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Operator geodesically convex functions and their applications

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ABSTRACT

In this paper, we introduce operator geodesically convex and operator convex-log functions and characterize some properties of them. Then we apply these classes of functions to present several operator Aczél and Minkowski-type inequalities extending some known results. The concavity counterparts are also considered.

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1. Introduction

It is known that the theory of matrix/operator convex functions introduced by Kraus [1] has many important applications in matrix analysis and quantum information and so on. Following this study, significant concepts of convexity have been extended elegantly to Hilbert space operators from scalar cases. The main aim of this paper is to establish an analogue of some convexity properties for operator functions. For this purpose, we first briefly review a survey on convex functions and operator convex functions. Throughout this paper, we use subintervals J and J_+ in the sense that $J \subset \mathbb{R}$ and $J_+ \subset (0, \infty)$.

Definition 1.1 ([2]): Let f be a positive function defined on J or J_+ .

(AA) The function f is said to be (usual) convex iff

$$f((1 - \nu)a + \nu b) \leq (1 - \nu)f(a) + \nu f(b) \quad \text{for any } \nu \in [0, 1] \text{ and } a, b \in J.$$

(AG) The function f is said to be log-convex iff

$$f((1 - \nu)a + \nu b) \leq f^{1-\nu}(a)f^\nu(b) \quad \text{for any } \nu \in [0, 1] \text{ and } a, b \in J.$$

(GA) The function f is said to be geodesically convex iff

$$f(a^{1-\nu}b^\nu) \leq (1-\nu)f(a) + \nu f(b) \quad \text{for any } \nu \in [0, 1] \text{ and } a, b \in J_+.$$

(GG) The function f is said to be geometrically convex iff

$$f(a^{1-\nu}b^\nu) \leq f^{1-\nu}(a)f^\nu(b) \quad \text{for any } \nu \in [0, 1] \text{ and } a, b \in J_+.$$

If the inequalities are reversed, then we have the corresponding types of concave functions.

Definition 1.2 ([3]): A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called convex-log if it can be written in the form $f(t) = h(\log t)$, $t > 0$, where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function.

We give a remark on the basic properties of the above function.

Remark 1.3: (i) A convex-log function satisfies the inequality

$$f(a^{1-\nu}b^\nu) \leq (1-\nu)f(a) + \nu f(b), \quad a, b > 0,$$

for $\nu \in [0, 1]$. Indeed,

$$\begin{aligned} f(a^{1-\nu}b^\nu) &= h(\log(a^{1-\nu}b^\nu)) = h((1-\nu)\log(a) + \nu\log(b)) \\ &\leq (1-\nu)h(\log(a)) + \nu h(\log(b)) = (1-\nu)f(a) + \nu f(b). \end{aligned}$$

So, we can say every convex-log function is a geodesically convex function. Precisely, the equivalence between a convex-log function and a geodesically convex function is shown in [3, Lemma 3.2].

- (ii) For a continuous positive function f , if $\log f$ is convex, then it is natural to say that f is log-convex. If f is a positive increasing log-convex function, then it is a geometrically convex function. Indeed, it follows $f(a^{1-\nu}b^\nu) \leq f((1-\nu)a + \nu b) \leq f^{1-\nu}(a)f^\nu(b)$ from the monotone increasingness of f and the arithmetic-geometric mean inequality. We note that a non-increasing log-convex function is not a geometrically convex function in general since the first inequality above is not valid. We can give a counter-example. Take the function $f(t) := -\log t$ on $(0, 1/e]$. Then the function $f(t)$ is decreasing and log-convex since $\frac{d^2}{dt^2}(\log(-\log t)) = \frac{-\log t - 1}{t^2(\log t)^2} \geq 0$ for $t \in (0, 1/e]$. Then we have $(-\log a)^{1-\nu}(-\log b)^\nu - (-\log(a^{1-\nu}b^\nu)) = \sqrt{2} - 3/2 < 0$ when $a = 1/e$, $b = 1/e^2$ and $\nu = 1/2$, so that this function does not satisfy the definition of a geometrically convex function.

In addition, a geometrically convex function is not always a log-convex function. It is known that the function $\sinh t$ on $(0, \infty)$ is geometrically convex [2]. However, the calculation $\frac{d^2(\sinh t)}{dt^2} = -\frac{1}{\sinh^2 t} < 0$ shows that the function $\sinh t$ is not log-convex.

- (iii) A geometrically convex function is a geodesically convex function by the arithmetic-geometric mean inequality. However, an increasing convex-log function is not necessarily a geometrically convex function. There are some examples that show the difference between these two classes of functions. For instance, the function $f(t) = t^p$, $p \in \mathbb{R}$, is convex-log, by letting $h(t) = \exp(pt)$, but it is not log-convex, since $\log(f(t)) = p \log(t)$ is not convex.

For a real-valued function f and a self-adjoint operator $A \in B(\mathcal{H})$, the value $f(A)$ is understood by means of the functional calculus. For each $\alpha \in [0, 1]$ and strictly positive operators A, B , $A \nabla_\alpha B := (1 - \alpha)A + \alpha B$, $A !_\alpha B := ((1 - \alpha)A^{-1} + \alpha B^{-1})^{-1}$ and $A \sharp_\alpha B := A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$ are the α -arithmetic, α -harmonic and α -geometric means, respectively. It is known that for any $A, B > 0$, we have $A !_\alpha B \leq A \sharp_\alpha B \leq A \nabla_\alpha B$. Some of the above definitions of convexity have been extended to the operator case as follows.

Definition 1.4: Let f be a continuous real function on J_+ , A, B be strictly positive operators with spectra contained in J_+ .

(i) The function f is said to be operator convex iff

$$f((1 - \nu)A + \nu B) \leq (1 - \nu)f(A) + \nu f(B) \quad \text{for any } \nu \in [0, 1] \text{ and } A, B > 0.$$

(ii) The nonnegative function f is said to be operator log-convex iff

$$f((1 - \nu)A + \nu B) \leq f(A) \sharp_\nu f(B) \quad \text{for any } \nu \in [0, 1] \text{ and } A, B > 0.$$

The concept of operator convexity was delicately introduced by Kraus [1]. Ando and Hiai [4] obtained a full characterization of operator log-convex functions. Also, a variant of the geometrically convexity property is presented in [5] as follows:

$$f(A \sharp_\nu B) \leq M(f(A) \sharp_\nu f(B))$$

with constant $M \geq 1$. In this note, we extend the definition of geodesically convex and convex-log functions to the operator space. In the second section, we first introduce operator geodesically convex (concave) functions. We present some properties of them and show that the class of such functions is fairly rich. Then we obtain an operator Aczél inequality, including operator geodesically convex functions. In the third section, we give the definition of an operator log-convex function and investigate some properties of that. Further, a variant of the operator Aczél inequality involving operator concave-log functions is given. The last section is devoted to studying another type of geodesically convex function, which leads to getting some Minkowski-type inequalities. The obtained results generalize the corresponding Minkowski and Aczél inequalities in [6,7], respectively.

2. Operator geodesically convex function

In 1956, Aczél [8] proved that if $a_i, b_i (1 \leq i \leq n)$ are positive real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then

$$\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^n a_i^2\right) \left(b_1^2 - \sum_{i=2}^n b_i^2\right).$$

Popoviciu [9] presented an exponential extension of the Aczél inequality, so that if $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$, then

$$a_1 b_1 - \sum_{i=2}^n a_i b_i \geq \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}}.$$

Aczél's and Popoviciu's inequalities were sharpened and some generalizations and variants of these inequalities are presented. See [10] and references therein. An operator version of the classical Aczél inequality was given in [7]. Furthermore, some reverses of the operator Aczél inequality were given in [11], and a variant of them was provided in [12].

In this section, we introduce an operator geodesically convex (concave) function and present an operator Aczél inequality involving this class of functions.

Definition 2.1: A nonnegative continuous function f on J_+ is said to be operator geodesically convex iff

$$f(A \sharp_\nu B) \leq f(A) \nabla_\nu f(B), \quad (1)$$

for any strictly positive operators A, B with spectra contained in J_+ and $\nu \in [0, 1]$. The function f is also said to be operator geodesically concave iff $-f$ is operator geodesically convex.

We first aim to show that the class of functions satisfying (1) is fairly rich. For this purpose, the following lemmas are provided. We also recall that a continuous real function f defined on an interval J is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all A, B with spectra in J .

Lemma 2.2: Let f, f_1 and f_2 be nonnegative continuous functions on J_+ .

- (i) If f is operator monotone and operator convex, then f is operator geodesically convex.
- (ii) If f_1 is operator monotone and operator convex and f_2 is operator geodesically convex, then $f_1 \circ f_2$ is operator geodesically convex.
- (iii) If f_1 and f_2 are two operator geodesically convex functions, then so is $\alpha f_1 + f_2$ for $\alpha > 0$.

Proof: For strictly positive operators A, B , we have the well-known Young inequality $A \sharp_\nu B \leq A \nabla_\nu B$. Now, (i) clearly holds by the assumptions on f and applying the Young inequality. For (ii) we have

$$\begin{aligned} f_1 \circ f_2(A \sharp_\nu B) &= f_1(f_2(A \sharp_\nu B)) \\ &\leq f_1(f_2(A) \nabla_\nu f_2(B)) \quad (\text{op. monotonicity of } f_1 \text{ and (1)}) \\ &\leq f_1(f_2(A)) \nabla_\nu f_1(f_2(B)) \quad (\text{op. convexity of } f_1) \\ &= f_1 \circ f_2(A) \nabla_\nu f_1 \circ f_2(B). \end{aligned}$$

Now, let f_1 and f_2 be two operator geodesically convex functions and $\alpha > 0$. Then

$$\begin{aligned} (\alpha f_1 + f_2)(A \sharp_\nu B) &= \alpha f_1(A \sharp_\nu B) + f_2(A \sharp_\nu B) \\ &\leq \alpha (f_1(A) \nabla_\nu f_1(B)) + (f_2(A) \nabla_\nu f_2(B)) \\ &= (\alpha f_1(A) + f_2(A)) \nabla_\nu (\alpha f_1(B) + f_2(B)) \\ &= (\alpha f_1 + f_2)(A) \nabla_\nu (\alpha f_1 + f_2)(B). \end{aligned}$$

That is, $\alpha f_1 + f_2$ is a geodesically convex function as well. ■

Lemma 2.3: Let f and g be continuous functions from $(0, \infty)$ into itself.

- (i) If $f(x)$ is an operator geodesically convex function so is $f(\frac{1}{x})$.
- (ii) If $g(x)$ is an operator geodesically concave function so is $g(\frac{1}{x})$.

Proof: Let A and B be strictly positive operators. Thanks to the geometric mean property $(A\sharp_\nu B)^{-1} = A^{-1}\sharp_\nu B^{-1}$, we have the first result as follows:

$$f((A\sharp_\nu B)^{-1}) = f(A^{-1}\sharp_\nu B^{-1}) \leq f(A^{-1})\nabla_\nu f(B^{-1}).$$

The second one is obtained similarly. ■

In the above lemma, if we let f and g be nonnegative continuous functions from $J \subset (0, \infty)$, we will assume that J contains both $Sp(A)$ and $Sp(A^{-1})$, where $Sp(A)$ represents the spectrum of A . The next theorem presents a connection between operator geodesically concavity and convexity. We recall that $g^*(x) := \frac{1}{g(\frac{1}{x})}$ is called the adjoint of function g .

Theorem 2.4: Let g be an operator geodesically concave function. Then the functions $1/g$ and g^* are operator geodesically convex.

Proof: Let A, B be strictly positive operators. For the operator geodesically concave function g , we have

$$g(A\sharp_\nu B) \geq g(A)\nabla_\nu g(B). \quad (2)$$

Therefore,

$$(g(A\sharp_\nu B))^{-1} \leq (g(A)\nabla_\nu g(B))^{-1} = g(A)^{-1}\sharp_\nu g(B)^{-1} \leq g(A)^{-1}\nabla_\nu g(B)^{-1},$$

which shows $1/g$ is operator geodesically convex. Combining this result with part (i) of Lemma 2.3 easily yields g^* is a operator geodesically convex function. However, in the sequel, we give a direct proof providing a refined inequality. Rewriting inequality (2) with the operators A^{-1}, B^{-1} and taking the inverse, we have

$$(g(A^{-1}\sharp_\nu B^{-1}))^{-1} \leq (g(A^{-1})\nabla_\nu g(B^{-1}))^{-1}. \quad (3)$$

Hence

$$\begin{aligned} g^*(A\sharp_\nu B) &= (g(A^{-1}\sharp_\nu B^{-1}))^{-1} \\ &\leq (g(A^{-1})\nabla_\nu g(B^{-1}))^{-1} \quad (\text{by the inequality (3)}) \\ &= (g^*(A)^{-1}\nabla_\nu g^*(B)^{-1})^{-1} \\ &= g^*(A)\sharp_\nu g^*(B) \\ &\leq g^*(A)\nabla_\nu g^*(B), \end{aligned}$$

as desired. ■

- Example 2.5:** (i) The simplest example of operator geodesically convex functions is $f(t) = t - a, a \geq 0$ on (a, ∞) . For $a = 0, f(t) = t$ leads to the Young inequality $A \sharp_\nu B \leq A \nabla_\nu B$.
- (ii) Another example is $f(t) = \frac{1}{1-t}$ on $(0, 1)$ due to its operator convexity and operator monotonicity [13], hence so is $f(1/t) = \frac{t}{1-t}$ on $(0, 1)$.
- (iii) Let $f(t) = \frac{1}{t}$ on $(0, \infty)$. Then

$$f(A \sharp_\nu B) = (A \sharp_\nu B)^{-1} = A^{-1} \sharp_\nu B^{-1} \leq A^{-1} \nabla_\nu B^{-1} = f(A) \nabla_\nu f(B).$$

This function is an instance of an operator geodesically convex function which is not operator monotone.

- Example 2.6:** (i) Every operator monotone decreasing and operator concave function g on J is an operator geodesically concave function.
- (ii) It can be seen that the Young inequality $A \sharp_\nu B \leq A \nabla_\nu B$ is equivalent to $I - A \sharp_\nu B \geq (I - A) \nabla_\nu (I - B)$. This means the function $g(t) = 1 - t$ on $(0, 1)$ is an operator geodesically concave function. A similar statement holds for $g(t) = a - t$ on $(0, a)$. By applying Theorem 2.4, it is deduced that the functions $g(t)^{-1} = \frac{1}{a-t}$ on $(0, a)$ and $g^*(t) = \frac{t}{at-1}$ on $(\frac{1}{a}, \infty)$ are operator geodesically convex.
- (iii) Let $g(t) = a - \frac{1}{t}, t \in (\frac{1}{a}, \infty)$. Then

$$\begin{aligned} g(A \sharp_\nu B) &= aI - (A \sharp_\nu B)^{-1} = aI - (A^{-1} \sharp_\nu B^{-1}) \\ &\geq (aI - A^{-1}) \nabla_\nu (aI - B^{-1}) \\ &= g(A) \nabla_\nu g(B). \end{aligned}$$

Hence, $g(t)$ is operator geodesically concave but not operator monotone decreasing.

The corresponding results of Lemma 2.2 hold for operator geodesically concave functions as well. The next result provides an operator Aczél inequality involving this class of functions.

Theorem 2.7: Let $g : J_+ \rightarrow [0, \infty)$ be an operator geodesically concave function, and $p, q > 1$ with $1/p + 1/q = 1$. For strictly positive operators A and B with spectra contained in J_+ , we have

$$g(A^p \sharp_{1/q} B^q) \geq g(A^p) \sharp_{1/q} g(B^q) \quad (4)$$

and

$$\langle g(A^p \sharp_{1/q} B^q) x, x \rangle \geq \langle g(A^p) x, x \rangle^{1/p} \langle g(B^q) x, x \rangle^{1/q}, \quad (5)$$

for all $x \in \mathcal{H}$.

Proof: Since g is an operator geodesically concave function, we have

$$g(A \sharp_\nu B) \geq g(A) \nabla_\nu g(B) \geq g(A) \sharp_\nu g(B). \quad (6)$$

Replacing A, B with A^p, B^q , respectively, and putting $v = 1/q$, then we obtain the desired inequality (4). From the first inequality of (6) and the arithmetic-geometric mean inequality, we have, for all $x \in \mathcal{H}$,

$$\langle g(A^p \sharp_{1/q} B^q) x, x \rangle \geq \frac{1}{p} \langle g(A^p) x, x \rangle + \frac{1}{q} \langle g(A^q) x, x \rangle \geq \langle g(A^p) x, x \rangle^{1/p} \langle g(B^q) x, x \rangle^{1/q}.$$

■

By letting $g(t) = 1 - t$ on $(0, 1)$, we have the following result.

Corollary 2.8: *Let $1/p + 1/q = 1$ with $p, q > 1$. For commuting positive invertible operators A and B with spectra contained in $(0, 1)$,*

$$1 - \|(AB)^{1/2} x\|^2 \geq (1 - \|A^{p/2} x\|^2)^{1/p} (1 - \|B^{q/2} x\|^2)^{1/q},$$

for all $x \in \mathcal{H}$ with $\|x\| = 1$.

3. Operator convex-log functions

As stated in Definition 1.2, a function $f : (0, \infty) \rightarrow \mathbb{R}$ is called convex-log if it can be written in the form $f(t) = h(\log t)$, $t > 0$, where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function. In this section, we are going to present a corresponding definition for operator functions and investigate some properties of that.

Definition 3.1: Let $J_1, J_2 \subset \mathbb{R}$. We call a function $f : J_+ \rightarrow \mathbb{R}$ operator convex-log if it can be written in the form $f(t) = h(\log t)$, $t > 0$, where $h : J_1 \rightarrow J_2$ is an operator convex function. Also, a function $g : J_+ \rightarrow \mathbb{R}$ is called an operator concave-log function if it can be written in the form $g(t) = \varphi(\log t)$, $t > 0$, where $\varphi : J_1 \rightarrow J_2$ is an operator concave function.

Remark 3.2: (i) We stated in Remark 1.3 that every convex-log function is a geodesically convex function. However, this is not valid for the operator case. So we use two terms, ‘convex-log’ and ‘geodesically convex’, in this article.

(ii) We set $J_1 = J_2 = (0, \infty)$ in Definition 3.1. It is known that the function $\log(t)$ is operator concave on $(0, \infty)$. Further, every operator concave function $\varphi : (0, \infty) \rightarrow (0, \infty)$ is also operator monotone [14]. By applying these facts to the operator concave-log function g , we have

$$\begin{aligned} g(A \nabla_v B) &= \varphi(\log(A \nabla_v B)) \\ &\geq \varphi(\log(A) \nabla_v \log(B)) \\ &\geq \varphi(\log(A)) \nabla_v \varphi(\log(B)) \\ &= g(A) \nabla_v g(B). \end{aligned}$$

This means any operator concave-log function $g : J_+ \rightarrow (0, \infty)$ is an operator concave function.

- (iii) There is a wide range of this class of functions. The simplest examples are the functions $(\log(t))^p$ on $[1, \infty)$, which for $p \in [-1, 0] \cup [1, 2]$ are operator convex-log and for $p \in [0, 1]$ are operator concave-log.

In the rest of this section, we will use the following definition considered with gentle restrictions on the domains. These restrictions enable us to provide some results on the operator log-convex functions involving operator means.

Definition 3.3: We say a function $f : [1, \infty) \rightarrow [0, \infty)$ is operator convex-log if it can be written in the form $f(t) = h(\log t)$, $t \geq 1$, where $h : [0, \infty) \rightarrow [0, \infty)$ is an operator convex function. Similarly, a function $g : [1, \infty) \rightarrow [0, \infty)$ is called an operator concave-log function if it can be written in the form $g(t) = \varphi(\log t)$, $t \geq 1$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an operator concave function.

In what follows, the capital letters A, B denote $n \times n$ matrices or bounded linear operators on an n -dimensional complex Hilbert space $\mathcal{H}$. For positive operators A and B , the weak majorization $A \prec_w B$ means that

$$\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B), \quad k = 1, 2, \dots, n,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ are the eigenvalues of A listed in decreasing order. If equality holds when $k = n$, we have the majorization $A \prec B$. See [15] for more details. Also, the notation $\leq_{ols}$ is used for the so-called Olson order. For positive operators, $A \leq_{ols} B$ if and only if $A^r \leq B^r$ for every $r \geq 1$ [16].

Lemma 3.4 ([17, Corollary 2.3]): Let A and B be positive operators acting on a Hilbert space of finite dimension. Then for every $v \in [0, 1]$

$$\log(A \sharp_v B) \prec \log(A) \nabla_v \log(B).$$

Theorem 3.5: Let $f : [1, \infty) \rightarrow [0, \infty)$ be an operator convex-log function. Then for every $A, B \succ I$ and $v \in [0, 1]$

$$f(A \sharp_v B) \prec_w f(A) \nabla_v f(B). \quad (7)$$

Proof: Since f is operator convex-log, then there is an operator convex function $h : [0, \infty) \rightarrow [0, \infty)$ such that $f(t) = h(\log t)$, $t \geq 1$. Since $A, B \succ I$, so $A \sharp_v B \succ I$ and we have

$$\begin{aligned} f(A \sharp_v B) &= h(\log(A \sharp_v B)) \\ &\prec_w h(\log(A) \nabla_v \log(B)) \quad (\text{by Lemma 3.4}) \\ &\leq h(\log(A)) \nabla_v h(\log(B)) \quad (\text{by op. convexity of } h) \\ &= f(A) \nabla_v f(B). \end{aligned}$$

In the second inequality, we use the fact that for every convex function h , $A \prec B$ implies $h(A) \prec_w h(B)$ [14, Proposition 4.1.4]. ■

Remark 3.6: Inequality (7) can be considered as a variant of the operator geodesically convexity property for expansive operators. Also, it provides an elegant extension of Lemma 3.4.

In the sequel, we use the notation $\mu(s, t) := \max\{S(s), S(t)\}$, where $S(t) = \frac{t^{\frac{1}{t-1}}}{e^{\log(t^{\frac{1}{t-1}})}}$ for $t > 0$ is the so-called Specht ratio. Note that $\lim_{t \rightarrow 1} S(t) = 1$ and $S(t) = S(1/t) > 1$ for $t \neq 1$, $t > 0$. For more details, see [18]. We first give a reverse of Lemma 3.4 and then apply it to show the next main result. The following lemmas are needed.

Lemma 3.7 ([5, Lemma 1]): Let $0 < sA \leq B \leq tA$, $0 < s \leq t$ and $v \in [0, 1]$. Then

$$A \nabla_v B \leq \mu(s, t)(A \sharp_v B). \quad (8)$$

Lemma 3.8 ([19, Theorem 1]): Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and $v \in [0, 1]$. Then for all $r > 0$ and $k = 1, 2, \dots, n$

$$\lambda_k(e^{(1-v)H+vK}) \leq \mu^{\frac{1}{r}}(e^{rs}, e^{rt}) \lambda_k(e^{rH} \sharp_v e^{rK})^{\frac{1}{r}},$$

where $\preceq_{ols}$ is the so-called Olson order.

Lemma 3.9 ([19, Lemma 1]): Let A and B be positive definite matrices such that $sA \leq B \leq tA$ for some scalars $0 < s \leq t$ and $v \in [0, 1]$. Then

$$A^r \sharp_v B^r \leq \mu^r(s, t)(A \sharp_v B)^r \quad 0 < r \leq 1.$$

Proposition 3.10: Let A and B be positive definite matrices such that $e^s A \preceq_{ols} B \preceq_{ols} e^t A$ for some scalars $s \leq t$, and $v \in [0, 1]$. Then

$$\lambda_k(\log A \nabla_v \log B) \leq \lambda_k(\log(MN(A \sharp_v B))), \quad (9)$$

and so

$$\log A \nabla_v \log B \prec_w \log(MN(A \sharp_v B)),$$

where $M := \mu^{\frac{1}{r}}(e^{rs}, e^{rt})$, $N := \mu(e^s, e^t)$, and $0 < r \leq 1$.

Proof: Considering the condition $e^s A \preceq_{ols} B \preceq_{ols} e^t A$ in the form of $e^s e^{\log A} \preceq_{ols} e^{\log B} \preceq_{ols} e^t e^{\log A}$, we can apply Lemma 3.8 by setting $H = \log A$, $K = \log B$, $M = \mu^{\frac{1}{r}}(e^{rs}, e^{rt})$ and $r > 0$ as follows:

$$\lambda_k(e^{(1-v)\log A + v\log B}) \leq M \lambda_k(A^r \sharp_v B^r)^{\frac{1}{r}}. \quad (10)$$

On the other hand, since the sandwich condition $e^s A \preceq_{ols} B \preceq_{ols} e^t A$ implies $e^s A \leq B \leq e^t A$, we can use Lemma 3.9 for $0 < r \leq 1$ as follows:

$$A^r \sharp_v B^r \leq \mu^r(e^s, e^t)(A \sharp_v B)^r.$$

So

$$\begin{aligned}\lambda_k(A^r \sharp_\nu B^r)^{\frac{1}{r}} &\leq (\mu^r(e^s, e^t) \lambda_k(A \sharp_\nu B)^r)^{\frac{1}{r}} \\ &= \mu(e^s, e^t) \lambda_k(A \sharp_\nu B).\end{aligned}\tag{11}$$

Let $N = \mu(e^s, e^t)$. Combining inequalities (10) and (11) implies

$$\lambda_k\left(e^{(1-\nu)\log A + \nu\log B}\right) \leq MN\lambda_k((A \sharp_\nu B)) = \lambda_k(MN(A \sharp_\nu B)).$$

Thereupon

$$\log\left(\lambda_k\left(e^{(1-\nu)\log A + \nu\log B}\right)\right) \leq \log(\lambda_k(MN(A \sharp_\nu B))),$$

and hence

$$\lambda_k((1-\nu)\log A + \nu\log B) \leq \lambda_k(\log(MN(A \sharp_\nu B))).$$

■

Theorem 3.11: Let $g : [1, \infty) \rightarrow [0, \infty)$ be an operator concave-log function, A and B be positive definite matrices such that $e^s I \prec_{ols} e^s A \preceq_{ols} B \preceq_{ols} e^t A$ for some scalars $0 < s \leq t$, and $\nu \in [0, 1]$. Then for every $0 < r \leq 1$ and $k = 1, 2, \dots, n$, we have

$$\lambda_k(g(A) \nabla_\nu g(B)) \leq S(e^{rt})^{\frac{1}{r}} S(e^t) \lambda_k(g(A \sharp_\nu B)).\tag{12}$$

Proof: Since g is an operator concave-log function, then there is an operator concave function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $g(t) = \varphi(\log t)$, $t \geq 1$. Also, according to Remark 3.2, g is an operator concave function. On the other hand, the sandwich condition $e^s I \prec_{ols} e^s A \preceq_{ols} B \preceq_{ols} e^t A$ with $0 < s < t$ implies $A, B \succ I$. Compute

$$\begin{aligned}\lambda_k(g(A) \nabla_\nu g(B)) &= \lambda_k(\varphi(\log(A)) \nabla_\nu \varphi(\log(B))) \\ &\leq \lambda_k(\varphi(\log(A) \nabla_\nu \log(B))) \quad (\text{op. concavity of } \varphi) \\ &= \varphi(\lambda_k(\log(A) \nabla_\nu \log(B))) \\ &\leq \varphi(\lambda_k(\log MN(A \sharp_\nu B))) \quad ((9) \text{ and monotonicity of } \varphi) \\ &= \lambda_k(\varphi(\log MN(A \sharp_\nu B))) \\ &= \lambda_k(g(MN(A \sharp_\nu B))) \\ &\leq MN\lambda_k(g(A \sharp_\nu B)) \quad (\text{concavity of } g),\end{aligned}$$

where constants M and N are defined in Proposition 3.10. On the other hand, since $S(h)$ is an increasing function on $[1, \infty)$ and $1 < e^s \leq e^t$ for $0 < s \leq t$, therefore $MN = S(e^{rt})^{\frac{1}{r}} S(e^t)$ as desired. For the last inequality, given that $MN \geq 1$, we use the fact for every nonnegative concave function g and every $z > 1$, $g(zx) \leq zg(x)$. ■

Remark 3.12: Under the assumptions of Theorem 3.11, we immediately have

$$g(A) \nabla_\nu g(B) \prec_w \mu g(A \sharp_\nu B),$$

where $\mu = S(e^{rt})^{\frac{1}{r}} S(e^t)$. This inequality provides a variant of the geodesically concavity property

$$g(a) \nabla_v g(b) \leq g(a \sharp_v b),$$

for operator concave-log functions. Also, inequality (12) is equivalent to the existence of a unitary operator U satisfying

$$g(A) \sharp_v g(B) \leq g(A) \nabla_v g(B) \leq \mu U g(A \sharp_v B) U^*. \quad (13)$$

By applying Theorem 3.11, we can get a variant of the operator Aczél inequality involving operator concave-log functions as follows:

Corollary 3.13: *Let $g : [1, \infty) \rightarrow [0, \infty)$ be an operator concave-log function, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$ and $e^s I \prec_{ols} e^s A^p \preceq_{ols} B^q \preceq_{ols} e^t A^p$ for some scalars $0 < s \leq t$. Then, there is a unitary operator U such that for all $x \in \mathcal{H}$*

$$\begin{aligned} g(A^p) \sharp_{\frac{1}{q}} g(B^q) &\leq \mu \cdot U g(A^p \sharp_{\frac{1}{q}} B^q) U^*, \\ \langle g(A^p) Ux, Ux \rangle^{1/p} \langle g(B^q) Ux, Ux \rangle^{1/q} &\leq \mu \cdot \langle g(A^p \sharp_{1/q} B^q) Ux, Ux \rangle, \end{aligned}$$

where $\mu := S(e^{rt})^{\frac{1}{r}} S(e^t)$ and $0 < r \leq 1$.

Proof: Putting $A := A^p$, $B := B^q$ and $v := 1/q$ in inequality (13), we have the first alleged inequality. For the second, we first note that the condition $e^s I \prec_{ols} e^s A^p \preceq_{ols} B^q \preceq_{ols} e^t A^p$ implies $e^s I \leq e^s A^p \leq B^q \leq e^t A^p$. So, by applying Lemma 3.7 for the operators A^p and B^q , we will get

$$A^p \nabla_v B^q \leq \mu (e^s, e^t) (A^p \sharp_v B^q) = S(e^t) (A^p \sharp_v B^q) \leq \mu \cdot (A^p \sharp_v B^q). \quad (14)$$

As it is shown in Remark 3.2, g is an operator concave function. Also, it is the composition of two operator monotone functions. So, we can write

$$\begin{aligned} \mu \cdot \langle g(A^p \sharp_{1/q} B^q) Ux, Ux \rangle &\geq \langle g(\mu (A^p \sharp_{1/q} B^q)) Ux, Ux \rangle \quad (\text{concavity of } g) \\ &\geq \langle g(A^p \nabla_{1/q} B^q) Ux, Ux \rangle \quad (\text{op. monotonicity of } g \text{ with (14)}) \\ &\geq \left\langle \left(\frac{1}{p} g(A^p) + \frac{1}{q} g(B^q) \right) Ux, Ux \right\rangle \quad (\text{op. concavity of } g) \\ &= \frac{1}{p} \langle g(A^p) Ux, Ux \rangle + \frac{1}{q} \langle g(B^q) Ux, Ux \rangle \\ &\geq \langle g(A^p) Ux, Ux \rangle^{1/p} \langle g(B^q) Ux, Ux \rangle^{1/q} \quad (\text{AM-GM inequality}). \end{aligned}$$

■

4. Another type of geodesically convex function

Definition 4.1 ([3]): A function $F : B(\mathcal{H})^+ \rightarrow \mathbb{R}$ is defined in the set $B(\mathcal{H})^+$ of positive operators on a finite-dimensional Hilbert space $\mathcal{H}$, where $\dim \mathcal{H} = n$ is said to be n -geodesically convex if

$$F(A \sharp_\nu B) \leq F(A) \nabla_\nu F(B), \quad \text{for any } \nu \in [0, 1]. \quad (15)$$

If inequality (15) holds for every n , then the function F is called geodesically convex.

The functions $F(A) = \text{tr}(e^A)$, $F(A) = \text{tr}(A^\alpha)$, $\alpha \geq 1$, $\lambda_1(e^A)$ and $\lambda_1(A^\alpha)$, $\alpha \geq 1$ are examples of geodesically convex functions. For more results and examples, see [20].

Bourin and Hiai [6, Proposition 3.5] showed that for every $A, B > 0$, $\nu \in [0, 1]$ and $k = 1, 2, \dots, n$,

$$\prod_{j=1}^k \lambda_j(A \sharp_\nu B) \leq \left\{ \prod_{j=1}^k \lambda_j(A) \right\} \sharp_\nu \left\{ \prod_{j=1}^k \lambda_j(B) \right\} \quad (16)$$

and

$$\prod_{j=n+1-k}^n \lambda_j(A \sharp_\nu B) \geq \left\{ \prod_{j=n+1-k}^n \lambda_j(A) \right\} \sharp_\nu \left\{ \prod_{j=n+1-k}^n \lambda_j(B) \right\}. \quad (17)$$

It is deduced from inequality (16) that $F(A) = \prod_{j=1}^k \lambda_j(A)$ and $F(A) = \det(A)$ are also geodesically convex functions. In this section, we investigate the geodesically convexity properties of some new functions involved with operator functions and achieve a generalization of the above Minkowski-type inequalities, simultaneously.

It is shown in [20, Theorem 2.3] if h is an increasing convex function on $(0, \infty)$, then $\sum_{j=1}^k h(\lambda_j(A))$ is geodesically convex. In the following, we give a corresponding result for increasing geometrically convex functions on $(0, \infty)$.

Lemma 4.2: *Let g be an increasing geometrically convex function on $(0, \infty)$. Then the function $F(A) = \sum_{j=1}^k g(\lambda_j(A))$, $k = 1, 2, \dots, n$, is a geodesically convex function.*

Proof: First, note that inequality (16) is equivalent to the following one:

$$\prod_{j=1}^k \lambda_j(A \sharp_\nu B) \leq \prod_{j=1}^k \lambda_j(A)^\nu \lambda_j(B)^{1-\nu} = \prod_{j=1}^k \lambda_j(A^\nu) \lambda_j(B^{1-\nu}). \quad (18)$$

Also, since $g(t)$ is a geometrically convex function, $g(e^t)$ is a convex function due to the following inequality

$$g(e^{x \nabla_\nu y}) = g(e^{(1-\nu)x} e^{\nu y}) = g(e^x \sharp_\nu e^y) \leq g(e^x) \sharp_\nu g(e^y) \leq g(e^x) \nabla_\nu g(e^y).$$

Now, by applying a classical result on the function g [14, Proposition 4.1.6] and inequality (18), we have

$$\sum_{j=1}^k g(\lambda_j(A \sharp_\nu B)) \leq \sum_{j=1}^k g(\lambda_j^{1-\nu}(A) \lambda_j^\nu(B)).$$

Hence, we can write

$$\begin{aligned}
 \sum_{j=1}^k g(\lambda_j(A \sharp_\nu B)) &\leq \sum_{j=1}^k g(\lambda_j^{1-\nu}(A) \lambda_j^\nu(B)) \\
 &\leq \sum_{j=1}^k (g(\lambda_j(A)))^{1-\nu} g((\lambda_j(B))^\nu) \quad (\text{geometrical convexity of } g) \\
 &\leq \left\{ \sum_{j=1}^k g(\lambda_j(A)) \right\} \sharp_\nu \left\{ \sum_{j=1}^k g(\lambda_j(B)) \right\} \quad (\text{Cauchy-Schwarz inequality}) \\
 &\leq \left\{ \prod_{j=1}^k g(\lambda_j(A)) \right\} \nabla_\nu \left\{ \prod_{j=1}^k g(\lambda_j(B)) \right\}, \quad (\text{AM-GM inequality}).
 \end{aligned}$$

■

In the sequel, we present some results involving operator functions.

Lemma 4.3 ([11]): *Let g be a nonnegative operator monotone decreasing function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some constants $0 < s \leq t$. Then, for all $\nu \in [0, 1]$*

$$g(A \sharp_\nu B) \leq \mu(s, t)(g(A) \sharp_\nu g(B)).$$

Theorem 4.4: *Let g be a nonnegative operator monotone decreasing function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $s, t > 0$. Then for all $\nu \in [0, 1]$ and $k = 1, 2, \dots, n$,*

$$\prod_{j=1}^k \lambda_j(g(A \sharp_\nu B)) \leq \mu^k(s, t) \left(\left\{ \prod_{j=1}^k \lambda_j(g(A)) \right\} \sharp_\nu \left\{ \prod_{j=1}^k \lambda_j(g(B)) \right\} \right).$$

Proof: We compute

$$\begin{aligned}
 \prod_{j=1}^k \lambda_j(g(A \sharp_\nu B)) &\leq \prod_{j=1}^k \lambda_j(\mu(s, t)(g(A) \sharp_\nu g(B))) \quad (\text{by Lemma 4.3}) \\
 &= \mu^k(s, t) \prod_{j=1}^k \lambda_j(g(A) \sharp_\nu g(B)) \\
 &\leq \mu^k(s, t) \left(\left\{ \prod_{j=1}^k \lambda_j(g(A)) \right\} \sharp_\nu \left\{ \prod_{j=1}^k \lambda_j(g(B)) \right\} \right) \quad (\text{by (16)}).
 \end{aligned}$$

■

Corollary 4.5: *Let g be a nonnegative operator monotone decreasing function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $s, t > 0$. Then for all $\nu \in [0, 1]$*

$$\det g(A \sharp_\nu B) \leq \mu^n(s, t) (\det g(A) \sharp_\nu \det g(B)).$$

Remark 4.6: According to Theorem 4.4, by letting $F(A) = \prod_{j=1}^k \lambda_j(g(A))$ where g is an operator monotone decreasing function on $(0, \infty)$, we have

$$F(A \sharp_\nu B) \leq \mu^k(s, t) (F(A) \nabla_\nu F(B)).$$

This inequality gives a variant of the geodesically convexity property of (15) for the function $F(A) = \prod_{j=1}^k \lambda_j(g(A))$. Further, it provides an extension of a Minkowski-type inequality (16) to the operator functions.

In the next, we will see an extension of a Minkowski-type inequality (17).

Theorem 4.7: Let f be an operator monotone function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $s, t > 0$. Then for all $\nu \in [0, 1]$ and $k = 1, 2, \dots, n$,

$$\prod_{j=n+1-k}^n \lambda_j(f(A \sharp_\nu B)) \geq \mu^k(s, t) \left(\left\{ \prod_{j=n+1-k}^n \lambda_j(f(A)) \right\} \sharp_\nu \left\{ \prod_{j=n+1-k}^n \lambda_j(f(B)) \right\} \right).$$

Proof: Since f is operator monotone on $(0, \infty)$, so $1/f$ is operator monotone decreasing on $(0, \infty)$ and we can apply Theorem 4.4 for $g = 1/f$ as follows:

$$\prod_{j=1}^k \lambda_j((f(A \sharp_\nu B))^{-1}) \leq \mu^k(s, t) \left\{ \prod_{j=1}^k \lambda_j((f(A))^{-1}) \right\} \sharp_\nu \left\{ \prod_{j=1}^k \lambda_j((f(B))^{-1}) \right\},$$

and hence

$$\begin{aligned} & \prod_{j=n+1-k}^n (\lambda_j(f(A \sharp_\nu B)))^{-1} \\ & \leq \mu^k(s, t) \left\{ \prod_{j=n+1-k}^n (\lambda_j(f(A)))^{-1} \right\} \sharp_\nu \left\{ \prod_{j=n+1-k}^n (\lambda_j(f(B)))^{-1} \right\}. \end{aligned}$$

By using the property $X^{-1} \sharp_\nu Y^{-1} = (X \sharp_\nu Y)^{-1}$ and reversing the inequality, we get the desired result. ■

Corollary 4.8: Let f be an operator monotone function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $s, t > 0$. Then for all $\nu \in [0, 1]$

$$\det f(A \sharp_\nu B) \geq \mu^n(s, t) (\det f(A) \sharp_\nu \det f(B)).$$

Remark 4.9: The appeared constant $\mu(s, t) = \max\{S(s), S(t)\}$ in all the results of the preceding sections can be replaced by $\max\{K(s)^R, K(t)^R\}$, where $K(h) = \frac{(h+1)^2}{4h}$, $h > 0$ is the Kantorovich constant and $R = \max\{\nu, 1 - \nu\}$, with no ordering between them (see [11]).

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