Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

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LINEAR AND MULTILINEAR ALGEBRA
https://doi.org/10.1080/03081087.2019.1664385

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On the dilation of a conditional operator

M. R. Jabbarzadeh and H. Emamalipour

Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

ABSTRACT
Let E be the conditional expectation operator with respect to a σ-finite subalgebra $\mathcal{A}$ of Σ. This paper is concerned with the study of a conditional type operator $T_{u,v}$ on $L^2(\Sigma)$ of the form $T_{u,v} = EM_uE + EM_vQ$, where u and v are measurable functions and $Q = I - E$. We discuss matrix theoretic characterizations of several properties of $T_{u,v}$ with an application-oriented approach.

ARTICLE HISTORY
Received 28 November 2018
Accepted 29 July 2019

COMMUNICATED BY
Ngai-Ching Wong

KEYWORDS
Conditional expectation; multiplication operator; generalized inverse; Moore–Penrose inverse; Drazin inverse

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
Primary 47B20; Secondary 47B25.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $B(\mathcal{H})$ be the set of linear bounded operators on $\mathcal{H}$. We use $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, to denote the range space and the null space of $T \in B(\mathcal{H})$. For an operator $T \in B(\mathcal{H})$, the adjoint of T is denoted by T^*. T is self-adjoint if $T^* = T$ and T is normal if $T^*T = TT^*$. We write $T \geq 0$ if T is a positive operator, meaning that $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. An orthogonal projection is an operator $P \in B(\mathcal{H})$ such that $P^2 = P = P^*$. Let $CR(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For $T \in B(\mathcal{H})$, the Moore–Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(\mathcal{H})$ that satisfies the equations $TT^\dagger T = T$, $T^\dagger TT^\dagger = T^\dagger$, $(TT^\dagger)^* = TT^\dagger$ and $(T^\dagger T)^* = T^\dagger T$. We recall that $T^\dagger$ exists if and only if T has closed range. The Drazin inverse of $T \in B(\mathcal{H})$, denoted by T^D, is the unique solution to the equations $T^{k+1}S = T^k$, $STS = S$, $TS = ST$, for some $k \in \mathbb{N}$. The minimal such k is called the index, denoted by $\operatorname{ind}(T)$, of T. When $k = 1$, the Drazin inverse reduces to the group inverse and it is denoted by $T^\#$. Recall that $\operatorname{asc}(T)$ ($\operatorname{des}(T)$), the ascent (descent) of $T \in B(\mathcal{H})$, is the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ ($\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If no such n exists, then $\operatorname{asc}(T) = \infty$ ($\operatorname{des}(T) = \infty$). For $T \in B(\mathcal{H})$, T^D exists if and only

CONTACT M. R. Jabbarzadeh mjabbar@tabrizu.ac.ir Faculty of Mathematical Sciences, University of Tabriz, 5166615648 Tabriz, Iran

if T has finite ascent and descent. In this case, $\mathrm{ind}(T) = \mathrm{asc}(T) = \mathrm{des}(T) = n$. For other important properties of $T^\dagger$ and T^D, see e.g. [1–5].

Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a sub-σ-finite algebra of Σ. The linear space of all complex-valued Σ-measurable functions on X is denoted by $L^0(\Sigma)$. All statements about equality, inclusion and disjointness are to be understood to hold up to a set of μ-measure 0. We use the notation of [6] which is a basic reference. The support of a measurable function $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a sub-sigma algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ and μ is the mapping $f \to E_\mu^{\mathcal{A}} f$, defined for all μ-measurable non-negative f where $E_\mu^{\mathcal{A}} f$, by the Radon–Nikodym theorem, is the unique finite-valued $\mathcal{A}$-measurable function satisfying

$$\int_A f \mathrm{d}\mu = \int_A E_\mu^{\mathcal{A}}(f) \mathrm{d}\mu, \quad \forall A \in \mathcal{A}.$$

For simplicity, set $E_\mu^{\mathcal{A}} = E$. Let $u \in L^0(\Sigma)$ be real valued and consider the set $B_u = \{x \in X : E(u^+)(x) = E(u^-)(x) = \infty\}$, where $u^+ = \max\{f, 0\}$ and $u^- = \max\{-f, 0\}$. The function u is said to be conditionable with respect to $\mathcal{A}$ if $\mu(B_u) = 0$. Put $E(u) = E(u^+) - E(u^-)$. If $u = u_1 + iu_2 \in L^0(\Sigma)$, then u is said to be conditionable if u_1 and u_2 are conditionable. In this case, we set $E(u) = E(u_1) + iE(u_2)$. This defines a linear operator $E : \mathcal{D}(E) \to L^0(\mathcal{A}) \subseteq L^0(\Sigma)$, where the domain $\mathcal{D}(E)$ of E is defined by $\mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$. It follows that $\mathcal{D}(E)$ contains $\{L^p(\Sigma) : 1 \leq p \leq \infty\} \cup \{f \in L^0(\Sigma) : f \geq 0\}$ (see [6,7]). A conditional expectation operator E on $L^2(\Sigma)$ is an orthogonal projection onto $L^2(\mathcal{A})$. A detailed discussion and verification of most of the properties may be found in [6,8–10]. Those properties of E used in our discussion are summarized below. In all cases, we assume that $f, g, fg \in \mathcal{D}(E)$.

- (i) If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- (ii) $\sigma(|E(f)|) \subseteq \sigma(E(|f|))$ and $\chi_S f = f$ whenever $\sigma(f) \subseteq S \in \Sigma$.
- (iii) (conditional Hölder inequality) $|E(fg)| \leq (E(|f|^p))^{1/p}(E(|g|^q))^{1/q}$, where $f, g \in L^0(\Sigma)$ are finite valued functions and $1/p + 1/q = 1$. The case $p = 2$ is called the conditional Cauchy–Schwarz inequality.

For $u \in L^0(\Sigma)$, the multiplication operator $M_u : L^2(\Sigma) \to L^0(\Sigma)$ is defined as $M_u f = uf$. It is a classical fact that $M_u \in B(L^2(\Sigma))$ if and only if $u \in L^\infty(\Sigma)$, and in this case, $\|M_u\| = \|u\|_\infty$ [11]. Conditional expectation type operators are closely related to the multiplication operators, integral and averaging operators and to the operators called conditional type which has been introduced in [8,12]. Conditional operators and various types of generalized inverses have been widely used in practice. For u and v in $L^0(\Sigma)$, we discuss matrix theoretic characterizations of $T_{u,v} = EM_uE + EM_vQ$ on $L^2(\Sigma)$, where $Q = I - E$.

In the next section, first we review some basic results on EM_u and state some general assumptions. Then we obtain the Drazin and Moore–Penrose inverses of $T_{u,v}$ under certain conditions. In addition, we study several other properties of $T_{u,v}$ with an application-oriented approach.

2. Characterizations

We begin this section with a simple fact which will be applied in the sequel.

Lemma 2.1: *Let $f \in L^\infty(\mathcal{A})$. Then*

$$\|f\|_\infty = \sup\left\{\frac{1}{\mu(A)}\int_A |f|\mathrm{d}\mu,\ A \in \mathcal{A},\ 0 < \mu(A) < \infty\right\}. \tag{1}$$

Proof: Denote the right side of (1) by α. For arbitrary $\varepsilon > 0$, take $B = \{|f| > \|f\|_\infty - \varepsilon\}$. Since $\mathcal{A}$ is σ-finite, there is $A_0 \subseteq B$ with $0 < \mu(A_0) < \infty$ such that

$$\alpha \geq \frac{1}{\mu(A_0)}\int_{A_0} |f|\mathrm{d}\mu \geq \|f\|_\infty - \varepsilon.$$

The converse is obvious. ■

It is worth noting that $L^\infty(\Sigma)$ is invariant under E. Indeed, by Lemma 2.1 and (ii), $\|E(f)\|_\infty \leq \|E(|f|)\|_\infty \leq \|f\|_\infty < \infty$ for all $f \in L^\infty(\Sigma)$. Specially, if $\mathcal{A} = \{\emptyset, X\}$ and $\mu(X) < \infty$, then $E(f) = 1/\mu(X)\int_X f\mathrm{d}\mu$, for all $f \in L^1(\Sigma)$ and hence $\mu(X)|E(f)| \leq \int_X |f|\mathrm{d}\mu = \|f\|_1$ for all $f \in L^1(\Sigma)$. Thus, $E(L^1(\Sigma)) \subseteq L^\infty(\Sigma)$. Let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of Σ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \Sigma$. For $u \in L^0(\mathcal{B})$ with $u\mathcal{D}(E^{\mathcal{B}}) \subseteq \mathcal{D}(E^{\mathcal{B}})$, define $T_u : L^p(\mathcal{B}) \to L^p(\mathcal{A})$ as $T_u(f) = E^{\mathcal{A}}_{\mathcal{B}}(uf)$, where $E^{\mathcal{A}}_{\mathcal{B}}$ is the conditional expectation operator from $L^p(\mathcal{B})$ onto $L^p(\mathcal{A})$.

Proposition 2.2: *Let $u \in L^0(\mathcal{B})$, $E = E^{\mathcal{A}}_{\mathcal{B}}$, $1 < p < \infty$ and let q be the conjugate component to p. Then $T_u \in B(L^p(\mathcal{B}), L^p(\mathcal{A}))$ if and only if $E(|u|^q) \in L^\infty(\mathcal{A})$. In this case, the adjoint operator $T_u^* : L^q(\mathcal{A}) \to L^q(\mathcal{B})$ is given by $T_u^*(g) = ug$ and $\|T_u\| = \|E(|u|^q)\|_\infty^{1/q}$.*

Proof: The proof is given in [12, Proposition 2.1]. For the sake of completeness, we give the details here. Let $E(|u|^q) \in L^\infty(\mathcal{A})$. Then for each $f \in (L^p(\mathcal{B}))$, we have

$$\begin{aligned}\int_X |E(uf)|^p\mathrm{d}\mu &\leq \int_X (E(|u|^q))^{\frac{p}{q}} E(|f|^p)\mathrm{d}\mu \\ &\leq \|E(|u|^q)\|_\infty^{\frac{p}{q}} \int_X |f|^p\mathrm{d}\mu = \|E(|u|^q)\|_\infty^{\frac{p}{q}}\|f\|_p^p.\end{aligned}$$

It follows that

$$\|T_u\| = \sup_{\|f\|_p \leq 1} \|T_u(f)\|_p = \sup_{\|f\|_p \leq 1} (\int_X |E(uf)|^p d)^{\frac{1}{p}} \leq \|E(|u|^q)\|_\infty^{\frac{1}{q}}.$$

Thus T_u^* is bounded. Since the mapping $g \mapsto F_g : F_g(f) = \int_X fg\mathrm{d}\mu$ is an isometry from $L^q(\mathcal{A})$ onto $(L^p(\mathcal{A}))^*$, so for each $g \in L^q(\mathcal{A})$ and $B \in \mathcal{B}$ we have

$$\begin{aligned}(T_u^* F_g)(\chi_B) = F_g(T_u\chi_B) &= \int_X E(u\chi_B)g\mathrm{d}\mu \\ &= \int_X (u\chi_B)g\mathrm{d}\mu = F_{ug}(\chi_B).\end{aligned}$$

After identifying g with F_g we obtain $T_u^*(g) = ug$. Now, let T_u be bounded and let $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$. Then

$$\int_A E(|u|^q)\mathrm{d}\mu = \int_A |u|^q\mathrm{d}\mu = \int_X |T_u^*(\chi_A)|^q\mathrm{d}\mu \leq \|T_u\|^q\mu(A).$$

It follows from Lemma 2.1 that

$$\|E(|u|^q)\|_\infty^{\frac{1}{q}} = \left\{\sup \frac{1}{\mu(A)} \int_A E(|u|^q)\mathrm{d}\mu,\ A \in \mathcal{A}, 0 < \mu(A) < \infty\right\}^{\frac{1}{q}} \le \|T_u\|.$$

This completes the proof. ■

Let $p=1$ and let $f \in L^1(\mathcal{B})$. If $u \in L^\infty(\mathcal{B})$, then $\|T_u(f)\|_1 \le \|E(|uf|)\|_1 = \|uf\|_1 \le \|u\|_\infty \|f\|_1$. Conversely, if $\|T_u\| < \infty$, then $\|uf\|_1 = \|E(|uf|)\|_1 = \|E(\mathrm{sgn}(uf)uf)\|_1 = \|T_u(\mathrm{sgn}(uf)f)\|_1 \le \|T_u\|\ \ \|f\|_1$. Then $M_u \in B(L^1(\mathcal{B}))$, and hence $u \in L^\infty(\mathcal{B})$. Consequently, $T_u \in B(L^1(\mathcal{B}), L^1(\mathcal{A}))$ if and only if $u \in L^\infty(\mathcal{B})$, and in this case $\|T_u\| = \|u\|_\infty$. By a similar argument and using the fact that $(L^1)^* = L^\infty$, one can obtain that $T_u \in B(L^\infty(\mathcal{B}), L^\infty(\mathcal{A}))$ if and only if $E(|u|) \in L^\infty(\mathcal{A})$ [6].

For $p \in (1, \infty)$ with the conjugate component q, put $\mathcal{K}_q(\mathcal{B}) = \{u \in L^0(\mathcal{B}) : E(|u|^q) \in L^\infty(\mathcal{A})\}$. Then $L^\infty(\mathcal{B}) \subseteq \mathcal{K}_q(\mathcal{B})$. By Proposition 2.2, $T_u \in B(L^p(\mathcal{B}), L^p(\mathcal{A}))$ if and only if $u \in \mathcal{K}_q(\mathcal{B})$. Let $\alpha \in \mathbb{C}$ and $u, v \in \mathcal{K}_q(\mathcal{B})$. Then by the conditional Hölder inequality, we have $E(|\alpha u + v|^q)^{1/q} \le |\alpha| E(|u|^q)^{1/q} + E(|v|^q)^{1/q}$. Thus $\mathcal{K}_q(\mathcal{B})$ is a linear subspace of $L^0(\mathcal{B})$. For $u \in \mathcal{K}_q(\mathcal{B})$, set $\|u\|_{\mathcal{K}_q} = \|E(|u|^q)\|_\infty^{1/q}$. Then $\|\ \|_{\mathcal{K}_q}$ is a norm on $\mathcal{K}_q(\mathcal{B})$ and the mapping $u \mapsto T_u$ is an isometry from $(\mathcal{K}_q(\mathcal{B}), \|\cdot\|_{\mathcal{K}_q})$ onto $\mathcal{B}_p := \{T_u : u \in \mathcal{K}_q(\mathcal{B})\}$. Since $\mathcal{B}_p$ is a weakly closed subspace of $B(L^p(\mathcal{B}), L^p(\mathcal{A}))$ [6, Theorem 3.2.1], we see that $(\mathcal{K}_q(\mathcal{B}), \|\cdot\|_{\mathcal{K}_q})$ is a Banach space. In addition, if $w \in L^0(\mathcal{B})$, $u \in \mathcal{K}_q(\mathcal{B})$ and $|w| \le |u|$, the monotonicity of E implies that $w \in \mathcal{K}_q(\mathcal{B})$. So $\mathcal{K}_q(\mathcal{B})$ is an ordered ideal. Now, let $u, w \in \mathcal{K}_2(\mathcal{B})$. It follows from the conditional Cauchy–Schwarz inequality that $E(|T_u(w)|^2) = |E(uw)|^2 \le E(|u|^2)E(|w|^2) \in L^\infty(\mathcal{A})$, and so $T_u(w) \in \mathcal{K}_2(\mathcal{B})$. Hence $\mathcal{K}_2(\mathcal{B})$ is an invariant subspace for $\mathcal{B}_p$.

Set $\mathcal{B} = \Sigma$ and take $\mathcal{K} = \mathcal{K}_2(\Sigma)$. For $w, u \in \mathcal{D}(E)$, the mapping $T : L^2(\Sigma) \supseteq \mathcal{D}(T) \to L^2(\Sigma)$ given by $T(f) = wE(uf)$ for $f \in \mathcal{D}(T) = \{f \in L^2(\Sigma) : T(f) \in L^2(\Sigma)\}$ is well-defined and linear. Such an operator is called a Lambert (weighted) conditional operator induced by the pair (w, u). Set $a = \sqrt{E(|w|^2)}$ and $b = \sqrt{E(|u|^2)}$. It is easy to check that $\|M_w E M_u f\|_2 = \|M_a E M_u f\|_2 = \|E M_{au} f\|_2$ for all $f \in L^2(\Sigma)$. Then Proposition 2.2 implies that $T = M_w E M_u$ is bounded in $L^2(\Sigma)$ if and only if $au \in \mathcal{K}$. In this case, $\|T\| = \|ab\|_\infty$ ([13]).

Recall that $L^2(\Sigma) = \mathcal{R}(E) \oplus \mathcal{N}(E)$, where $\mathcal{R}(E) = L^2(\mathcal{A})$ and $\mathcal{N}(E) = \{f - Ef : f \in L^2(\Sigma)\}$. Put $Q = I - E$. So each element of $L^2(\Sigma)$ can be written as $f = f_1 + f_2 = E(f) + Q(f)$ such that $E(f_1) = f_1$ and $E(f_2) = 0$. Let $f \in L^0(\Sigma)$. Since $E(|f|^2) = E((f_1 + f_2)(\bar{f}_1 + \bar{f}_2)) = |f_1|^2 + E(|f_2|^2)$, we see that $\max\{|f_1|, E(|f_2|^2)^{1/2}\} \le E(|f|^2)^{1/2}$. Hence for $f \in L^\infty(\Sigma)$, we have

$$\max\{\|f_1\|_{\mathcal{K}_2}, \|f_2\|_{\mathcal{K}_2}\} = \max\{\|f_1\|_\infty, \|E(|f_2|^2)^{1/2}\|_\infty\} \le \|f\|_{\mathcal{K}_2}.$$

Let $f, g \in L^0(\Sigma)$. Then it may happen that $f_2 g_2 \in L^0(\mathcal{A})$ or $f_2 g_2 \in \mathcal{N}(E)$. In general, $f_2 g_2 = (f_2 g_2)_1 - (f_2 g_2)_2$.

Example 2.3: (a) Let $X = [-1, 1]$, let $\mathrm{d}\mu = \frac{\mathrm{d}x}{2}$, let Σ be the Lebesgue measurable sets, and let $\mathcal{A} = \{\emptyset, X\}$. Then $E(f) = \frac{1}{2}\int_{-1}^{1} f(x)\mathrm{d}x$, for all $f \in L^2([-1, 1])$. Put $f(x) = x$ and $g(x) = 1 - 3x^2$. Then $E(f) = E(g) = E(fg) = 0$. Therefore, $f_2 = f$, $g_2 = g$, and $f_2 g_2 = fg$ are in $\mathcal{N}(E)$.

(b) Let $X = [0,1]$, let $\mathrm{d}\mu = \frac{\mathrm{d}x}{2}$, let Σ be the Lebesgue sets, and let $\mathcal{A}$ be the sigma subalgebra of Σ consisting of sets symmetric about the point $\frac{1}{2}$. It is easy to check that $f_1(x) = \frac{f(x)+f(1-x)}{2}$ and $f_2(x) = \frac{f(x)-f(1-x)}{2}$, for all $f \in L^2([0,1])$. So, $L^2(\mathcal{A}) = \{f \in L^2([0,1]) : f(x) = f(1-x)\}$. Set $f(x) = g(x) = x - \frac{1}{2}$. Then $f, g \in \mathcal{N}(E)$, but $(fg)(x) = (x - \frac{1}{2})^2 \in L^2(\mathcal{A})$.

The matrix representation of $T = M_w E M_u$ with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$ is

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} M_{w_1 u_1} & EM_{w_1 u_2} \\ M_{w_2 u_1} & M_{w_2} EM_{u_2} \end{bmatrix} \tag{2}$$

(see [14]). Note that

$$\begin{aligned} \|ab\|_\infty &= \|E(|w|^2)E(|u|^2)\|_\infty^{\frac{1}{2}} = \|(|w_1|^2 + E(|w_2|^2))(|u_1|^2 + E(|u_2|^2))\|_\infty^{\frac{1}{2}} \\ &\leq \sqrt{\|T_{11}\|^2 + \|T_{12}\|^2 + \|T_{21}\|^2 + \|T_{22}\|^2}. \end{aligned}$$

Definition 2.4: Let $u, v \in \mathcal{K} = \mathcal{K}_2(\Sigma)$. An operator $T_{u,v}$ is called a conditional dilation of $T_u = EM_u$ if $T_{u,v} = EM_u E + EM_v Q$, where $Q = I - E$. In particular, $T_{u,u} = T_u$ whenever $u = v$.

Note that $T_{u,v} = M_{E(u-v)}E + EM_v$. Since any $f \in L^2(\Sigma)$ can be written uniquely as $f = f_1 + f_2$, where $f_1 = Ef \in L^2(\mathcal{A})$ and $f_2 = f - E(f) \in \mathcal{N}(E)$, it ensures that for any $u, v \in \mathcal{K}$,

$$\begin{aligned} T_{u,v}(f) &= E((u_1 + u_2)f_1) + E((v_1 + v_2)f_2) \\ &= u_1 f_1 + E(v_2 f_2) = M_{u_1} f_1 + EM_{v_2} f_2 = \begin{bmatrix} M_{u_1} & EM_{v_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \end{aligned}$$

Let $r, s \in \mathcal{K}$ and $\alpha \in \mathbb{F}$. Then

$$\alpha T_{u,v} + T_{r,s} = \begin{bmatrix} M_{\alpha u_1 + r_1} & EM_{\alpha v_2 + s_2} \\ 0 & 0 \end{bmatrix},$$

$$(T_{u,v})(T_{r,s}) = \begin{bmatrix} M_{u_1 r_1} & EM_{u_1 s_2} \\ 0 & 0 \end{bmatrix}.$$

In the following, we collect some elementary algebraic properties of the conditional dilation operators.

Proposition 2.5: *Let $u, v, r, s \in \mathcal{K}$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{F}$. The following hold.*

(a) $\alpha T_{u,v} + T_{r,s} = T_{\alpha u + r, \alpha v + s}$.
(b) $(T_{u,v})(T_{r,s}) = T_{rE(u), sE(u)}$.
(c) $T_{u,v}^n = M_{(E(u))^{n-1}} T_{u,v}$.
(d) $(T_{u,v}^n)^* = M_{(E(\bar{u}))^{n-1}} \{M_{(E(\bar{u}-\bar{v})+\bar{v})} E\}$.

Using Proposition 2.5(c), $\|T_{u,v}^n\| \leq \|E(u)\|_\infty^{n-1} \|T_{u,v}\|$. It follows that $r(T_{u,v}) \leq \|E(u)\|_\infty$, where $r(T_{u,v})$ is the spectral radius of $T_{u,v}$. Using [15, Theorem 2.8.], we have $\mathrm{sp}(T_{u,v}) \cup$

$\{0\} = \mathrm{sp}(T_u) \cup \{0\} = \text{ess range}\{E(u)\} \cup \{0\}$, where $\mathrm{sp}(T_u)$ is the spectrum of T_u. Consequently, $r(T_{u,v}) = \|E(u)\|_\infty$.

Theorem 2.6: *Let $u, v \in \mathcal{K}$ and let $T_{u,v} \neq 0$. Then the following hold.*

(a) *$T_{u,v}$ is self-adjoint if and only if $E(u)$ is real valued and $v_2 \in L^0(\mathcal{A})$.*
(b) *$T_{u,v}$ is positive if and only if $E(u) \geq 0$ and $v_2 \in L^0(\mathcal{A})$.*
(c) *$T_{u,v}$ is an orthogonal projection if and only if $T_{u,v} = E$.*
(d) *$T_{u,v}$ is normal if and only if v is an $\mathcal{A}$-measurable function.*
(c) *$T_{u,v}$ is quasinormal if and only if T is normal.*

Proof: (a) Using the matrix representation of $T_{u,v}$ with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, we have

$$T_{u,v} = T^*_{u,v} \Longleftrightarrow \begin{bmatrix} M_{u_1} & EM_{v_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_{\bar{u}_1} & 0 \\ M_{\bar{v}_2} & 0 \end{bmatrix} \Longleftrightarrow \bar{u}_1 = u_1 \quad \text{and} \quad v \in L^0(\mathcal{A}).$$

So $T_{u,v}$ is self-adjoint if and only if $T_{u,v} = T_{u,0}$ with $E(u) = E(\bar{u})$.

(b) Using (a), $T_{u,v} \geq 0$ if and only if $T_{u,v} = EM_uE \geq 0$ with $\bar{u}_1 = u_1$. But $\int_X u|E(f)|^2 d\mu = \langle M_u Ef, Ef\rangle = \langle EM_u Ef, f\rangle \geq 0$ if and only if $u \geq 0$.

(c) We see from (a) and Proposition 2.5(c) that $T^*_{u,v} = T_{u,v} = T^2_{u,v}$ if and only if $T_{u,v} = EM_uE = (EM_uE)^2 = M_{E(u)}EM_uE$ if and only if $E(u) = 1$. Thus $T_{u,v}$ is an orthogonal projection if and only if $T_{u,v} = E$.

$$(d)\ T_{u,v}T^*_{u,v} = T^*_{u,v}T_{u,v} \Longleftrightarrow \begin{bmatrix} M_{|u_1|^2} + EM_{|v_2|^2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_{|u_1|^2} & M_{\bar{u}_1}EM_{v_2} \\ M_{u_1\bar{v}_2} & M_{\bar{v}_2}EM_{v_2} \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} EM_{|v_2|^2} = 0; \\ M_{\bar{u}_1}EM_{v_2} = 0; \\ M_{u_1\bar{v}_2} = 0; \\ M_{\bar{v}_2}EM_{v_2} = 0. \end{cases} \Longleftrightarrow v_2 = 0 \Longleftrightarrow v \in L^0(\mathcal{A}).$$

(e) Direct computation shows that $(T^*_{u,v}T_{u,v})T_{u,v} = T_{u,v}(T^*_{u,v}T_{u,v})$ if and only if

$$\begin{bmatrix} M_{u_1|u_1|^2} & EM_{|u_1|^2 v_2} \\ M_{u_1^2\bar{v}_2} & M_{\bar{v}_2}EM_{u_1v_2} \end{bmatrix} = \begin{bmatrix} M_{|u_1|^2u_1} + M_{|v_2|^2u_1} & EM_{|u_1|^2v_2} + EM_{|v_2|^2}EM_{v_2} \\ 0 & 0 \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} M_{|v_2|^2u_1} = 0; \\ EM_{|v_2|^2}EM_{v_2} = 0; \\ M_{u_1^2\bar{v}_2} = 0; \\ M_{\bar{v}_2}EM_{u_1v_2} = 0. \end{cases} \Longleftrightarrow v_2 = 0 \Longleftrightarrow v \in L^0(\mathcal{A}).$$

■

Proposition 2.7: *Let $u, v \in \mathcal{K}$. Then the following hold.*

(a) $\mathcal{N}(T_{u,v}|_{L^2(\mathcal{A})}) = (\bar{u}L^2(\mathcal{A}))^\perp \cap L^2(\mathcal{A})$.

(b) $\mathcal{N}(T_{u,v}|_{\mathcal{N}(E)}) = (\bar{v}L^2(\mathcal{A}))^{\perp} \cap \mathcal{N}(E)$.
(c) $T_{u,v} = 0$ *if and only if* $u_1 = 0$ *and* $v_2\mathcal{N}(E) \subseteq \mathcal{N}(E)$.
(d) $\mathcal{N}(T_{u,v}) = \mathcal{N}(T_{u,v}|_{L^2(\mathcal{A})}) \oplus \mathcal{N}(T_{u,v}|_{\mathcal{N}(E)}) = \mathcal{N}(EM_uE) \cap \mathcal{N}(EM_vQ)$.

Proof: The proof of (a) and (b) are similar. We prove (a) only. Put $T_1 = T_{u,v}|_{L^2(\mathcal{A})}$ and $T_2 = T_{u,v}|_{\mathcal{N}(E)}$. Then for each $g \in L^2(\mathcal{A})$, we have

$$\begin{aligned} f \in (\bar{u}L^2(\mathcal{A}))^{\perp} \cap L^2(\mathcal{A}) &\iff \langle g, EM_uEf\rangle = \langle g, EM_uf\rangle = \langle Eg, uf\rangle \\ &= \langle g, uf\rangle = \langle \bar{u}g, f\rangle = 0 \iff f \in \mathcal{N}(T_1). \end{aligned}$$

Thus $\mathcal{N}(T_1) = \{f \in L^2(\mathcal{A}) : f \perp \bar{u}L^2(\mathcal{A})\}$.

(c) $T_{u,v} = 0$ if and only if $EM_uE = T_{u,v}E = 0$ and $EM_vQ = T_{u,v}Q = 0$. But this is equivalent to $u_1f_1 = 0$ and $E(u_2f_2) = 0$, for all $f = f_1 + f_2 \in L^2(\Sigma)$. This yields the result.

(d) $\mathcal{N}(T_{u,v}) = (\mathcal{N}(T_{u,v}) \cap L^2(\mathcal{A})) \oplus (\mathcal{N}(T_{u,v}) \cap \mathcal{N}(E)) = \mathcal{N}(T_1) \oplus \mathcal{N}(T_2)$. Moreover,

$$\mathcal{N}(EM_uE) = \mathcal{N}\left(\begin{bmatrix} M_{u_1} & 0 \\ 0 & 0 \end{bmatrix}\right) = \{(\bar{u}_1L^2(\mathcal{A}))^{\perp} \cap L^2(\mathcal{A})\} \oplus \mathcal{N}(E);$$

$$\mathcal{N}(EM_vQ) = \mathcal{N}\left(\begin{bmatrix} 0 & EM_{v_2} \\ 0 & 0 \end{bmatrix}\right) = L^2(\mathcal{A}) \oplus \{(\bar{v}_2L^2(\mathcal{A}))^{\perp} \cap \mathcal{N}(E)\}$$

and

$$\mathcal{N}\left(\begin{bmatrix} M_{u_1} & EM_{v_2} \\ 0 & 0 \end{bmatrix}\right) = \{(\bar{u}_1L^2(\mathcal{A}))^{\perp} \cap L^2(\mathcal{A})\} \oplus \{(\bar{v}_2L^2(\mathcal{A}))^{\perp} \cap \mathcal{N}(E)\}.$$

It follows that $\mathcal{N}(T_{u,v}) = \mathcal{N}(EM_uE) \cap \mathcal{N}(EM_vQ)$. ■

Let $T \in B(H)$ have closed range. We recall that the unique operator $S \in B(L^2(\Sigma))$ satisfying

$$(1)\ TST = T, \quad (2)\ STS = S, \quad (3)\ (TS)^* = TS, \quad (4)\quad (ST)^* = ST,$$

is called the Moore–Penrose inverse of T and is denoted by $T^{\dagger}$. Let $T\{i, \ldots, j\}$ denote the set of all operators S satisfying condition (k) for all labels k in the list $\{i, \ldots, j\}$. In this case, $S \in T\{i, \ldots, j\}$ is an $\{i, \ldots, j\}$-inverse of T and is denoted by $T^{(i,\ldots,j)}$. Note that $T^{(1,2,3,4)} = T^{\dagger}$. For other important properties of $T^{\dagger}$, see [2,3].

Let $a = E(|u|^2) \in L^{\infty}(\mathcal{A})$ be bounded away from zero on X. Set $S = M_{\frac{\bar{u}}{a}}E$. Then $S \in B(L^2(\Sigma))$ and $T_1ST_1 = (EM_u)(M_{\frac{\bar{u}}{a}}E)(EM_u) = EM_u = T_1$. It follows that T_1 has closed range. Also, it is easy to check that T_1 satisfies the other three equations above. Thus $S = T_1^{\dagger}$.

Recall that $T_{u,v} = T_1 + T_2$, where $T_1 = (EM_u)E$ and $T_2 = (EM_v)Q$. Let $c = |u_1|^2$ and $d = E(|v_2|^2)$ be bounded away from zero X. Then the block matrices of T_1 and $T_1^{\dagger}$ with respect to the decomposition $L^2(\mathcal{A}) \oplus \mathcal{N}(E)$ are

$$T_1 = \begin{bmatrix} M_{u_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad T_1^\dagger = \begin{bmatrix} M_{\frac{\bar{u}_1}{c}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly,

$$T_2 = \begin{bmatrix} 0 & EM_{v_2} \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad T_2^\dagger = \begin{bmatrix} 0 & 0 \\ M_{\frac{\bar{v}_2}{d}} & 0 \end{bmatrix}.$$

Now, put

$$S = \begin{bmatrix} M_{\frac{\bar{u}_1}{2c}} & 0 \\ M_{\frac{\bar{v}_2}{2d}} & 0 \end{bmatrix}.$$

Since $(M_{\frac{|u_1|^2}{2c}} + EM_{\frac{|v_2|^2}{2d}})_{|L^2(\mathcal{A})} = I$, then

$$T_{u,v}S = \begin{bmatrix} M_{u_1} & EM_{v_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{\frac{\bar{u}_1}{2c}} & 0 \\ M_{\frac{\bar{v}_2}{2d}} & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $T_{u,v}S$ is self-adjoint and $T_{u,v}ST_{u,v} = T_{u,v}$. Also, we have

$$ST_{u,v} = \begin{bmatrix} \frac{1}{2}I & M_{\frac{\bar{u}_1}{2c}} EM_{v_2} \\ M_{\frac{u_1 \bar{v}_2}{2d}} & M_{\frac{\bar{v}_2}{2d}} EM_{v_2} \end{bmatrix}. \tag{3}$$

It follows that

$$ST_{u,v}S = \begin{bmatrix} A_{|L^2(\mathcal{A})} & 0 \\ B_{|L^2(\mathcal{A})} & 0 \end{bmatrix},$$

where

$$\begin{aligned} A_{|L^2(\mathcal{A})} &= (M_{\frac{\bar{u}_1}{4c}} + M_{\frac{\bar{u}_1}{2c}} EM_{\frac{|v_2|^2}{2d}})_{|L^2(\mathcal{A})} = M_{\frac{\bar{u}_1}{2c}}; \\ B_{|L^2(\mathcal{A})} &= (M_{\frac{|u_1|^2 \bar{v}_2}{4cd}} + M_{\frac{\bar{v}_2}{2d}} EM_{\frac{|v_2|^2}{2d}})_{|L^2(\mathcal{A})} \\ &= \{M_{\frac{\bar{v}_2}{2d}} (M_{\frac{|u_1|^2}{2c}} + EM_{\frac{|v_2|^2}{2d}})\}_{|L^2(\mathcal{A})} = M_{\frac{\bar{v}_2}{2d}}. \end{aligned}$$

Consequently, $ST_{u,v}S = S$. Note that $ST_{u,v}$ in (3) is not self-adjoint. But, if $c = d$ then $(ST_{u,v})^* = ST_{u,v}$. These observations establish the following result.

Theorem 2.8: *Let $T_{u,v} \in B(L^2(\Sigma))$, and $c = |u_1|^2$. If $d = E(|v_2|^2)$ is bounded away from zero on X, then*

$$T_{u,v}^{(1,2,3)} = \begin{bmatrix} M_{\frac{\bar{u}_1}{2c}} & 0 \\ M_{\frac{\bar{v}_2}{2d}} & 0 \end{bmatrix}.$$

Moreover, if $c = d$ then $T_{u,v}^\dagger = T_{u,v}^{(1,2,3)}$.

Let $T_1 \in B(L^2(\Sigma))$ and let $|E(u)| \geq \delta$ on X, for some $\delta > 0$. Let $n \geq 2$ and $f \in \mathcal{N}(T_1^n)$. Then $(E(u))^{n-1}E(uf) = 0$, and so $T_1 f = E(uf) = 0$. Thus $\mathcal{N}(T_1^n) = \mathcal{N}(T_1)$. Now, let $g \in \mathcal{R}(T_1)$. Then $g = T_1 f$ for some $f \in L^2(\Sigma)$ and

$$\int_X \left| \frac{f}{(E(u))^{n-1}} \right|^2 \mathrm{d}\mu \leq \frac{1}{\delta^{n-1}} \|f\|_2^2.$$

It follows that

$$\begin{aligned} g = E(uf) &= \frac{1}{(E(u))^{n-1}}((E(u))^{n-1}E(uf)) \\ &= (E(u))^{n-1}E(u\frac{f}{(E(u))^{n-1}}) \in \mathcal{R}(T_1^n). \end{aligned}$$

Consequently, $\mathrm{ind}(T_1) = \mathrm{asc}(T_1) = \mathrm{des}(T_1) = 1$. Put $S = EM_{\frac{u}{(E(u))^2}}$. Then $S \in B(L^2(\Sigma))$, $ST_1S = S$, $T_1S = EM_{\frac{u}{E(u)}} = ST_1$ and

$$T_1^{k+1}S = ((E(u))^k EM_u)(EM_{\frac{u}{(E(u))^2}}) = (E(u))^{k-1}EM_u = T_1^k.$$

Consequently, $T_1^D = S = T_1^{\#}$.

Suppose $u_1 = E(u)$ is bounded away from zero on its support. Take $A = \sigma(u_1)$ and put

$$S = \begin{bmatrix} M_{\frac{\chi_A}{u_1}} & EM_{\frac{\chi_A \bar{v}_2}{u_1^2}} \\ 0 & 0 \end{bmatrix},$$

where $u_1^2 = |u_1|^2 \operatorname{sgn}(u_1)$. This implies that

$$\begin{aligned} ST_{u,v} &= \begin{bmatrix} M_{\chi_A} & EM_{\frac{\chi_A \bar{v}_2}{u_1}} \\ 0 & 0 \end{bmatrix} = T_{u,v}S; \\ ST_{u,v}S &= \begin{bmatrix} M_{\chi_A} & EM_{\frac{\chi_A \bar{v}_2}{u_1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{\frac{\chi_A}{u_1}} & EM_{\frac{\chi_A \bar{v}_2}{u_1^2}} \\ 0 & 0 \end{bmatrix} = S; \\ T_{u,v}^{k+1}S &= \begin{bmatrix} M_{u_1^{k+1}} & EM_{v_2 u_1^k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{\frac{\chi_A}{u_1}} & EM_{\frac{\chi_A v_2}{u_1^2}} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} M_{u_1^k} & EM_{v_2 u_1^{k-1}} \\ 0 & 0 \end{bmatrix} = T_{u,v}^k, \quad k \in \mathbb{N}. \end{aligned}$$

Thus $S = T_{u,v}^D$ is the Drazin inverse of $T_{u,v}$. Moreover, if u_1 is bounded away from zero on X, then by Proposition 2.5(c) and (d) we have

$$\begin{aligned} \mathcal{N}(T_{u,v}^k) &= \mathcal{N}(u_1^{k-1}T_{u,v}) = \mathcal{N}(T_{u,v}); \\ \mathcal{R}(T_{u,v}^k) &= \mathcal{N}(T_{u,v}^{*k})^{\perp} = \mathcal{N}(T_{u,v}^*)^{\perp} = \mathcal{R}(T_{u,v}), \end{aligned}$$

for all $k \in \mathbb{N}$. Thus, $\mathrm{ind}(T_{u,v}) = 1$ and so $T_{u,v}^D = T^{\#}$. These observations establish the following result.

Proposition 2.9: *Let u_1 be bounded away from zero on $A = \sigma(u_1)$. Then*

$$T^D_{u,v} = \begin{bmatrix} M_{\frac{\chi_A}{u_1}} & EM_{\frac{\chi_A \bar{v}_2}{u_1^2}} \\ 0 & 0 \end{bmatrix}.$$

Moreover, if $|u_1| \geq \alpha$ on X for some $\alpha \geq 0$, then $T^D_{u,v} = T^{\#}$.

Let $w, w' \in L^0(\Sigma)$, $u, v \in L^2(\Sigma)$ and let $\{au, bv\} \subset \mathcal{K}$, where $a = \sqrt{E(|w|^2)}$ and $b = \sqrt{E(|w'|^2)}$. Put $\mathcal{T}_{u,v} = M_w EM_u E + M_{w'} EM_v Q$. The conditional type operator $\mathcal{T}_{u,v}$ is a generalization of $T_{u,v}$. Indeed, if $w = w' = 1$, then $\mathcal{T}_{u,v} = T_{u,v}$. Using (2), the matrix representation of $\mathcal{T}_{u,v}$ with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$ is

$$\begin{aligned} \mathcal{T}_{u,v} &= \begin{bmatrix} M_{w_1 u_1} & EM_{w_1 u_2} \\ M_{w_2 u_1} & M_{w_2} EM_{u_2} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} M_{w'_1 v_1} & EM_{w'_1 v_2} \\ M_{w'_2 v_1} & M_{w'_2} EM_{v_2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} M_{w_1 u_1} & EM_{w'_1 v_2} \\ M_{w_2 u_1} & M_{w'_2} EM_{v_2} \end{bmatrix}. \end{aligned}$$

Now, let $K = E(|u|^2)E(|w|^2) \in L^\infty(\Sigma)$ and suppose K is bounded away from zero on its support. It follows that $M_w EM_u \in B(L^2(\Sigma))$ has closed range and $(M_w EM_u)^\dagger = M_{\frac{\bar{u}}{K}} EM_{\bar{w}}$ [16]. Using (2), the matrix representation of $M_w EM_u$ is

$$(M_w EM_u)^\dagger = \begin{bmatrix} M_{\frac{\bar{u}_1 \bar{w}_1}{K}} & EM_{\frac{\bar{u}_1 \bar{w}_2}{K}} \\ M_{\frac{\bar{u}_2 \bar{w}_1}{K}} & M_{\frac{\bar{u}_2}{K}} EM_{\bar{w}_2} \end{bmatrix}.$$

Set $c = |u_1|^2 > 0$, $b = |w_1|^2 > 0$ and $d = E(|w_2|^2) > 0$. Then we have

$$\mathcal{T}_1 := M_w EM_u E = \begin{bmatrix} M_{w_1 u_1} & 0 \\ M_{w_2 u_1} & 0 \end{bmatrix}$$

and

$$\mathcal{T}_1^{(1,2,4)} = \begin{bmatrix} M_{\frac{\bar{u}_1 \bar{w}_1}{2bc}} & EM_{\frac{\bar{u}_1 \bar{w}_2}{2cd}} \\ 0 & 0 \end{bmatrix}.$$

Note that

$$\mathcal{T}_1 \mathcal{T}_1^{(1,2,4)} = \begin{bmatrix} \frac{I}{2} & EM_{\frac{w_1}{2d}} EM_{\bar{w}_2} \\ M_{\frac{\bar{w}_1 w_2}{2b}} & 0 \end{bmatrix}$$

is not self-adjoint. But, if $b = d$, then we have

$$\mathcal{T}_1^\dagger = \begin{bmatrix} M_{\frac{\bar{u}_1 \bar{w}_1}{2bc}} & EM_{\frac{\bar{u}_1 \bar{w}_2}{2bc}} \\ 0 & 0 \end{bmatrix}.$$

In a similar way, we have

$$\mathcal{T}_2 := M_{w'} EM_v Q = \begin{bmatrix} 0 & EM_{w'_1 v_2} \\ 0 & M_{w'_2} EM_{v_2} \end{bmatrix}$$

and

$$\mathcal{T}_2^{(1,2,4)} = \begin{bmatrix} 0 & 0 \\ M_{\frac{\bar{w}_1'\bar{v}_2}{2em}} & M_{\frac{\bar{v}_2}{2mn}} EM_{\bar{w}_2'} \end{bmatrix},$$

where $e = |w_1'|^2 > 0$, $m = E(|v_2|^2) > 0$ and $n = E(|w_2'|^2) > 0$. Moreover, if $e = n$, then

$$\mathcal{T}_2^{\dagger} = \begin{bmatrix} 0 & 0 \\ M_{\frac{\bar{w}_1'\bar{v}_2}{2em}} & M_{\frac{\bar{v}_2}{2em}} EM_{\bar{w}_2'} \end{bmatrix}.$$

Theorem 2.10: *The following assertions hold.*

(a) *Let $u, v \in \mathcal{K}_2 = \mathcal{K}_2(\Sigma)$. If $u_n \to u$ and $v_n \to v$ in $\mathcal{K}_2$-norm and $w_n \to w$ in L^∞-norm, then $\mathcal{T}_{u_n,v_n} \to \mathcal{T}_{u,v}$ in the operator norm.*
(b) *Let $a = \sqrt{E(|w|^2)}$, $a' = \sqrt{E(|w'|^2)}$ and let $au_1, a'v_2 \in \mathcal{K}_2$. Then $\mathcal{T}_{u,v} \in B(L^2(\Sigma))$ and $\max\{c, \mathrm{d}\} \leq \|\mathcal{T}_{u,v}\| \leq \sqrt{\|au_1\|_{\mathcal{K}_2}^2 + \|a'v_2\|_{\mathcal{K}_2}^2}$, where*

$$c = \sqrt{\|w_1u_1\|_\infty^2 + \|\,|u_1|^2E(|w_2|^2)\|_\infty},$$
$$d = \sqrt{\|\,|w_1'|^2E(|v_2|^2)\|_\infty + \|E(|w_2'|^2)E(|v_2|^2)\|_\infty}.$$

(c) *$L^2(\mathcal{A})$ is a reducing subspace of $\mathcal{T}_{u,v}$ if and only if $w\chi_{\sigma(E(u))}$ and $v\chi_{\sigma(E(w'))}$ are $\mathcal{A}$-measurable functions.*

Proof: (a) First note that

$$\begin{aligned}\|T_{u_n,v_n} - T_{u,v}\| &= \|E(M_{u_n} - M_u)E + E(M_{v_n} - M_v)Q\| \\ &\leq \|EM_{u_n-u}\| + \|EM_{v_n-v}\| \\ &= \|E(|u_n - u|^2)\|_\infty^{\frac{1}{2}} + \|E(|v_n - v|^2)\|_\infty^{\frac{1}{2}} \\ &= \|u_n - u\|_{\mathcal{K}} + \|v_n - v\|_{\mathcal{K}} \to 0,\end{aligned}$$

as $n \to \infty$. Put $M = \sup_n \|T_{u_n,v_n}\|$. Then we have

$$\begin{aligned}\|\mathcal{T}_{u_n,v_n} - \mathcal{T}_{u,v}\| &= \|M_{w_n}T_{u_n,v_n} - M_wT_{u,v}\| \\ &\leq M\|w_n - w\|_\infty + \|M_w\|\,\|T_{u_n,v_n} - T_{u,v}\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

(b) Since $au, a'v \in \mathcal{K}_2$, then

$$\begin{aligned}\|\mathcal{T}_{u,v}\| &\leq \|M_wEM_u\| + \|M_{w'}EM_v\| = \|EM_{au}\| + \|EM_{a'v}\| \\ &= \|au\|_{\mathcal{K}_2} + \|a'v\|_{\mathcal{K}_2} < \infty.\end{aligned}$$

Thus $\mathcal{T}_{u,v}$ is bounded. Now let $f \in L^2(\Sigma)$. Then we have

$$\begin{aligned}\|M_wEM_uEf\|^2 &= \int_X E(|w|^2)|E(uE(f))|^2\mathrm{d}\mu = \int_X |E(auE(f))|^2\mathrm{d}\mu \\ &\leq \int_X |au_1|^2|E(f)|^2\mathrm{d}\mu \leq \|au_1\|_{\mathcal{K}_2}^2\|E(f)\|_2^2.\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|M_{w'}EM_vQf\|^2 &= \|EM_{a'v}Qf\|^2 = \int_X |E(a'vQ(f))|^2\mathrm{d}\mu \le \int_X |a'v_2|^2|Q(f)|^2\mathrm{d}\mu \\ &\le \|a'v_2\|^2_{\mathcal{K}_2}\|Q(f)\|_2^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|\mathcal{T}_{u,v}f\|_2 &\le \|au_1\|_{\mathcal{K}_2}\|E(f)\|_2 + \|a'v_2\|_{\mathcal{K}_2}\|Q(f)\|_2 \\ &\le (\|au_1\|^2_{\mathcal{K}_2} + \|a'v_2\|^2_{\mathcal{K}_2})^{\frac{1}{2}}(\|E(f)\|_2^2 + \|Q(f)\|_2^2)^{\frac{1}{2}} \\ &= (\|au_1\|^2_{\mathcal{K}_2} + \|a'v_2\|^2_{\mathcal{K}_2})^{\frac{1}{2}}\|f\|_2.\end{aligned}$$

Hence, $\|\mathcal{T}_{u,v}\| \le \sqrt{\|au_1\|^2_{\mathcal{K}_2} + \|a'v_2\|^2_{\mathcal{K}_2}}$. Now, let $f = f_1 + f_2 \in L^2(\Sigma)$ with $\|f\|_2 \le 1$. Then we have

$$\|\mathcal{T}_{u,v}\| \ge \left\|\mathcal{T}_{u,v}\begin{bmatrix} f_1 \\ 0\end{bmatrix}\right\| = \left\|\begin{bmatrix} M_{w_1u_1}f_1 \\ M_{w_2u_1}f_1\end{bmatrix}\right\|.$$

But

$$\begin{aligned}\|M_{w_1u_1\,|L^2(\mathcal{A})}\| &= \|M^*_{w_1u_1\,|L^2(\mathcal{A})}\| = \|EM_{\bar{w}_1\bar{u}_1\,|L^2(\mathcal{A})}\| \\ &= \|EM_{\bar{w}_1\bar{u}_1}\| = \|E(|w_1u_1|^2)\|^{\frac{1}{2}}_\infty = \|w_1u_1\|_\infty; \\ \|M_{w_2u_1\,|L^2(\mathcal{A})}\| &= \|M^*_{w_2u_1\,|\mathcal{N}(E)}\| = \|EM_{\bar{w}_2\bar{u}_1\,|\mathcal{N}(E)}\| \\ &= \|EM_{\bar{w}_2\bar{u}_1}\| = \|E(|w_2u_1|^2)\|^{\frac{1}{2}}_\infty = \|\,|u_1|^2E(|w_2|^2)\|^{\frac{1}{2}}_\infty.\end{aligned}$$

We obtain that

$$\|\mathcal{T}_{u,v}\| \ge \left\|\begin{bmatrix} M_{w_1u_1} \\ M_{w_2u_1}\end{bmatrix}\right\| = \sqrt{\|w_1u_1\|^2_\infty + \|\,|u_1|^2E(|w_2|^2)\|_\infty}.$$

In a similar way, we have

$$\begin{aligned}\|EM_{w'_1v_2\,|\mathcal{N}(E)}\| &= \|EM_{w'_1v_2}\| = \|\,|w'_1|^2E(|v_2|^2)\|^{\frac{1}{2}}_\infty; \\ \|M_{w'_2}EM_{v_2\,|\mathcal{N}(E)}\| &= \|M_{w'_2}EM_{v_2}\| = \|E(|w'_2|^2)E(|v_2|^2)\|^{\frac{1}{2}}_\infty,\end{aligned}$$

and so

$$\|\mathcal{T}_{u,v}\| \ge \left\|\begin{bmatrix} EM_{w'_1v_1} \\ M_{w'_2}EM_{v_2}\end{bmatrix}\right\| = \sqrt{\|\,|w'_1|^2E(|v_2|^2)\|_\infty + \|E(|w'_2|^2)E(|v_2|^2)\|_\infty}.$$

Consequently, $\max\{c, \mathrm{d}\} \le \|\mathcal{T}_{u,v}\|$.

(c) Using the matrix representation of $\mathcal{T}_{u,v}$, $L^2(\mathcal{A})$ is an invariant subspace of $\mathcal{T}_{u,v}$ if and only if $M_{w_2u_1} = 0$ on $L^2(\mathcal{A})$. But in this case, we have

$$\begin{aligned} w_2u_1 = 0 &\iff (w - E(w))E(u) = 0 \\ &\iff E(wE(u)) = wE(u) \\ &\iff wE(u) \in L^0(\mathcal{A}) \\ &\iff w\chi_{\sigma(E(u))} \in L^0(\mathcal{A}). \end{aligned}$$

In a similar way,

$$\begin{aligned} \mathcal{T}_{u,v}(L^2(\mathcal{A})) \subseteq L^2(\mathcal{A}) &\iff M_{\bar{w}_1'\bar{v}_2} = 0 \\ &\iff w_1'v_2 = 0 \\ &\iff E(w')(v - E(v)) = 0 \\ &\iff vE(w') = E(vE(w')) \\ &\iff vE(w') \in L^0(\mathcal{A}) \\ &\iff v\chi_{\sigma(E(w'))} \in L^0(\mathcal{A}). \end{aligned}$$

This completes the proof. ■

Note that if $w = w' = 1$, then $w_1 = w_1' = 1$ and $w_2 = w_2' = 0$. We have the following corollary.

Corollary 2.11:

(a) *Let $u, v \in \mathcal{K}_2$. If $\|u_n - u\|_{\mathcal{K}} \to 0$ and $\|v_n - v\|_{\mathcal{K}} \to 0$, then $\|T_{u_n,v_n} - T_{u,v}\| \to 0$.*
(b) *Let $u_1, v_2 \in \mathcal{K}_2$. Then $\max\{\|u_1\|_{\mathcal{K}}, \|v_2\|_{\mathcal{K}}\} \le \|T_{u,v}\| \le \sqrt{\|u_1\|^2_{\mathcal{K}} + \|v_2\|^2_{\mathcal{K}}}$.*
(c) *$L^2(\mathcal{A})$ is a reducing subspace of $T_{u,v}$ if and only if v is $\mathcal{A}$-measurable function, i.e. $T_{u,v} = T_u$.*

Example 2.12: (a) Let $X = \{1,2\}$, $\Sigma = 2^X$, $\mu(\{1\}) = \mu(\{2\}) = 1/2$ and let $\mathcal{A} = \{\emptyset, X\}$. Then $L^2(\Sigma) \cong \mathbb{C}^2$ and

$$E(f) = \frac{1}{\mu(X)} \int_X f d\mu = \frac{f_1 + f_2}{2}.$$

Put $u = (-1,-2)$ and $v = (2,6)$. Then

$$EM_uE = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3}{4} & \frac{-3}{4} \\ \frac{-3}{4} & \frac{-3}{4} \end{bmatrix};$$

$$EM_vQ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Thus

$$T_{u,v} = EM_uE + EM_vQ = \begin{bmatrix} \frac{-7}{4} & \frac{1}{4} \\ \frac{-7}{4} & \frac{1}{4} \end{bmatrix}, \quad \text{and} \quad 1.75 \leq \|T_{u,v}\| = 2.50.$$

On the other hand, since $u_1 = (-3/2, -3/2)$ and $v_2 = v - E(v) = (-2, 2)$, then $\|u_1\|_{\mathcal{K}} = 3/2$, $\|v_2\|_{\mathcal{K}} = 2$. Now, by Corollary 2.11, $2 \leq \|T_{u,v}\| \leq 2.50$.

In this stage, we obtain the $\{1,2,3\}$-inverse of $T_{u,v}$. Using Theorem 2.8, we have

$$\begin{aligned} T_{u,v}^{(1,2,3)} &= EM_{\frac{\bar{u}_1}{2c}}E + (I - E)M_{\frac{\bar{v}_2}{2d}}E \\ &= EM_{(\frac{\bar{u}_1}{2c} - \frac{\bar{v}_2}{2d})}E + M_{\frac{\bar{v}_2}{2d}}E \\ &= M_{\frac{\bar{u}_1}{2c}}E + M_{\frac{\bar{v}_2}{2d}}E \\ &= M_{(\frac{\bar{u}_1}{2c} + \frac{\bar{v}_2}{2d})}E. \end{aligned}$$

Direct computations show that $2c = (9/2, 9/2)$, $2d = (8, 8)$, $\bar{u}_1/2c = (-1/3, -1/3)$, $\bar{v}_2/2d = (-1/4, 1/4)$ and so $\frac{\bar{u}_1}{2c} + \frac{\bar{v}_2}{2d} = (-7/12, -1/12)$.

$$T_{u,v}^{(1,2,3)} = \begin{bmatrix} \frac{-7}{12} & 0 \\ 0 & \frac{-1}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{-7}{24} & \frac{-7}{24} \\ \frac{-1}{24} & \frac{-1}{24} \end{bmatrix}.$$

Note that

$$T_{u,v}^{(1,2,3)} T_{u,v} = \begin{bmatrix} 1 & \frac{-7}{48} \\ \frac{7}{48} & \frac{-1}{48} \end{bmatrix}$$

is not self-adjoint. However, if we take $u = (-3, -1)$, we obtain $c = d = (4, 4)$ and

$$T_{u,v} = \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad T_{u,v}^{\dagger} = \begin{bmatrix} \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 \end{bmatrix}.$$

(b) Let $X = [-1, 1]$, $\mathrm{d}\mu = \frac{\mathrm{d}x}{2}$, Σ the Lebesgue sets, and $\mathcal{A}$ the sigma subalgebra of Σ consisting of sets symmetric about the origin. Then for $f \in L^2(\Sigma)$, $E(f)$ is the even part of f. Let $\mathcal{T}_{u,v} = M_{2\sin x}EM_{\cos x}E + M_{x^2}EM_{x^2+x}Q$. Since for each $f \in L^2(\Sigma)$, $E(f)(x) = \frac{f(x)+f(-x)}{2}$, we have

$$\begin{aligned} \mathcal{T}_{u,v}(f) &= 2\sin x E(\cos x)E(f) + x^2E((x^2 + x)f) - x^2E(x^2 + x)E(f) \\ &= \sin 2x \frac{f(x) + f(-x)}{2} + x^2 \frac{(x^2 + x)f(x) + (x^2 - x)f(-x)}{2} - x^4 \frac{f(x) + f(-x)}{2} \\ &= \frac{\sin 2x + x^3}{2} f + \frac{\sin 2x - x^3}{2} f(-x), \\ w_1 &= w_2' = 0, \quad w_2 = 2\sin x, \quad u_1 = \cos x, \quad \text{and} \quad v_2 = x. \end{aligned}$$

Also, $a = \sqrt{E(|w|^2)} = \sqrt{E(4\sin^2 x)} = 2|\sin x|$ and $a' = \sqrt{E(|w'|^2)} = x^4$, $\|au_1\|_{\mathcal{K}_2} = \|\sin 2x\|_\infty = 1$ and $\|a'v_2\|_{\mathcal{K}_2} = \|x^3\|_\infty = 1$. Thus, by Theorem 2.10 we have $1 \leq \|\mathcal{T}_{u,v}\| \leq \sqrt{2}$.

Let us consider $T_{u,v} = EM_{\cos x}E + EM_{x^2+x}Q$. Then we have

$$T_{u,v}(f) = \left(\frac{\cos x + x}{2}\right) f + \left(\frac{\cos x - x}{2}\right) f(-x),$$

$u_1/2c = 1/2\cos x$ and $v_2/2d = 1/2x$. Hence the $\{1,2,3\}$-inverse of $T_{u,v}$ is

$$T_{u,v}^{(1,2,3)}(f) = M_{\left(\frac{u_1}{2c}+\frac{v_2}{2d}\right)}E(f) = \left(\frac{1}{4\cos x} + \frac{1}{4x}\right)(f + f(-x)).$$

Now, let $u = |\sin x| + x$ and $v = x^2 + \sin x$. Then $u_1 = |\sin x|$, $v_2 = \sin x$, and so $c = d = \sin^2 x$. It follows that

$$T_{u,v}(f) = \left(\frac{|\sin x| + \sin x}{2}\right) f + \left(\frac{|\sin x| - \sin x}{2}\right) f(-x)$$

and

$$T_{u,v}^{\dagger}(f) = M_{\left(\frac{u_1+v_2}{2c}\right)}E(f) = \frac{1 + \operatorname{sgn}(\sin x)}{4|\sin x|}(f + f(-x)).$$

Acknowledgments

The authors are very grateful to the referee(s) for careful reading of the paper and for a number of helpful comments and corrections which improved the presentation of this paper.

Disclosure statement

M. R. Jabbarzadeh was supported by a grant from University of Tabriz [grant number S/3480].

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