 \nu} E M_{u_2} \end{pmatrix}.$$

Since ν is an $\mathcal{A}$-measurable function, then $TS = ST$. Using (2.3), we have

$$T^2 = \begin{pmatrix} M_{w_1 u_1 (w_1 u_1 + E(w_2 u_2))} & E M_{w_1 u_2 (w_1 u_1 + E(w_2 u_2))} \\ M_{w_2 u_1 (w_1 u_1 + E(w_2 u_2))} & M_{w_2 (w_1 u_1 + E(w_2 u_2))} E M_{u_2} \end{pmatrix}.$$

Put $a = w_1 u_1 + E(w_2 u_2)$. Then $a = E(uw) \in L^\infty(\mathcal{A})$ and

$$T^2 = \begin{pmatrix} M_{w_1 u_1 a} & E M_{w_1 u_2 a} \\ M_{w_2 u_1 a} & M_{w_2} E M_{u_2 a} \end{pmatrix}.$$

By induction on $k \in \mathbb{N}$, it is easy to check that

$$T^k = \begin{pmatrix} M_{w_1 u_1 a^{k-1}} & E M_{w_1 u_2 a^{k-1}} \\ M_{w_2 u_1 a^{k-1}} & M_{w_2 a^{k-1}} E M_{u_2} \end{pmatrix},$$

and so

$$\begin{aligned} T^{k+1} S &= \begin{pmatrix} M_{w_1 u_1 a^k} & E M_{w_1 u_2 a^k} \\ M_{w_2 u_1 a^k} & M_{w_2 a^k} E M_{u_2} \end{pmatrix} \begin{pmatrix} M_{w_1 u_1 \nu} & E M_{w_1 u_2 \nu} \\ M_{w_2 u_1 \nu} & M_{w_2 \nu} E M_{u_2} \end{pmatrix} \\ &= \begin{pmatrix} M_{w_1 u_1 a^k \nu (w_1 u_1 + E(w_2 u_2))} & E M_{w_1 u_2 a^k \nu (w_1 u_1 + E(w_2 u_2))} \\ M_{w_2 u_1 a^k \nu (w_1 u_1 + E(w_2 u_2))} & M_{w_2 a^k \nu (w_1 u_1 + E(w_2 u_2))} E M_{u_2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} M_{w_1u_1a^{k+1}v} & EM_{w_1u_2a^{k+1}v} \\ M_{w_2u_1a^{k+1}v} & M_{w_2a^{k+1}v}EM_{u_2} \end{pmatrix}.$$

Thus, $T^{k+1}S = T^k$ whenever $a^{k+1}v = a^{k-1}$. Hence

$$S = \begin{pmatrix} M_{\frac{w_1u_1\chi_{\sigma(a)}}{a^2}} & EM_{\frac{w_1u_2\chi_{\sigma(a)}}{a^2}} \\ M_{\frac{w_2u_1\chi_{\sigma(a)}}{a^2}} & M_{\frac{w_2\chi_{\sigma(a)}}{a^2}}EM_{u_2} \end{pmatrix}.$$

In fact, $S = M_{\frac{\chi_{\sigma(a)}}{a^2}}T$, and so $STS = S$. Consequently, $S = T^D$. □

Recall that if $T = U|T|$ is the polar decomposition of $T \in B(\mathcal{H})$, then $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the Aluthge transformation of T. Put $g = \frac{E(uw)\bar{u}}{E(|u|^2)}$. By a similar argument used in Theorem 2.2, we have

$$\widetilde{T} = \begin{pmatrix} M_{g_1u_1} & EM_{g_1u_2} \\ M_{g_2u_1} & M_{g_2}EM_{u_2} \end{pmatrix} = M_{\frac{E(uw)\bar{u}}{E(|u|^2)}}EM_u. \tag{2.4}$$

Theorem 2.4 *Let $T = M_wEM_u \in \mathcal{L}$ and let $|E(uw)| \geq \delta$ for some $\delta > 0$ on $\sigma(E(uw))$. Then T is Drazin invertible with $ind(T) = 2$.*

Proof Put $a = E(uw)$ and let $g \in \mathcal{R}(T^2)$. Then $g = waE(uf)$ for some $f \in L^2(\Sigma)$. Since

$$\int_X |\frac{f\chi_{\sigma(a)}}{a^{n-1}}|^2 d\mu = \int_X \frac{|f|^2}{|a|^{2(n-1)}}d\mu \leq \frac{\|f\|_2^2}{\delta^{2(n-1)}} < \infty,$$

then

$$g = wa^nE\left(u\frac{f\chi_{\sigma(a)}}{a^{n-1}}\right) = T^{n+1}\left(\frac{f\chi_{\sigma(a)}}{a^{n-1}}\right) \in \mathcal{R}(T^{n+1}).$$

Thus, $\mathcal{R}(T^2) = \mathcal{R}(T^{n+1})$ for all $n \in \mathbb{N}$. Now, let $T^{n+1}f = 0$. Then $wa^nE(uf) = 0$. Put $h = \frac{\chi_{\sigma(a)}}{a^{n-1}}$. Then $\|h\|_\infty \leq \frac{1}{\delta^{n-1}} < \infty$, and so

$$T^2f = waE(uf) = \frac{\chi_{\sigma(a)}}{a^{n-1}}wa^nE(uf) = M_hT^{n+1}f = 0.$$

Hence, $\mathcal{N}(T^{n+1}) = \mathcal{N}(T^2)$ for all $n \in \mathbb{N}$. Consequently, $\mathrm{ind}(T) = \mathrm{asc}(T) = \mathrm{des}(T) = 2$. □

Lemma 2.5 *Let $\{u, w, uw\} \subseteq \mathcal{D}(E)$.Then the following assertions hold.*

(a) $E(uw) = uE(w)$ *iff* $u\chi_{E(w)} \in L^0(\mathcal{A})$.
(b) $M_{w_2u_1} = 0$ *iff* $w_2u_1 = 0$ *iff* $w\chi_{E(u)} \in L^0(\mathcal{A})$.
(c) $EM_{w_1u_2} = 0$ *iff* $w_1u_2 = 0$ *iff* $u\chi_{E(w)} \in L^0(\mathcal{A})$.

Proof (a) Let $u\chi_{E(w)} \in L^0(\mathcal{A})$. Then $uE(w) = u\chi_{E(w)}E(w) = E(uw\chi_{E(w)}) = E(uw)$. Conversely, if $E(uw) = uE(w)$ then $u\chi_{E(w)} = \frac{E(uw)}{E(w)}\chi_{E(w)} \in L^0(\mathcal{A})$.

(b) Let $M_{w_2u_1}(f) = 0$ for all $f \in L^2(\mathcal{A})$. Since $\mathcal{A}$ is sigma-finite, there exists $\{A_n\}_n \subseteq \mathcal{A}$ such that $X = \cup_n A_n$, $A_n \subseteq A_{n+1}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. In this case $\chi_{A_n} \nearrow \chi_X$. Put $f = \chi_{A_n}$. Then $w_2u_1\chi_{A_n} = 0$. It follows that $w_2u_1 = 0$ as $n \to \infty$. The converse is obvious. On the other hand,

$$\begin{aligned} w_2u_1 = 0 &\Longleftrightarrow (w - E(w))E(u) = 0 \\ &\Longleftrightarrow wE(u) = E(w)E(u) \\ &\Longleftrightarrow wE(u) = E(wE(u)) \\ &\Longleftrightarrow wE(E(u)) = E(wE(u)) \\ &\Longleftrightarrow w\chi_{E(u)} \in L^0(\mathcal{A}), \quad \text{(by part (a))}. \end{aligned}$$

(c) Let $M_{w_1}EM_{u_2} = EM_{w_1u_2} = 0$ on $\mathcal{N}(E)$. Then $w_1u_2E(u_2f) = 0$ for all $f \in \mathcal{N}(E)$. Put $f = w_1\bar{u}_2\chi_{A_n}$ as in the previous part of the proof. Then $\|f\|^2 = \int_{A_n} |w_1|^2|u_2|^2 d\mu \leq \int_{A_n} E(|w|^2)E(|u|^2)d\mu \leq \|T\|^2\mu(A_n) < \infty$, and so $f \in \mathcal{N}(E) \cap L^2(\Sigma)$. By hypothesis, we have $w_1u_2E(|u_2|^2)w_1\chi_{A_n} = 0$. Since $\sigma(u_2) \subseteq \sigma(E(|u_2|^2))$, then $w_1u_2 = 0$ as $n \to \infty$. Moreover,

$$\begin{aligned} w_1u_2 = 0 &\Longleftrightarrow E(w)(u - E(u)) = 0 \\ &\Longleftrightarrow uE(w) = E(uE(w)) \\ &\Longleftrightarrow u\chi_{E(w)} \in L^0(\mathcal{A}), \quad \text{(by part (a))}. \end{aligned}$$

This completes the proof. □

Note that $\frac{w_1u_2\chi_{\sigma(a)}}{a^2} = 0 = \frac{w_2u_1\chi_{\sigma(a)}}{a^2}$ iff $w_1u_2 = 0 = w_2u_1$ iff $\frac{\bar{u}_1\bar{w}_2}{E(|w|^2)E(|u|^2)} = 0 = \frac{\bar{u}_2\bar{w}_1}{E(|w|^2)E(|u|^2)}$. So, by (2.1), (2.2), (2.3) and Lemma 2.5 we have the following corollary.

Corollary 2.6 *$L^2(\mathcal{A})$ is a reducing subspace for $T \in \mathcal{L}$ iff it is a reducing subspace for T^D iff it is a reducing subspace for $T^\dagger$.*

Lemma 2.7 [10] *Let $T = M_wEM_u \in \mathcal{L}$. Then T is normal if and only if $T = M_{g\bar{u}}EM_u$ for some $g \in L^0(\mathcal{A})$. In this case $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.*

Take $K := E(|u|^2)E(|w|^2)$ and let $a := E(uw)$ be bounded away from zero on its support. Put

$$r = \frac{\chi_{\sigma(K)}}{K}, \qquad s = \frac{\chi_{\sigma(a)}}{a^2}, \qquad t = \frac{a}{E(|u|^2)}.$$

Recall from (2.1), (2.2) and (2.4) that $T^\dagger = M_rT^*$, $T^D = M_sT$ and $\widetilde{T} = M_{t\bar{u}}EM_u$. So, if $K = 1 = a$ on X, then $T^\dagger = T^*$ and $(T^D)^n = T^n = M_{a^{n-1}}T = T$ for all $n \in \mathbb{N}$. Now, suppose a is bounded below on X. Then by the conditional Cauchy-Schwarz inequality, $\sigma(K) = X$, $\|a\|_\infty^2 \le \|K\|_\infty = \|T\|^2 < \infty$ and hence $\{a, \frac{1}{a}\} \subseteq L^\infty(\mathcal{A})$ and $M_a^{-1} = M_{1/a} \in B(L^2(\Sigma))$. So we have the following corollary.

Corollary 2.8 *Let $T = M_w E M_u \in \mathcal{L}$ and let $a = E(uw)$ be bounded below on X. Then the following assertions hold.*

(a) *For each $n \in \mathbb{N}$, $(T^n)^D = M_{a^{-(n+1)}\chi_{\sigma(a)}} T = (T^D)^n$.*
(b) *If $(T^D)^n = T^D = T$, then $a = 1$ on $\sigma(a)$.*
(c) *If $T^\dagger = T^*$, then $r = 1$ on $\sigma(r)$.*
(d) *Let $\sigma(w) \subseteq \sigma(a)$. Then T^D is normal iff T is normal iff $T^\dagger$ is normal iff $T^\dagger T = TT^\dagger$.*

Proof (a) Since $T^n = M_{a^{n-1}w} E M_u$, then we have

$$\begin{aligned}(T^D)^n &= (M_s T)^n = M_{S^n} T^n = M_{(\frac{\chi_{\sigma(a)}}{a^2})^n} M_{a^{n-1}} T \\ &= M_{\frac{wa^{n-1}\chi_{\sigma(a)}}{a^{2n}}} E M_u = (M_{a^{n-1}w} E M_u)^D = (T^n)^D.\end{aligned}$$

(b) Let $T^D = T$. Then $M_{(a^{-2}\chi_{\sigma(a)}-1)} T(f) = 0$, for all $f \in L^2(\Sigma)$. Put $f = w\sqrt{E(|u|^2)}\chi_{A_n} \in L^2(\Sigma)$, as in the proof of Lemma 2.5. Then we have

$$\begin{aligned}&(a^{-2}\chi_{\sigma(a)} - 1)(wa\sqrt{E(|u|^2)}\chi_{A_n} = 0 \\ &\xrightarrow[E]{\times u} (a^{-2}\chi_{\sigma(a)} - 1)a^2\sqrt{E(|u|^2)} = 0, \quad \text{as } n \to \infty \\ &\Longrightarrow (a^{-2} - 1)\chi_{\sigma(a)} = 0. \Longrightarrow a = 1, \quad \text{on } \sigma(a).\end{aligned}$$

(c) Let $T^\dagger = T^*$. Then $M_{r-1}T^*(f) = (r-1)\bar{u}E(\bar{w}f) = 0$, for all $f \in L^2(\Sigma)$. Again put $f = w\sqrt{E(|u|^2)}\chi_{A_n}$. Then $(r-1)\bar{u}E(|w|^2)\sqrt{E(|u|^2)} = 0$ as $n \to \infty$. Multiplying by u and then taking E, we obtain $(r-1)\frac{1}{r}\sqrt{E(|u|^2)} = 0$, and so $r = 1$ on $\sigma(r)$.

(d) Set $\mathcal{N} = \{T = M_w E M_u \in B(L^2(\Sigma)) \setminus \{0\} : T \text{ is normal}\}$. As it turns out, $M_w E M_u \in \mathcal{N}$ if and only if $w = g\bar{u}$ for some $g \in L^0(\mathcal{A})$. Let $T \in \mathcal{N}$. Then $\{M_r T^*, M_s T\} \subseteq \mathcal{N}$ and $T^\dagger T = M_r T^* T = T M_r T^* = TT^\dagger$. Let $T^D = M_{sw} E M_u \in \mathcal{N}$, then $sw = g\bar{u}$ for some $g \in L^0(\mathcal{A})$. Since $\sigma(a) = \sigma(s)$, it follows that $w = \chi_{\sigma(a)} w = (\frac{g\chi_{\sigma(a)}}{s})\bar{u}$ and hence $\{T, T^\dagger\} \subseteq \mathcal{N}$. Similarly, if $T^\dagger \in \mathcal{N}$, then $M_r T = M_{rw} E M_u \in \mathcal{N}$ and hence $rw = g_1\bar{u}$ for some $g_1 \in L^0(\mathcal{A})$. Since $\sigma(a) \subseteq \sigma(K) = \sigma(r)$, then $w = \chi_{\sigma(r)} w = (\frac{g_1\chi_{\sigma(r)}}{r})\bar{u}$, and so $\{T, T^D\} \subseteq \mathcal{N}$. Now, let $T^\dagger T = TT^\dagger$. Then $M_r(T^*T - TT^*) = 0$, and so

$$M_{\frac{\bar{u}\chi_{\sigma(K)}}{E(|u|^2)}} E M_u - M_{\frac{w\chi_{\sigma(K)}}{E(|w|^2)}} E M_{\bar{w}} = 0.$$

Put $f = \bar{u}\sqrt{E(|w|^2)}\chi_{A_n} \in L^2(\Sigma)$, as in the proof of Lemma 2.5. Then we have

$$\bar{u}\chi_{\sigma(K)}\sqrt{E(|w|^2)} = \frac{\bar{a}}{\sqrt{E(|w|^2)}}w, \quad \text{as } n \to \infty.$$

So $w = \chi_{\sigma(a)}w = \frac{\chi_{\sigma(a)}\sqrt{E(|w|^2)}}{\bar{a}}\bar{u}$, and hence $T \in \mathcal{N}$. □

For $T \in B(\mathcal{H})$, the spectrum of T is denoted by $\sigma(T)$ and $r(T)$ its spectral radius. In [10] it was proved that the spectrum of $T = M_w E M_u \in \mathcal{L}$ is the essential range of $E(uw)$. Recall from (2.1), (2.2) and the conditional Cauchy-Schwarz inequality, we have the following corollary.

Corollary 2.9 *Let $T = M_w E M_u \in \mathcal{L}$. Then $\sigma(\widetilde{T}) \setminus \{0\} = \sigma(T) \setminus \{0\}$,*

$$\sigma(T^\dagger) \setminus \{0\} = ess\ range\left(\frac{E(\overline{uw})}{E(|u|^2)E(|w|^2)}\right) \setminus \{0\};$$
$$\sigma(T^D) \setminus \{0\} = ess\ range\left(\frac{\chi_{E(uw)}}{E(uw))}\right) \setminus \{0\},$$

and so $\max\{\frac{1}{r(T)}, r(T^\dagger)\} \leq r(T^D)$. *Moreover, if* $k = E(|u|^2)E(|w|^2) \geq 1$ *then* $r(T^\dagger) \leq r(T)$ *and if* $k \leq 1$, *then* $r(T) \leq r(T^\dagger)$.

For $u, w \in L^2(\Sigma) \setminus \{0\}$, the rank-one operator $u \otimes w$ on $L^2(\Sigma)$ is defined by $(u \otimes w)f = \langle f, w\rangle u$, for all $f \in L^2(\Sigma)$. Let $\mu(X) = 1$ and $\mathcal{A}_0 = \{\emptyset, X\}$. Put $E^{\mathcal{A}_0} = E_0$. Then by (1.2) we have $\int_X E_0(f)d\mu = \int_X f d\mu$, for all $f \in L^2(\Sigma)$. Since X is an $\mathcal{A}_0$-atom, then the $\mathcal{A}_0$-measurable function $E_0(f)$ is constant on X. It follows that $E_0(f) = \int_X f d\mu$, for all $f \in L^2(\Sigma)$. In this case the nonzero operator $T = M_w E_0 M_u$ is bounded on $L^2(\Sigma)$ if and only if

$$E_0(|u|^2)E_0(|w|^2) = \left(\int_X |u|^2 d\mu)(\int_X |w|^2 d\mu\right) = \|u\|_2^2\ \|w\|_2^2 < \infty.$$

Note that for all $f \in L^2(\Sigma)$, $Tf = wE_0(uf) = w\int_X uf d\mu = \langle f, \bar{u}\rangle w = (w \otimes \bar{u})f$. Thus, T is a rank-one operator with $\|T\| = \|u\|_2\|w\|_2$. Since $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*) = \mathcal{R}(\bar{u} \otimes w)$, then $T^\dagger$ is also a rank-one operator. Put $T^\dagger = \bar{u} \otimes w'$, for some $w' \in L^2(\Sigma)$. To obtain $T^\dagger$ it is enough to find the element w' in $L^2(\Sigma)$. Since $TT^\dagger = (w \otimes \bar{u})(\bar{u} \otimes w') = \|u\|^2(w \otimes w')$, then we have $TT^\dagger T = \|u\|^2(w \otimes w')(w \otimes \bar{u}) = \|u\|^2\langle w, w'\rangle(w \otimes \bar{u})$. It follows that $TT^\dagger T = T$ if and only if $\|u\|^2\langle w, w'\rangle = 1$. Hence $w' = \frac{w}{\|u\|_2^2\|w\|_2^2}$, and so $\|T^\dagger\| = \|T\|^{-1}$ (see [12]). From this, it is easy to check that $T^\dagger$ is satisfy the other equations in (1.1). Thus,

$$T^\dagger f = (\bar{u} \otimes \frac{w}{\|u\|_2^2\ \|w\|_2^2})f = \left(\frac{1}{\|u\|_2^2\ \|w\|_2^2}\int_X \bar{w} f d\mu\right)\bar{u}$$
$$= \frac{\bar{u}}{E_0(|u|^2)E_0(|w|^2)}E(\bar{w}f) = M_{\frac{1}{K}}T^* f,$$

where $f \in L^2(\Sigma)$ and $K = E_0(|u|^2)E_0(|w|^2) = \|u\|_2^2 \ \|w\|_2^2 < \infty$. Put $C = \sigma(E_0(uw))$ and set $S = M_v(w \otimes \bar{u})$, where $v = \frac{\chi_C}{(E_0(uw))^2}$. Since for $n \in \mathbb{N}$, $(w \otimes \bar{u})^{n+1} = \langle w, \bar{u}\rangle^n (w \otimes \bar{u})$ and

$$v\langle w, \bar{u}\rangle^{n+1} = v\left(\int_X uw d\mu\right)^{n+1} = \frac{(E_0(uw))^{n+1}}{(E_0(uw))^2} = \langle w, \bar{u}\rangle^{n-1},$$

then $T^{n+1}S = M_{v\langle w,\bar{u}\rangle^{n+1}}(w \otimes \bar{u}) = M_{\langle w,\bar{u}\rangle^{n-1}}(w \otimes \bar{u}) = T^n$. Also, it is easy to check that $TS = ST$ and $STS = S$. Thus, $T^D = S$.

Now, fix any $n \in \mathbb{N}$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{A_1, \cdots, A_n\}$ of X. Then

$$E(f) = \sum_{i=1}^{n} \frac{1}{\mu(A_i)} \left(\int_{A_i} f d\mu\right) \chi_{A_i}, \quad f \in L^2(\Sigma).$$

It follows that

$$\begin{aligned} Tf = (M_w E M_u) f &= \sum_{i=1}^{n} \frac{w}{\mu(A_i)} \left(\int_{A_i} uf d\mu\right) \chi_{A_i} \\ &= \left\{\frac{\chi_{A_1} w}{\mu(A_1)} \otimes (\chi_{A_1}\bar{u}) + \cdots + \frac{\chi_{A_n} w}{\mu(A_n)} \otimes (\chi_{A_n}\bar{u})\right\} f. \end{aligned}$$

Put $\bar{u}_i = \chi_{A_i}\bar{u}$, $w_i = \chi_{A_i} w$ and $\chi_{A_i} L^2(\Sigma) = L^2(A_i)$. Then the matrix representation T with respect to the decomposition $L^2(\Sigma) = L^2(A_1) \oplus \cdots \oplus L^2(A_n)$ is $T = \mathrm{diag}(\frac{w_1 \otimes \bar{u}_1}{\mu(A_1)}, \cdots, \frac{w_n \otimes \bar{u}_n}{\mu(A_n)})$, and so

$$T^\dagger = \mathrm{diag}\left(\frac{(\bar{u}_1 \otimes w_1)\mu(A_1)}{\|u_1\|_2^2 \ \|w_1\|_2^2}, \cdots, \frac{(\bar{u}_n \otimes w_n)\mu(A_n)}{\|u_n\|_2^2 \ \|w_n\|_2^2}\right),$$

where $\|u_i\|^2 = \int_{A_i} |u|^2 d\mu$ and $\|w_i\|^2 = \int_{A_i} |w|^2 d\mu$. By a similar argument as above, we obtain $T^D = \sum_{i=1}^n M_{v_i}(w_i \otimes \bar{u}_i)$, where $v_i = \frac{\chi_{C_i}}{(E(u_i w_i))^2}$ and $C_i = \sigma(E_n(u_i w_i))$. These observations establish the following result.

Theorem 2.10 *Let $T_0 = M_w E_0 M_u$ and $T_n = M_w E_n M_u$ be nonzero elements in $\mathcal{L}$. Put $C = \sigma(E(uw))$. Then $T_0 = w \otimes \bar{u}$ is a rank-one operator and*

$$T_0^\dagger = \bar{u} \otimes \frac{w}{\|u\|_2^2 \ \|w\|_2^2}, \qquad T_0^D = M_{\frac{\chi_C}{(E_0(uw))^2}}(w \otimes \bar{u}).$$

Moreover, if $\bar{u}_i = \bar{u}_{|A_i}$, $w_i = w_{|A_i}$ and $C_i = \sigma(E_n(u_i w_i))$, then

$$T_n^\dagger = \sum_{i=1}^{n} \frac{\mu(A_i)}{\|u_i\|_2^2 \ \|w_i\|_2^2} (\bar{u}_i \otimes w_i);$$

$$T_n^D = \sum_{i=1}^{n} M_{\frac{\chi_{C_i}}{(E(u_i w_i))^2}}(w_i \otimes \bar{u}_i).$$

Example 2.11 Let $X = [0,1]$, $d\mu = dx$, Σ be the Lebesgue measurable sets and let $\mathcal{A}_0 = \{\emptyset, X\}$ and $E^{\mathcal{A}_0} = E_0$. Set $u(x) = 3x^2$ and $w(x) = x^2$. Then for each $f \in L^2([0,1])$ we have

$$\begin{aligned}
Tf(x) &= w(x)E_0(uf)(x) = w(x)\int_0^1 u(x)f(x)dx \\
&= \left(\int_0^1 3x^2 f(x)dx\right)x^2 = \left(\int_0^1 x^2 f(x)dx\right)3x^2 = T^* f(x).
\end{aligned}$$

It is easy to check that $T^{k+1} = (\frac{3}{5})^k T$, $\|u\|_2^2 = \frac{9}{5}$, $\|w\|_2^2 = \frac{1}{5}$ and $E_0(uw) = \frac{3}{5} = \sigma(T)$. Then by Theorem 2.10, $T^\dagger(f) = \frac{25}{9}(\bar{u} \otimes \bar{w})(f) = \frac{25}{3}(\int_0^1 x^2 f(x)dx)x^2$. It follows that $T^\dagger = \frac{25}{9}T = T^D$.

For $T, S \in B(\mathcal{H})$, the equality $(T_1T_2)^D = T_2^D T_1^D$ is called the reverse order low for the Drazin inverse whenever both sides of equality are well defined. Since the reverse order law does not hold for various classes of generalized inverses, so a significant number of papers investigated the sufficient or equivalent conditions such that the reverse order law holds (see [3] and reference therein). In the following we first prove a result concerning the reverse order law $(TS)^\dagger = S^\dagger T^\dagger$ for $S, T \in B(\mathcal{H})$ under a certain condition.

Lemma 2.12 *Let $T \in B(\mathcal{H}, \mathcal{K})$ and $S \in B(\mathcal{K}, \mathcal{H})$ be operators such that T is invertible and T, S and ST have closed ranges. Then $T^{-1}S^\dagger \in ST\{1,2,3\}$. Furthermore, if $T^{-1} = T^*$, then $(ST)^\dagger = T^{-1}S^\dagger$.*

Proof Since $ST(T^{-1}S^\dagger)ST = ST$, $(T^{-1}S^\dagger)ST(T^{-1}S^\dagger) = T^{-1}S^\dagger$ and $ST(T^{-1}S^\dagger) = SS^\dagger$ is self adjoint, so $T^{-1}S^\dagger \in ST\{1,2,3\}$. Now, let $T^{-1} = T^*$. Since $S^\dagger S$ is self-adjoint then $((T^{-1}S^\dagger)ST)^* = T^*S^\dagger S(T^{-1})^* = (T^{-1}S^\dagger)ST)$, and so $(ST)^\dagger = T^{-1}S^\dagger$. □

Theorem 2.13 *Let $T \in B(\mathcal{H}, \mathcal{K})$ and $S \in B(\mathcal{K}, \mathcal{H})$ be operators such that T, S and ST have closed ranges. If $T_1 = T_{|\mathcal{R}(T^*)} : \mathcal{R}(T^*) \to R(T)$ is an isometry, then $(ST)^\dagger = T^\dagger S^\dagger$ if and only if $S_2^* S_1 = 0$, where $S_1 = S_{|\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(S)$ and $S_2 = S_{|\mathcal{N}(T^*)} : \mathcal{N}(T^*) \to \mathcal{R}(S)$.*

Proof Since $\mathcal{H} = \mathcal{R}(T^*) \oplus \mathcal{N}(T) = \mathcal{R}(S) \oplus \mathcal{N}(S^*)$ and $\mathcal{K} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$, the matrix decompositions of T and S with $\mathcal{R}(T^*) \oplus \mathcal{N}(T) \xrightarrow{T} \mathcal{R}(T) \oplus \mathcal{N}(T^*) \xrightarrow{S} \mathcal{R}(S) \oplus \mathcal{N}(S^*)$ are:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}; \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & 0 \end{pmatrix},$$

where T_1 is invertible. It follows, see [3], that

$$T^\dagger = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}; \qquad S^\dagger = \begin{pmatrix} S_1^* D^{-1} & 0 \\ S_2^* D^{-1} & 0 \end{pmatrix},$$

where $D = S_1 S_1^* + S_2 S_2^*$ is invertible and positive in $B(\mathcal{H})$. Thus, $T^\dagger S^\dagger = T_1^{-1} S_1^* D^{-1} \oplus 0$. On the other hand, by Lemma 2.12, we obtain $(ST)^\dagger = (S_1 T_1 \oplus 0)^\dagger = (S_1 T_1)^\dagger \oplus 0 = T_1^{-1} S_1^\dagger \oplus 0$. Consequently,

$$\begin{aligned}
(ST)^\dagger = T^\dagger S^\dagger &\iff S_1^* D^{-1} = S_1^\dagger. \\
&\iff S_1^* = S_1^\dagger D \\
&\iff S_1^* = S_1^\dagger (S_1 S_1^* + S_2 S_2^*) = S_1^\dagger S_1 S_1^* + S_1^\dagger S_2 S_2^* \\
&\iff S_1 = S_1 S_1^\dagger S_1 + S_2 S_2^* (S_1^*)^\dagger = S_1 + S_2 S_2^* (S_1^*)^\dagger \\
&\iff S_2 S_2^* (S_1^*)^\dagger = 0 \\
&\iff R(S_1) = R((S_1^*)^\dagger) \subseteq N(S_2 S_2^*) = N(S_2^*) \\
&\iff S_2^* S_1 = 0.
\end{aligned}$$

This complets the proof. □

It is a worth nothing that under assumptions of Theorem 2.13, $(ST)^\dagger = T^\dagger S^\dagger$ if and only if $\mathcal{R}(ST) = \mathcal{R}(S_1) \subseteq \mathcal{N}(S_2^*) = (\mathcal{R}(S_2))^\perp = \{Sx : x \in \mathcal{N}(T^*)\}^\perp$. Also, using matrix representation of $T : \mathcal{R}(T^*) \oplus \mathcal{N}(T) \to \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ and $T^\dagger$, $T^\dagger = T^*$ if and only if $T_1^{-1} = T_1^*$ if and only if T_1 is unitary.

The expressions for the generalized Drazin inverse of the product and the sum were studied by many authors (see e.g. [15]). In the following we consider the product and additive Drazin problem for conditional operators.

Lemma 2.14 *Let $T_1 = M_w E M_u$ and $T_2 = M_s E M_r$ be in $\mathcal{L}$. If $E(rw)E(us)$ is bounded away from zero on its support, then $T_1 T_2$ is Derazin invertible.*

Proof Put $C = \sigma(E(rw)E(us))$. Since $T_1 T_2 = M_{wE(us)} E M_r$, then by Theorem 2.2 we have

$$(T_1 T_2)^D = M_{\frac{\chi_C}{(E(rw)E(us))^2}} T_1 T_2 = M_{\frac{\chi_C}{(E(rw))^2 E(us)}} M_w E M_r.$$

This completes the proof. □

Theorem 2.15 *Let $T_1 = M_w E M_u$ and $T_2 = M_s E M_r$ be in $\mathcal{L}$. If $E(uw)$, $E(rs)$ and $E(rw)E(us)$ are bounded away from zero on their supports, then $(T_1 T_2)^D = T_2^D T_1^D$ if and only if T_1 commute with T_2 and*

$$(E(rw))^2 (E(us))^2 \chi_{A \cap B} = (E(uw))^2 (E(rs))^2 \chi_C,$$

where $A = \sigma(E(uw))$, $B = \sigma(E(rs))$ and $C = \sigma(E(rw)) \cap \sigma(E(us))$.

Proof By Theorem 2.3, $T_1^D = M_\alpha T_1$ and $T_2^D = M_\beta T_2$, where $\alpha = \frac{\chi_A}{(E(uw))^2}$ and $\beta = \frac{\chi_B}{(E(rs))^2}$. Since $T_1T_2 = M_{wE(us)}EM_r$, then by Lemma 2.14, T_1T_2 is Drazin invertible and $(T_1T_2)^D = M_\gamma T_1T_2$, where $\gamma = \frac{\chi_C}{(E(rw))^2(E(us))^2}$. Thus, the reverse order low $(T_1T_2)^D = T_2^DT_1^D$ holds if and only if T_1 and T_2 commute and $\alpha\beta = \gamma$. This completes the proof. □

Theorem 2.16 *Let $T_1 = M_wEM_u$ and $T_2 = M_sEM_r$ be in $\mathcal{L}$. Then the following assertions hold.*

(a) *If $s = w$ and $E(uw) + E(rw)$ is bounded away from zero on its support, then $T_1 + T_2$ is Drazin invertible.*
(b) *If $E(uw) = 0 = E(rs)$ and $E(ur)E(sw)$ is bounded away from zero on its support, then $T_1 + T_2$ is Drazin invertible with $ind(T_1 + T_2) = 1$.*

Proof (a) Put $S = E(uw) + E(rw)$. Using Theorem 2.2 we have

$$(T_1 + T_2)^D = (M_wEM_{u+r})^D = M_{\frac{\chi_{\sigma(S)}}{S^2}} M_wEM_{u+r}.$$

(b) Put $K = E(ur)E(sw)$ and define $S = M_{\frac{\chi_{\sigma(K)}}{K}}(T_1 + T_2)$. Since the $\mathcal{A}$-measurable function K is bounded away from zero on its support, then $S \in B(L^2(\Sigma))$ and

$$S(T_1 + T_2) = M_{\frac{\chi_{\sigma(K)}}{K}}(T_1 + T_2)^2 = (T_1 + T_2)S.$$

Since $T_1^2 = M_{wE(uw)}EM_u$=0 and $T_2^2 = M_{sE(rs)}EM_r = 0$, then

$$\begin{aligned}(T_1 + T_2)^2S &= (T_1T_2 + T_2T_1)S \\ &= M_{\frac{\chi_{\sigma(K)}}{K}}\{T_1T_2T_1 + T_2T_1T_2\} \\ &= M_{\frac{\chi_{\sigma(K)}}{K}}\{M_{E(ur)E(sw)}T_1 + M_{E(ur)E(sw)}T_2\} \\ &= M_{\chi_{\sigma(K)}}(T_1 + T_2).\end{aligned}$$

Since $S = 0$ on $X\backslash\sigma(K)$, then $(T_1+T_2)^2S = T_1+T_2$. Notice that $(T_1+T_2)^{n+1}S = (T_1 + T_2)^2S(T_1 + T_2)^{n-1} = (T_1 + T_2)^n$, for all $n \geq 1$. Moreover,

$$\begin{aligned}S(T_1 + T_2)S &= M_{\frac{\chi_{\sigma(K)}}{K^2}}(T_1 + T_2)^3 \\ &= M_{\frac{\chi_{\sigma(K)}}{K^2}}\{T_1T_2T_1 + T_2T_1T_2\} \\ &= M_{\frac{\chi_{\sigma(K)}}{K}}(T_1 + T_2) = S.\end{aligned}$$

These ensure that S is the group inverse of $T_1 + T_2$. □

In the following we shall use our results to calculate the Moore–Penrose inverse and the Drazin inverse of $M_wEM_u \in \mathcal{L}$.

Example 2.17 (a) Let $\varepsilon > 0$, $X = [-\pi/4+\varepsilon, \pi/4-\varepsilon]$, $d\mu = dx$, Σ be the Lebesgue sets and let $\mathcal{A}$ be the σ-subalgebra generated by the symmetric sets about the origin. Now any real valued function on X can be written uniquely as a sum of an even function and an odd function, one simply uses the functions $f_e(x) = (f(x)+f(-x))/2$ and $f_o(x) = (f(x)-f(-x))/2$. Put $0 < a < \pi/4-\varepsilon$. Then for each $f \in L^2(\Sigma)$ we have $\int_{-a}^{a} E(f)(x)dx = \int_{-a}^{a} f_e(x)dx$ and consequently, $Ef = f_e$. This example is due to Alan Lambert [13]. Now let $u(x) = \sin x+\cos x$ and $w(x) = \sin x - \cos x$. Then $u_1 = -w_1 = \cos x$, $u_2 = w_2 = \sin x$ and $a = w_1u_1 + E(w_2u_2) = -\cos^2 x + E(\sin^2 x) = -\cos^2 x + \sin^2 x = -\cos 2x$. Moreover, $E(|u|^2) = E(1+sin2x) = 1$ and $E(|w|^2) = E(1-sin2x) = 1$. Thus, $T = M_w EM_u \in \mathcal{L}$ with $\|T\| = 1$. Now, using (2.3) and Theorem 2.3 we get that

$$T = \begin{pmatrix} M_{-\cos^2 x} & EM_{-\frac{1}{2}\sin 2x} \\ M_{\frac{1}{2}\sin 2x} & M_{\sin x}EM_{\sin x} \end{pmatrix};$$
$$T^D = \begin{pmatrix} M_{-\frac{\cos^2 x}{\cos^2 2x}} & EM_{-\frac{\sin 2x}{2\cos^2 2x}} \\ M_{\frac{\sin 2x}{2\cos^2 2x}} & M_{\frac{\sin x}{\cos^2 2x}}EM_{\sin x} \end{pmatrix} = M_{\frac{1}{\cos^2 2x}}T.$$

Put $k = \frac{\bar{u}}{E(|u|^2)E(|w|^2)}$. Then by (2.1) we have

$$T^\dagger = \begin{pmatrix} M_{k_1\bar{w}_1} & EM_{k_1\bar{w}_2} \\ M_{k_2\bar{w}_1} & M_{k_2}EM_{\bar{w}_2} \end{pmatrix} = \begin{pmatrix} M_{-\cos^2 x} & EM_{\frac{1}{2}\sin 2x} \\ M_{-\frac{1}{2}\sin 2x} & M_{\sin x}EM_{\sin x} \end{pmatrix} = T^*.$$

Moreover, since $g = \frac{E(uw)\bar{u}}{E(|u|^2)} = -\cos 2x(\sin x + \cos x)$, $g_1 = -\cos 2x \cos x$ and $g_2 = -\cos 2x \sin x$, so we get that

$$\begin{aligned}\widetilde{T} &= -\begin{pmatrix} M_{\cos 2x\cos^2 x} & EM_{\cos 2x\cos x\sin x} \\ M_{\cos 2x\cos x\sin x} & M_{\cos 2x\sin x}EM_{\sin x} \end{pmatrix} \\ &= -M_{\cos 2x(\sin x+\cos x)}EM_{\sin x+\cos x}.\end{aligned}$$

Now, set $u = \frac{1}{\cos x\sin x}$, $w = \sin x$, $\alpha = \cos^2 x\sin x$ and $T = M_w EM_u$. Then $E(uw) = \frac{1}{\cos x}$ and $E(u\alpha) = \cos x$. Then by Proposition 2.2 we have

$$T^D = M_{\cos^2 x\sin x}EM_{\frac{1}{\cos x\sin x}} = M_{\sin x}EM_{\coth x}.$$

Note that, in this setting, $\{u, w, \alpha\} \subset \mathcal{N}(E)$.

(b) Let $X = \{1, 2, 3\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/3$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{\{1\}, \{2, 3\}\}$. Then

$$E(f) = \left(\frac{1}{\mu(A_1)}\int_{A_1} f d\mu\right)\chi_{A_1} + \left(\frac{1}{\mu(A_2)}\int_{A_2} f d\mu\right)\chi_{A_2}$$

$$= f(1)\chi_{A_1} + \frac{f(2)+f(3)}{2}\chi_{A_2},$$

where $A_1 = \{1\}$ and $A_2 = \{2, 3\}$. Then matrix representation of $E = E^{\mathcal{A}}$ with respect to the standard orthonormal basis is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Let $w = (w_1, w_2, w_3)$ and $u = (u_1, u_2, u_3)$ be nonzero elements of $\mathbb{C}^3$. Then

$$\begin{aligned} T = M_w E M_u &= \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} \\ &= \begin{bmatrix} w_1 u_1 & 0 & 0 \\ 0 & \frac{w_2 u_2}{2} & \frac{u_3 w_2}{2} \\ 0 & \frac{u_2 w_3}{2} & \frac{u_3 w_3}{2} \end{bmatrix}; \end{aligned}$$

$$\begin{aligned} uw &= (u_1 w_1, u_2 w_2, u_3 w_3); \\ |u|^2 &= (|u_1|^2, |u_2|^2, |u_3|^2); \\ |w|^2 &= (|w_1|^2, |w_2|^2, |w_3|^2); \\ E(uw) &= \left(u_1 w_1, \frac{u_2 w_2 + u_3 w_3}{2}, \frac{u_2 w_2 + u_3 w_3}{2} \right); \\ E(|u|^2) &= \left(|u_1|^2, \frac{|u_2|^2 + |u_3|^2}{2}, \frac{|u_2|^2 + |u_3|^2}{2} \right); \\ E(|w|^2) &= \left(|w_1|^2, \frac{|w_2|^2 + |w_3|^2}{2}, \frac{|w_2|^2 + |w_3|^2}{2} \right). \end{aligned}$$

Put $a = u_1 w_1$, $b = 1/2(u_2 w_2 + u_3 w_3)$ and $c = 1/4(|u_2|^2 + |u_3|^2)(|w_2|^2 + |w_3|^2)$. Then $(E(uw))^2 = (a^2, b^2, b^2)$ and $E(|u|^2)E(|w|^2) = (|a|^2, c, c)$. For $x \in \{a, b, c\}$, we take $\frac{1}{x} = 0$ whenever $x = 0$. Then we have

$$\begin{aligned} T^\dagger = M_{\left(\frac{1}{|a|^2}, \frac{1}{c}, \frac{1}{c}\right)} T^* &= \begin{bmatrix} \frac{1}{|a|^2} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} \bar{w}_1 \bar{u}_1 & 0 & 0 \\ 0 & \frac{\bar{w}_2 \bar{u}_2}{2} & \frac{\bar{u}_2 \bar{w}_3}{2} \\ 0 & \frac{\bar{u}_3 \bar{w}_2}{2} & \frac{\bar{u}_3 \bar{w}_3}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\bar{w}_1 \bar{u}_1}{|a|^2} & 0 & 0 \\ 0 & \frac{\bar{w}_2 \bar{u}_2}{2c} & \frac{\bar{u}_2 \bar{w}_3}{2c} \\ 0 & \frac{\bar{u}_3 \bar{w}_2}{2c} & \frac{\bar{u}_3 \bar{w}_3}{2c} \end{bmatrix}; \end{aligned}$$

$$T^D = M_{\left(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{b^2}\right)} T = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} w_1u_1 & 0 & 0 \\ 0 & \frac{w_2u_2}{2} & \frac{u_3w_2}{2} \\ 0 & \frac{u_2w_3}{2} & \frac{u_3w_3}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{w_1u_1}{a^2} & 0 & 0 \\ 0 & \frac{w_2u_2}{2b^2} & \frac{u_3w_2}{2b^2} \\ 0 & \frac{u_2w_3}{2b^2} & \frac{u_3w_3}{2b^2} \end{bmatrix}.$$

Now, put $u = (1, 2i, -4)$ and $w = (2i, -2i, -4)$. Then $E(uw) = E(2i, 4, 16) = (2i, 10, 10)$, $E(|u|^2) = E(1, 4, 16) = (1, 10, 10)$ and $E(|w|^2) = E(4, 4, 16) = (4, 10, 10)$. It follows that $a = 2i$, $b = 10$ and $c = 100$ and

$$T = \begin{bmatrix} 2i & 0 & 0 \\ 0 & 2 & 4i \\ 0 & -4i & 8 \end{bmatrix}, \quad T^\dagger = \begin{bmatrix} \frac{-i}{2} & 0 & 0 \\ 0 & \frac{1}{50} & \frac{i}{25} \\ 0 & \frac{-i}{25} & \frac{2}{25} \end{bmatrix} = T^D.$$

Note that $w = (2i, 1, 1)\bar{u}$, where $(2i, 1, 1)$ is $\mathcal{A}$-measurable. So, T is normal but it is not self-adjoint. Moreover, by Corollary 2.9, $\sigma(T^\dagger) = \sigma(T^D) = \{\frac{-i}{2}, \frac{1}{10}\}$ and $\sigma(T) = \{2i, 10\}$.

Now, put $u = (0, i, -1)$, $w = (1, 1, i)$, $r = (0, 1, -1)$ and $s = (1, -1, -1)$. Then $E(uw) = (0, 0, 0) = E(rs)$ and

$$E(ur) = (0, \frac{1+i}{2}, \frac{1+i}{2});$$
$$E(sw) = (1, -\frac{1+i}{2}, -\frac{1+i}{2});$$
$$E(ur)E(sw) = (0, \frac{-i}{2}, \frac{-i}{2});$$
$$\frac{1}{E(ur)E(sw)} = (0, 2i, 2i).$$

Moreover,

$$T_1 = M_w E M_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{i}{2} & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-i}{2} \end{bmatrix}, \quad T_2 = M_s E M_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

It follows that

$$(T_1 + T_2)^D = (T_1 + T_2)^\# = M_{(0,2i,2i)}(T_1 + T_2)$$
$$= 2i(T_1 + T_2) = i \begin{bmatrix} 0 & 0 & 0 \\ 0 & i-1 & -2 \\ 0 & 0 & 1-i \end{bmatrix}.$$

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