Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

Inequalities for accretive-dissipative block matrices involving convex and concave functions

Mohammad Reza Jabbarzadeh & Venus Kaleibary

To cite this article: Mohammad Reza Jabbarzadeh & Venus Kaleibary (2022) Inequalities for accretive-dissipative block matrices involving convex and concave functions, Linear and Multilinear Algebra, 70:3, 395-410, DOI: 10.1080/03081087.2020.1726277

To link to this article: https://doi.org/10.1080/03081087.2020.1726277

Published online: 13 Feb 2020.

LINEAR AND MULTILINEAR ALGEBRA
2022, VOL. 70, NO. 3, 395–410
https://doi.org/10.1080/03081087.2020.1726277

Inequalities for accretive-dissipative block matrices involving convex and concave functions

Mohammad Reza Jabbarzadeh and Venus Kaleibary

Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

ABSTRACT

We present some unitarily invariant norm and Schatten p-norm inequalities relevant to accretive-dissipative matrices involving convex and concave functions. In particular, we show that if $T = (T_{ij})_{2\times 2}$ is an accretive-dissipative block matrix with $T_{ij} \in M_n(\mathbb{C})$, $i, j = 1, 2$, and f is an increasing convex function on $[0, \infty)$ with $f(0) = 0$, then

$$2\left\| f\left(\frac{|T|}{2}\right)\right\|_u \leq \|f(\sqrt{2}\,|T_{11}|)\|_u + \|f(\sqrt{2}\,|T_{22}|)\|_u,$$

for every unitarily invariant norm $\|\cdot\|_u$. We also extract several inequalities for accretive-dissipative $n \times n$ operator matrices. The obtained inequalities extend some known results.

ARTICLE HISTORY
Received 24 August 2019
Accepted 2 February 2020

COMMUNICATED BY
N.-C. Wong

KEYWORDS
Block matrix; accretive-dissipative matrix; function; majorization; unitarily invariant norm; Schatten p-norm

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
15A60; 47A30; 15A18

1. Introduction

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. The matrix $T \in M_{2n}(\mathbb{C})$ can be represented as an 2×2 block matrix, i.e. $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, with $T_{jk} \in M_n(\mathbb{C}), j, k = 1, 2$. On the other hand, every $T \in M_{2n}(\mathbb{C})$ can be written uniquely as

$$T = A + iB, \tag{1}$$

where $A = (T + T^*)/2$ and $B = (T - T^*)/2i$ are Hermitian matrices. This is the Cartesian decomposition of T, and A and B are called the real and imaginary parts of T, respectively. In this paper, the decomposition (1) is represented as

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} + i\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}, \tag{2}$$

where $T_{jk}, B_{jk}, A_{jk} \in M_n(\mathbb{C}), j, k = 1, 2$.

For a matrix $T \in M_n(\mathbb{C})$, we always denote the eigenvalues of $|T| = (T^*T)^{1/2}$ by $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T)$ and put $\{s_j(T)\}$ as a vector of eigenvalues of $|T|$. These are called the singular values of T. For $1 \leq p < \infty$, the Schatten p-norm of T is defined

CONTACT V. Kaleibary v.kaleibary@gmail.com Faculty of Mathematical Sciences, University of Tabriz, Tabriz 5166615648, Iran

by $\|T\|_p = (\sum_{j=1}^{n} s_j^p(T))^{1/p}$. The Schatten p-norms are important examples of unitarily invariant norms $\|\cdot\|_u$, i.e. norms satisfying $\|T\|_u = \|UTV\|_u$ for all unitaries $U, V \in M_n(\mathbb{C})$.

A matrix $T \in M_n(\mathbb{C})$ is called accretive-dissipative if in its Cartesian decomposition (1), the matrices A and B are positive. In recent years, considerable attention has been given to the accretive-dissipative operators or matrices [8,11,13].

Recently, Lin and Zhou [14] considered the accretive-dissipative block matrix (2) and established several norm inequalities between the whole block matrix and its diagonal blocks.

After that, Gumus et al. [8] investigated another type of inequalities involving accretive-dissipative operators (2) interfering with some matrix functions. They showed [8, Theorem 2.5] if f is an increasing convex function on $[0,\infty)$ such that $f(0) = 0$, then

$$\left\|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right\|_u \le \left\|f\left(|T|^2\right)\right\|_u. \tag{3}$$

In this paper, we are interested in some inequalities for accretive-dissipative block matrices that involve matrix functions. In Section 2, we present an inequality between the norm of T to its diagonal blocks providing an upper bound for the inequality (3). The obtained result extends the main theorem in [14] to all convex functions, simultaneously. Section 3 is devoted to studying Schatten p-norm inequalities, including sums of accretive-dissipative matrices and convex functions. Among other inequalities, we prove that if T and S are accretive-dissipative matrices and f is an increasing convex function on $[0,\infty)$ with $f(0) = 0$, then for every $\alpha \in [0,1]$ and $p \ge 1$,

$$\left\|f\left(|\alpha T + (1-\alpha)S|\right)\right\|_p^p \le 2^{p-1}\left(\left\|\alpha f(\sqrt{2}\,|T|)\right\|_p^p + \left\|(1-\alpha)f(\sqrt{2}\,|S|)\right\|_p^p\right).$$

We also provide some corresponding inequalities related to concave functions. In the special case $f(t) = t$ and $\alpha = \frac{1}{2}$, the results reduce to an inequality presented by Kittaneh and Sakkijha [11, Theorem 2.7] as follows:

$$2^{-(p/2)}\left(\|T\|_p^p + \|S\|_p^p\right) \le \|T+S\|_p^p \le 2^{(3p/2)-1}\left(\|T\|_p^p + \|S\|_p^p\right). \tag{4}$$

In the last two sections, we deal with a class of functions on $[0,\infty)$ which preserve weak log-majorization. In Section 4, we first present some majorizations for this class of functions comparing diagonal and off-diagonal blocks of T. Eventually, an application of the results in Section 4 leads to an elegant unitarily invariant norm inequality for $n \times n$ operator matrices. The obtained results extend some Schatten p-norm inequalities in [11] to all unitarily invariant norms.

We note the statements in Section 4 and 5 are held not only for matrices but also for operators on any infinite-dimensional Hilbert spaces $\mathcal{H}$. Also, throughout the paper, all functions are assumed to be continuous.

2. Unitarily invariant norm inequalities

In what follows, capital letters A, B, C means $n \times n$ matrices or bounded linear operators on an n-dimensional complex Hilbert space $\mathcal{H}$. In addition, all partitioned matrices are in

$M_{2n}(\mathbb{C})$ with matrix entries in $M_n(\mathbb{C})$. We start this section with the following definition of majorization. For the matrices A, B, the weak log-majorization $\{s_j(A)\} \prec_{wlog} \{s_j(B)\}$ means

$$\prod_{j=1}^{k} s_j(A) \leq \prod_{j=1}^{k} s_j(B), \quad k = 1, 2, \ldots, n.$$

There is a close relation between the (log-)majorization and the unitarily invariant norm inequalities that will be constructive in our proofs. In the following, we state some related lemmas and use them frequently.

Lemma 2.1 ([3, Theorem 1.1])**:** *Let A, B be positive matrices. Then*

$$s_j(A+B) \leq \sqrt{2}\, s_j(A+iB), \quad j = 1, 2, \ldots, n. \tag{5}$$

Lemma 2.2 ([9, Theorem 6.23])**:** *Let A, B be positive and f be an increasing convex function on $[0, \infty)$. If $\{s_j(A)\} \prec_{wlog} \{s_j(B)\}$, then $\|f(A)\|_u \leq \|f(B)\|_u$ for every unitarily invariant norm $\|\cdot\|_u$.*

Lemma 2.3: *Let A, B be positive and f be an increasing convex function on $[0, \infty)$. Then for every unitarily invariant norm $\|\cdot\|_u$ and $r > 0$,*

$$\left\|f\left(|A+iB|^r\right)\right\|_u \leq \left\|f\left((A+B)^r\right)\right\|_u \leq \left\|f\left(2^{r/2}\,|A+iB|^r\right)\right\|_u.$$

Proof: It is shown in [16] if A, B are positive, then

$$\{s_j(A+iB)\} \prec_{wlog} \{s_j(A+B)\}, \quad j = 1, 2, \ldots, n. \tag{6}$$

Also, by Lemma 2.1 we have

$$\{s_j(A+B)\} \prec_{wlog} \{\sqrt{2} s_j(A+iB)\}, \quad j = 1, 2, \ldots, n. \tag{7}$$

Combining (6) and (7), taking rth power on all sides and using the spectral mapping theorem gives

$$\{s_j(|A+iB|^r)\} \prec_{wlog} \{s_j((A+B)^r)\} \prec_{wlog} \{2^{r/2} s_j(|A+iB|^r)\}.$$

Now the desired inequality deduces from Lemma 2.2. ■

Part (a) and (c) of the following Lemma has been given in [5]. Parts (b) and (d) can be found in [12] and [7], respectively.

Lemma 2.4: *Let $A, B \in B(\mathcal{H})$ be positive. Then for every unitarily invariant norm $\|\cdot\|_u$,*

(a) $\|f((A+B)/2)\|_u \leq \|(f(A)+f(B))/2\|_u$ *for every nonnegative convex function f on $[0, \infty)$.*
(b) $\|f(A)+f(B)\|_u \leq \|f(A+B)\|_u$ *for every nonnegative convex function f on $[0, \infty)$ with $f(0) = 0$.*

(c) $\|(f(A)+f(B))/2\|_u \leq \|f((A+B)/2)\|_u$ *for every nonnegative concave function* f *on* $[0,\infty)$.
(d) $\|f(A+B)\|_u \leq \|f(A)+f(B)\|_u$ *for every nonnegative concave function* f *on* $[0,\infty)$.

The last needed result is a remarkable matrix decomposition introduced by Bourin and Lee in [5] as follows.

Lemma 2.5: *For every positive block matrix in* $M_{2n}(\mathcal{C})$, *we have the decomposition*

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} = U\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}U^* + V\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}V^*$$

for some unitaries $U, V \in M_{2n}(\mathcal{C})$.

Now we are ready to state the first main result of this section.

Theorem 2.6: *Let* $T \in M_{2n}(\mathcal{C})$ *be accretive-dissipative partitioned as in* (2) *and* f *be an increasing convex function on* $[0,\infty)$ *with* $f(0) = 0$. *Then*

$$2\left\|f\left(\frac{|T|}{2}\right)\right\|_u \leq \left\|f(\sqrt{2}\,|T_{11}|)\right\|_u + \left\|f(\sqrt{2}\,|T_{22}|)\right\|_u, \tag{8}$$

for every unitarily invariant norm $\|\cdot\|_u$.

Proof: By applying Lemma 2.3 for $r = 1$, we have

$$\left\|f\left(\frac{|T|}{2}\right)\right\|_u = \left\|f\left(\frac{|A+iB|}{2}\right)\right\|_u \leq \left\|f\left(\frac{A+B}{2}\right)\right\|_u. \tag{9}$$

On the other hand, since

$$\frac{A+B}{2} = \frac{1}{2}\begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{12}^*+B_{12}^* & A_{22}+B_{22} \end{bmatrix}$$

is a positive block matrix, according to Lemma 2.5 there are unitaries U and V such that

$$\frac{A+B}{2} = \frac{1}{2}\left(U\begin{bmatrix} A_{11}+B_{11} & 0 \\ 0 & 0 \end{bmatrix}U^* + V\begin{bmatrix} 0 & 0 \\ 0 & A_{22}+B_{22} \end{bmatrix}V^*\right).$$

Now letting

$$M = U\begin{bmatrix} A_{11}+B_{11} & 0 \\ 0 & 0 \end{bmatrix}U^* \text{ and } N = V\begin{bmatrix} 0 & 0 \\ 0 & A_{22}+B_{22} \end{bmatrix}V^*,$$

we can write

$$\begin{aligned}
&\left\| f\left(\frac{A+B}{2}\right)\right\|_u \\
&= \left\| f\left(\frac{M+N}{2}\right)\right\|_u \\
&\leq \left\| \frac{f(M)+f(N)}{2}\right\|_u \quad \text{(by Lemma 2.4, (a))} \\
&\leq \frac{1}{2}\left(\|f(M)\|_u + \|f(N)\|_u\right) \\
&= \frac{1}{2}\left(\left\| U\begin{bmatrix} f(A_{11}+B_{11}) & 0 \\ 0 & f(0)\end{bmatrix}U^*\right\|_u + \left\| V\begin{bmatrix} f(0) & 0 \\ 0 & f(A_{22}+B_{22})\end{bmatrix}V^*\right\|_u\right) \\
&= \frac{1}{2}\left(\left\|f(A_{11}+B_{11})\right\|_u + \left\|f(A_{22}+B_{22})\right\|_u\right) \\
&\leq \frac{1}{2}\left(\left\|f(\sqrt{2}\,|A_{11}+iB_{11}|)\right\|_u + \left\|f(\sqrt{2}\,|A_{22}+iB_{22}|)\right\|_u\right) \quad \text{(by Lemma 2.2)} \\
&= \frac{1}{2}\left(\left\|f(\sqrt{2}\,|T_{11}|)\right\|_u + \left\|f(\sqrt{2}\,|T_{22}|)\right\|_u\right).
\end{aligned}$$

Combining this inequality with the inequality (9), we have

$$\left\| f\left(\frac{|T|}{2}\right)\right\|_u \leq \left\| f\left(\frac{A+B}{2}\right)\right\|_u \leq \frac{1}{2}\left(\left\|f(\sqrt{2}\,|T_{11}|)\right\|_u + \left\|f(\sqrt{2}\,|T_{22}|)\right\|_u\right),$$

as desired. ■

In the following, we will see that Theorem 2.6 can be considered as an upper bound of (3) as well.

Corollary 2.7: *Let $T \in M_{2n}(\mathcal{C})$ be accretive-dissipative partitioned as in (2) and f be an increasing convex function $[0,\infty)$ with $f(0)=0$. Then for every unitarily invariant norm $\|\cdot\|_u$ and $p \geq 1$,*

$$\left\| f\left(\left(\frac{|T|}{2}\right)^p\right)\right\|_u \leq \frac{1}{2}\left(\left\|f\left((\sqrt{2}\,|T_{11}|)^p\right)\right\|_u + \left\|f\left((\sqrt{2}\,|T_{22}|)^p\right)\right\|_u\right).$$

In particular

$$\left\| f\left(\frac{|T|^2}{4}\right)\right\|_u \leq \frac{1}{2}\left(\left\|f\left(2\,|T_{11}|^2\right)\right\|_u + \left\|f\left(2\,|T_{22}|^2\right)\right\|_u\right).$$

Proof: The results are obtained by applying Theorem 2.6 to the convex function $f(t^p)$, $p \geq 1$. ■

Corollary 2.8: *Let $T \in M_{2n}(\mathcal{C})$ be accretive-dissipative partitioned as in* (2). *Then*

$$\left\| e^{|T|^2/4} - I_{2n} \right\|_u \leq \frac{1}{2} \left(\left\| e^{2\,|T_{11}|^2} - I_n \right\|_u + \left\| e^{2\,|T_{22}|^2} - I_n \right\|_u \right),$$

for every unitarily invariant norm $\| \cdot \|_u$.

Proof: Applying Corollary 2.7 to the increasing convex function $f(t) = e^t - 1$ gives the result. ∎

Remark 2.1: Replacing the accretive-dissipative operator T with $2T$ in the above inequalities, one can get the reverses of Theorem 2.5, Corollary 2.6 and Corollary 2.7 in [8], immediately.

Remark 2.2: Lin and Zhou [14, Theorem 3.11] presented an interesting inequality for the accretive-dissipative block matrix (2) as follows:

$$\|T\|_u \leq \sqrt{2} \left(\left\| T_{11} \right\|_u + \left\| T_{22} \right\|_u \right). \tag{10}$$

A significant extension of (10) to all increasing convex functions is provided in Theorem 2.6. In fact, by letting $f(t) = t^r,\ r \geq 1$ we have

$$\||T|^r\|_u \leq 2^{(3r/2)-1)} \left(\left\| |T_{11}|^r \right\|_u + \left\| |T_{22}|^r \right\|_u \right),$$

which coincides with the inequality (10) in the case $r = 1$.

3. Schatten *p*-norm inequalities

In this section, we will present some new Schatten p-norm inequalities related to sums of two accretive-dissipative matrices that include convex and concave functions and extend some known results. We first start with the inequalities on concave functions.

Lemma 3.1 ([4, Corollary 2.2])**:** *Let $T = A + iB$ be a decomposition into real and imaginary parts, and let f be a nonnegative concave function on* $[0, \infty)$. *Then for all unitarily invariant norms* $\| \cdot \|_u$,

$$\|f(|T|)\|_u \leq \|f(|A|) + f(|B|)\|_u.$$

Lemma 3.2: *Let A, B be positive matrices and f be a nonnegative increasing concave function on* $[0, \infty)$. *Then for every unitarily invariant norm* $\| \cdot \|_u$,

$$\frac{1}{2} \|f(2|A + iB|)\|_u \leq \|f(A + B)\|_u \leq \|f(\sqrt{2}\,|A + iB|)\|_u.$$

Proof: Using Lemma 3.1 and part (c) of Lemma 2.4, respectively

$$\|f(|A + iB|)\|_u \leq \|f(A) + f(B)\|_u \leq 2 \left\| f\left(\frac{A + B}{2} \right) \right\|_u. \tag{11}$$

Replacing A and B by $2A$ and $2B$, we have the first alleged inequality. The second one deduces from Lemma 2.1 and the equality $f(s_j(A)) = s_j(f(A))$ for nonnegative increasing functions on $[0, \infty)$. ∎

Lemma 3.3 ([15, p. 14])**:** *Let A, B be positive matrices. Then for every $p \geq 1$,*

$$\|A\|_p^p + \|B\|_p^p \leq \|A+B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p). \tag{12}$$

The following is our first main result in this section.

Theorem 3.4: *Let $T, S \in M_n(\mathbb{C})$ be accretive-dissipative and f be a nonnegative increasing concave function on $[0,\infty)$. Then for every $p \geq 1$,*

$$\frac{1}{4^p}\left(\|f(\sqrt{2}|T|)\|_p^p + \|f(\sqrt{2}|S|)\|_p^p\right) \leq \left\|f\left(\frac{|T+S|}{2}\right)\right\|_p^p.$$

Proof: At first, let X, Y be two positive matrices. By applying the right-hand side of Lemma 3.2 for $1/2\sqrt{2}X$ and $1/2\sqrt{2}Y$, we have

$$\left\|f\left(\frac{X+Y}{2\sqrt{2}}\right)\right\|_u \leq \left\|f\left(\sqrt{2}\left|\frac{1}{2\sqrt{2}}X + i\frac{1}{2\sqrt{2}}Y\right|\right)\right\|_u = \left\|f\left(\frac{|X+iY|}{2}\right)\right\|_u. \tag{13}$$

Also, applying the left-hand side of Lemma 3.2 for matrices $1/\sqrt{2}X$ and $1/\sqrt{2}Y$ gives

$$\frac{1}{2}\left\|f\left(2\frac{|X+iY|}{\sqrt{2}}\right)\right\|_u \leq \left\|f\left(\frac{X+Y}{\sqrt{2}}\right)\right\|_u. \tag{14}$$

Now, considering the Cartesian decompositions $T = A + iB$ and $S = C + iD$ we can write

$$\begin{aligned}
\left\|f\left(\frac{|T+S|}{2}\right)\right\|_p^p &= \left\|f\left(\frac{|A+C+i(B+D)|}{2}\right)\right\|_p^p \\
&\geq \left\|f\left(\frac{A+C+B+D}{2\sqrt{2}}\right)\right\|_p^p \quad \text{(by (13))} \\
&\geq \left\|\frac{f(\frac{A+B}{\sqrt{2}}) + f(\frac{C+D}{\sqrt{2}})}{2}\right\|_p^p \quad \text{(by Lemma 2.4, (c))} \\
&= \frac{1}{2^p}\left\|f\left(\frac{A+B}{\sqrt{2}}\right) + f\left(\frac{C+D}{\sqrt{2}}\right)\right\|_p^p \\
&\geq \frac{1}{2^p}\left(\left\|f\left(\frac{A+B}{\sqrt{2}}\right)\right\|_p^p + \left\|f\left(\frac{C+D}{\sqrt{2}}\right)\right\|_p^p\right) \quad \text{(by Lemma 3.3)} \\
&\geq \frac{1}{2^p}\left(\frac{1}{2^p}\left\|f\left(\frac{2|A+iB|}{\sqrt{2}}\right)\right\|_p^p + \frac{1}{2^p}\left\|f\left(\frac{2|C+iB|}{\sqrt{2}}\right)\right\|_p^p\right) \quad \text{(by (14))} \\
&= \frac{1}{4^p}\left(\|f(\sqrt{2}\,|T|)\|_p^p + \|f(\sqrt{2}\,|S|)\|_p^p\right).
\end{aligned}$$

■

Remark 3.1: By letting $f(t) = t$ in Theorem 3.4, it reduces to the left-hand side of inequality (4) as follows:

$$\frac{1}{4^p}\left(\|\sqrt{2}|T|\|_p^p + \|\sqrt{2}|S|\|_p^p \right) \leq \left\| \frac{|T+S|}{2} \right\|_p^p,$$

and thereupon

$$2^{-(p/2)}\left(\|T\|_p^p + \|S\|_p^p\right) \leq \|T+S\|_p^p.$$

Thanks to the following lemma, we give the counterpart of Theorem 3.4 for all convex functions and $\alpha \in [0,1]$ in the next theorem.

Lemma 3.5 ([1, Corollary 2.6])**:** *Let A, B be positive matrices and f be a convex function on $[0,\infty)$. Then for every unitarily invariant norm $\|\cdot\|_u$ and $\alpha \in [0,1]$,*

$$\|f(\alpha A + (1-\alpha)B)\|_u \leq \|\alpha f(A) + (1-\alpha)f(B)\|_u.$$

Theorem 3.6: *Let $T, S \in M_n(\mathcal{C})$ be accretive-dissipative and f be an increasing convex function on $[0,\infty)$. Then for every $\alpha \in [0,1]$ and $p \geq 1$,*

$$\|f(|\alpha T + (1-\alpha)S|)\|_p^p \leq 2^{p-1}\left(\|\alpha f(\sqrt{2}\,|T|)\|_p^p + \|(1-\alpha)f(\sqrt{2}\,|S|)\|_p^p \right).$$

Proof: Considering the Cartesian decompositions $T = A + iB$ and $S = C + iD$, we have

$$\begin{aligned}
&\|f(|\alpha T + (1-\alpha)S|)\|_p^p \\
&\quad = \|f(|\alpha(A+iB) + (1-\alpha)(C+iD)|)\|_p^p \\
&\quad = \|f(|\alpha A + (1-\alpha)C + i(\alpha B + (1-\alpha)D)|)\|_p^p \\
&\quad \leq \|f(\alpha A + (1-\alpha)C + \alpha B + (1-\alpha)D)\|_p^p \quad \text{(by Lemma 2.3)} \\
&\quad = \|f(\alpha(A+B) + (1-\alpha)(C+D))\|_p^p \\
&\quad \leq \|\alpha f(A+B) + (1-\alpha)f(C+D)\|_p^p \quad \text{(by Lemma 3.5)} \\
&\quad \leq 2^{p-1}\left(\|\alpha f(A+B)\|_p^p + \|(1-\alpha)f(C+D)\|_p^p \right) \quad \text{(by Lemma 3.3)} \\
&\quad \leq 2^{p-1}\left(\|\alpha f(\sqrt{2}\,(A+iB))\|_p^p + \|(1-\alpha)f(\sqrt{2}\,(C+iD))\|_p^p \right) \quad \text{(by Lemma 2.3)} \\
&\quad = 2^{p-1}\left(\|\alpha f(\sqrt{2}\,|T|)\|_p^p + \|(1-\alpha)f(\sqrt{2}\,|S|)\|_p^p \right),
\end{aligned}$$

as desired. ■

By applying Theorem 3.6, the following subadditive inequality for accretive-dissipative operators is achieved.

Corollary 3.7: *Let $T, S \in M_n(\mathcal{C})$ be accretive-dissipative and f be an increasing convex function on $[0,\infty)$. Then for every $p \geq 1$,*

$$\|f(|\alpha T + (1-\alpha)S|)\|_p^p \leq 2^{p-1} \left\|\alpha f\big(\sqrt{2}\,|T|\big) + (1-\alpha) f\big(\sqrt{2}\,|S|\big)\right\|_p^p.$$

Particularly

$$\left\|f\left(\frac{|T+S|}{2}\right)\right\|_p^p \leq \frac{1}{2}\left(\|f(\sqrt{2}\,|T|)\|_p^p + \|f(\sqrt{2}\,|T|)\|_p^p\right)$$

$$\leq 2^{p-1}\left\|\frac{f\big(\sqrt{2}\,|T|\big) + f\big(\sqrt{2}\,|S|\big)}{2}\right\|_p^p.$$

Proof: The results are obtained by applying Theorem 3.6 and the left-hand side of the inequality (12). ■

Remark 3.2: The right-hand side of [11, Theorem 2.7] follows as a special case of Theorem 3.6 with $f(t) = t$ and $\alpha = \frac{1}{2}$.

Finally, one can reach an extension of Lemma 3.3 to all operator convex functions, with a similar proof sketch, as follows:

Corollary 3.8: *Let $A, B \in M_n(\mathbb{C})$ be positive and f be a nonnegative increasing convex function on $[0,\infty)$. Then for every $\alpha \in [0,1]$ and $p \geq 1$,*

$$\|f(\alpha A + (1-\alpha)B)\|_p^p \leq \alpha^p \|f(A)\|_p^p + (1-\alpha)^p \|f(B)\|_p^p.$$

4. Majorizations for special class of functions

In [8], it has been shown some unitarily invariant norm inequalities involving accretive-dissipative block matrices and a class of nonnegative increasing functions on $[0,\infty)$ which preserve weak log-majorization, i.e. the functions satisfying the following condition: if $\prod_{j=1}^k x_j \leq \prod_{j=1}^k y_j, k = 1,2,\ldots$, then $\prod_{j=1}^k f(x_j) \leq \prod_{j=1}^k f(y_j)$ for every real numbers $x_1 \geq x_2 \geq \cdots \geq 0$ and $y_1 \geq y_2 \geq \cdots \geq 0$. The simple example of such functions is $f(t) = t^p, p \geq 0$. For more examples, see [8]. In the next, for the sake of convenience, we show this class of functions with $\mathcal{C}$ and present several majorizations related to them. It is worthwhile to mention that a function $f(t)$ preserves weak-log majorization if and only if $\log(f(e^t))$ is a convex, nondecreasing function on the real line. Equivalently, the class $\mathcal{C}$ is the class of functions $f(t)$ which are nondecreasing and geometrically convex, $f(\sqrt{xy}) \leq \sqrt{f(x)f(y)}$ for all $x, y > 0$. See [6].

Forthcoming results are stated for all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Lemma 4.1: *Let P_i, Q_i be positive operators and let C_i be contractive, $i = 1,2,\ldots,m$. Then, for every submultiplicative $f \in \mathcal{C}$, $r > 0$ and $k = 1,2,\ldots$,*

$$\prod_{j=1}^k s_j\left(f\left(\left|\sum_{i=1}^m P_i C_i Q_i\right|^r\right)\right) \leq \prod_{j=1}^k f\left(s_j\left(\left(\sum_{i=1}^m P_i^2\right)^{r/2}\right)\right) \cdot f\left(s_j\left(\left(\sum_{i=1}^m Q_i^2\right)^r /2\right)\right). \quad (15)$$

Proof: For every $r > 0$, it is inferred from [19, Lemma 2] and The Spectral mapping Theorem that

$$\prod_{j=1}^{k} s_j\left(\left|\sum_{i=1}^{m} P_i C_i Q_i\right|^r\right) \le \prod_{j=1}^{k} s_j\left(\left(\sum_{i=1}^{m} P_i^2\right)^{r/2}\right) \cdot s_j\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{r}/2\right). \tag{16}$$

Consequently, for $f \in \mathcal{C}$ we have

$$\begin{aligned}
&\prod_{j=1}^{k} s_j\left(f\left(\left|\sum_{i=1}^{m} P_i C_i Q_i\right|^r\right)\right)\\
&\quad = \prod_{j=1}^{k} f\left(s_j\left(\left|\sum_{i=1}^{m} P_i C_i Q_i\right|^r\right)\right)\\
&\quad \le \prod_{j=1}^{k} f\left(s_j\left(\left(\sum_{i=1}^{m} P_i^2\right)^{r/2}\right) \cdot s_j\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{r/2}\right)\right) \quad \text{(by (16))}\\
&\quad \le \prod_{j=1}^{k} f\left(s_j\left(\left(\sum_{i=1}^{m} P_i^2\right)^{r/2}\right)\right) \cdot f\left(s_j\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{r/2}\right)\right),
\end{aligned}$$

in which the last inequality follows from submultiplicativity of f. ■

Proposition 4.2: *Let T be an accretive-dissipative operator partitioned as in (2) and $f \in \mathcal{C}$ be a submultiplicative function. Then for every $r > 0$ and $j = 1, 2, \ldots,$*

$$\left\{s_j\left(f\left(|T_{12}|^r\right)\right)\right\} \prec_{wlog} \left\{s_j\left(f\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right) s_j\left(f\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right)\right\} \tag{17}$$

and

$$\left\{s_j\left(f\left(|T_{21}|^r\right)\right)\right\} \prec_{wlog} \left\{s_j\left(f\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right) s_j\left(f\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right)\right\}. \tag{18}$$

Proof: Since in the Cartesian decomposition $T = A + iB$, the operators A and B are positive, by [17, Lemma 1. 21] there exist two contractions W_1 and W_2 such that

$$A_{12} = A_{11}^{1/2} W_1 A_{22}^{1/2}, \quad B_{12} = B_{11}^{1/2} W_2 B_{22}^{1/2}.$$

Now, for $k = 1, 2, \ldots$

$$\begin{aligned}
\prod_{j=1}^{k} s_j\left(f\left(|T_{12}|^r\right)\right) &= \prod_{j=1}^{k} s_j\left(f\left(|A_{12} + iB_{12}|^r\right)\right)\\
&= \prod_{j=1}^{k} s_j\left(f\left(|A_{11}^{1/2} W_1 A_{22}^{1/2} + B_{11}^{1/2}(iW_2) B_{22}^{1/2}|^r\right)\right)
\end{aligned}$$

$$\leq \prod_{j=1}^{k} \left[s_j\Big(f\big((A_{11}+B_{11})^{r/2}\big)\Big)\right]\left[s_j\Big(f\big((A_{22}+B_{22})^{r/2}\big)\Big)\right] \quad \text{(by (15))}$$

$$= \prod_{j=1}^{k} \left[f\Big(s_j\big((A_{11}+B_{11})^{r/2}\big)\Big)\right]\left[f\Big(s_j\big((A_{22}+B_{22})^{r/2}\big)\Big)\right]$$

$$\leq \prod_{j=1}^{k} \left[f\Big(s_j\big(2^{r/4}|A_{11}+iB_{11}|^{r/2}\big)\Big)\right]\left[f\Big(s_j\big(2^{r/4}|A_{22}+iB_{22}|^{r/2}\big)\Big)\right]$$

(by (5) and monotony of f)

$$= \prod_{j=1}^{k} f\Big(s_j\big(2^{r/4}|T_{11}|^{r/2}\big)\Big) f\Big(s_j\big(2^{r/4}|T_{22}|^{r/2}\big)\Big)$$

$$= \prod_{j=1}^{k} s_j\Big(f\big(2^{r/4}|T_{11}|^{r/2}\big)\Big) s_j\Big(f\big(2^{r/4}|T_{22}|^{r/2}\big)\Big).$$

This proves the first desired inequality. The second one is obtained in a similar way considering the fact $A_{21} = A_{12}^*$ and $B_{21} = B_{12}^*$. ■

Remark 4.1: Since $s_j(|X^*|) = s_j(|X|)$ for every $X \in B(\mathcal{H})$, one can substitute each of operators $|T_{ij}|$ with $|T_{ij}^*|$, $i, j = 1, 2$ in the above inequalities. This state also holds for all the following consequences.

Lemma 4.3 ([2, p. 54])**:** *Let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ be sequences of real numbers with the components arranged in decreasing order. Moreover, we assume the components of α are nonnegative. If $\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$ for all $k = 1, 2, \ldots$, then $\sum_{j=1}^{k} \alpha_j x_j \leq \sum_{j=1}^{k} \alpha_j y_j$ for all $k = 1, 2, \ldots$.*

Theorem 4.4: *Let T be an accretive-dissipative operator partitioned as in (2) and $f \in \mathcal{C}$ be a submultiplicative function. Then for all positive numbers r, s, t with $(1/s) + (1/t) = 1$ and unitarily invariant norms $\|\cdot\|_u$,*

$$\max\left\{ \left\| f\big(|T_{12}|^r\big)\right\|_u, \left\| f\big(|T_{21}|^r\big)\right\|_u \right\} \leq \left\| f^s\big(2^{r/4}\,|T_{11}|^{r/2}\big)\right\|_u^{1/s} \cdot \left\| f^t\big(2^{r/4}\,|T_{22}|^{r/2}\big)\right\|_u^{1/t}, \quad (19)$$

and thereupon

$$\left\| f\big(|T_{12}|^r\big)\right\|_u + \left\| f\big(|T_{21}|^r\big)\right\|_u \leq 2\,\left\| f^s\big(2^{r/4}\,|T_{11}|^{r/2}\big)\right\|_u^{1/s} \cdot \left\| f^t\big(2^{r/4}\,|T_{22}|^{r/2}\big)\right\|_u^{1/t}.$$

Proof: Since weak log-majorization implies weak majorization, from the inequality (17) we have

$$\sum_{j=1}^{k} s_j\Big(f\big(|T_{21}|^r\big)\Big) \leq \sum_{j=1}^{k} s_j\Big(f\big(2^{r/4}\,|T_{11}|^{r/2}\big)\Big) s_j\Big(f\big(2^{r/4}\,|T_{22}|^{r/2}\big)\Big), \quad k = 1, 2, \ldots. \quad (20)$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence with decreasing nonnegative entries. Define $\|X\|_\alpha = \sum_{j=1}^{k} \alpha_j s_j(X)$ for $X \in B(\mathcal{H})$. Compute

$$\begin{aligned}
&\left\|f\left(|T_{12}|^r\right)\right\|_\alpha \\
&\quad= \sum_{j=1}^{k} \alpha_j s_j\left(f\left(|T_{12}|^r\right)\right) \\
&\quad\le \sum_{j=1}^{k} \alpha_j s_j\left(f\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right) s_j\left(f\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right) \quad \text{(by (20) and Lemma 4.3)} \\
&\quad= \sum_{j=1}^{k} \alpha_j^{1/s} s_j\left(f\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right) \cdot \alpha_j^{1/t} s_j\left(f\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right) \\
&\quad\le \left(\sum_{j=1}^{k} \alpha_j s_j^s\left(f\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right)\right)^{1/s} \left(\sum_{j=1}^{k} \alpha_j s_j^t\left(f\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right)\right)^{1/t} \\
&\qquad \text{(by Hölder's inequality)} \\
&\quad= \left(\sum_{j=1}^{k} \alpha_j s_j\left(f^s\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right)\right)^{1/s} \left(\sum_{j=1}^{k} \alpha_j s_j\left(f^t\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right)\right)^{1/t} \\
&\qquad \text{(by the S.M. Theorem)} \\
&\quad= \left\|f^s\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right\|_\alpha^{1/s} \cdot \left\|f^t\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right\|_\alpha^{1/t}.
\end{aligned}$$

As α is arbitrarily chosen, by [10, Corollary 3.5.9] we deduce

$$\left\|f\left(|T_{12}|^r\right)\right\|_u \le \left\|f^s\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right\|_u^{1/s} \cdot \left\|f^t\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right\|_u^{1/t},$$

for any unitarily invariant norm $\|\cdot\|_u$. Repeating the same argument and using the inequality (18), one gets

$$\left\|f\left(|T_{21}|^r\right)\right\|_u \le \left\|f^s\left(2^{r/4}\,|T_{11}|^{r/2}\right)\right\|_u^{1/s} \cdot \left\|f^t\left(2^{r/4}\,|T_{22}|^{r/2}\right)\right\|_u^{1/t}.$$

■

Remark 4.2: It has been proved in [8, Theorem 3.7] if T is an accretive-dissipative matrix partitioned as in (2) and $f \in \mathcal{C}$ is a submultiplicative convex function with $f(0) = 0$, then for all positive numbers s, t with $(1/s) + (1/t) = 1$ and unitarily invariant norms $\|\cdot\|_u$,

$$\left\|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right\|_u \le \left\|f^s\left(2\,|T_{11}|\right)\right\|_u^{1/s} \cdot \left\|f^t\left(2\,|T_{22}|\right)\right\|_u^{1/t}. \tag{21}$$

A corresponding inequality [8, Theorem 3.8] for a submultiplicative concave function $f \in \mathcal{C}$ with $f(0) = 0$ has been shown as follows:

$$\left\|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right\|_u \le 4\,\left\|f^s\left(|T_{11}|\right)\right\|_u^{1/s} \cdot \left\|f^t\left(|T_{22}|\right)\right\|_u^{1/t}. \tag{22}$$

Here, we are going to compare the above inequalities with the obtained one in Theorem 4.4. Considering Remark 4.1 and letting $r = 2$ in Theorem 4.4, we have

$$\left\|f(|T_{21}|^2)\right\|_u + \left\|f(|T_{21}^*|^2)\right\|_u \leq 2 \left\|f^s\left(\sqrt{2}\,|T_{11}|\right)\right\|_u^{1/s} \cdot \left\|f^t\left(\sqrt{2}\,|T_{22}|\right)\right\|_u^{1/t}. \tag{23}$$

This provides a new relation between the diagonal blocks and off diagonal blocks of T involving a submultiplicative function $f \in \mathcal{C}$, with no constraint of convexity or concavity on f. It also refines the appeared constants in (21) and (22) simultaneously, as follows:

(a) Let f be a concave function. Then for every number $a \geq 1$ we have $f(az) \leq af(z)$ and so

$$\begin{aligned}\left\|f(|T_{21}|^2) + f(|T_{21}^*|^2)\right\|_u &\leq \left\|f(|T_{21}|^2)\right\|_u + \left\|f(|T_{21}^*|^2)\right\|_u \\ &\leq 2 \left\|f^s\left(\sqrt{2}\,|T_{11}|\right)\right\|_u^{1/s} \cdot \left\|f^t\left(\sqrt{2}\,|T_{22}|\right)\right\|_u^{1/t} \quad \text{(by 23)} \\ &\leq 4 \left\|f^s\left(|T_{11}|\right)\right\|_u^{1/s} \cdot \left\|f^t\left(|T_{22}|\right)\right\|_u^{1/t}.\end{aligned}$$

This says in the case f is concave function, the inequality (23) is always more optimal than (22).

(b) Let f be the convex function $f(t) = t^r, r \geq 1$. Rewriting the inequalities (21) and (23) respectively, we have

$$\left\||T_{12}|^{2r} + |T_{21}^*|^{2r}\right\|_u \leq 2^{2r} \left\||T_{11}|^{rs}\right\|_u^{1/s} \cdot \left\||T_{22}|^{rt}\right\|_u^{1/t} \tag{24}$$

and

$$\left\||T_{12}|^{2r}\right\|_u + \left\||T_{21}^*|^{2r}\right\|_u \leq 2^{r+1} \left\||T_{11}|^{rs}\right\|_u^{1/s} \cdot \left\||T_{22}|^{rt}\right\|_u^{1/t}. \tag{25}$$

Since $r \geq 1$, then $2^{r+1} \leq 2^{2r}$ and hence the second inequality is a sharper one.

Remark 4.3: Lin and Zhou [14] showed that if T be an accretive-dissipative operator partitioned as in (2), then for any unitarily invariant norm $\|\cdot\|_u$,

$$\max\{\|T_{12}\|_u^2, \|T_{21}\|_u^2\} \leq 4\|T_{11}\|_u\|T_{22}\|_u.$$

Zhang [19] optimized the factor 4 to 2. Also, it has been obtained in [18] independently. Our result in Theorem 4.4 is a considerable extension of Zhang's refinement to some functions $f \in \mathcal{C}$.

5. An application for $n \times n$ operator matrices

In the next, we are going to present an elegant application of Theorem 4.4 for $n \times n$ operator matrices. Let $\mathbf{H} := \bigoplus_{i=1}^n \mathcal{H}$ and $T \in B(\mathbf{H})$ be accretive-dissipative represented in

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}, \tag{26}$$

in which $T_{i,j} \in B(\mathcal{H}), i, j = 1, 2, \ldots, n$. We provide a norm inequality between the positive powers of the operators $|T_{ij}|$ as follows.

Theorem 5.1: *Let $T \in B(\mathbf{H})$ be accretive-dissipative partitioned as in* (26) *and $f \in \mathcal{C}$ be a submultiplicative function. Then for all positive numbers r, s, t with $(1/s) + (1/t) = 1$ and unitarily invariant norms $\|\cdot\|_u$,*

$$\sum_{i \neq j} \left\| f\left(|T_{ij}|^r\right) \right\|_u \leq 2 \sum_{i=1}^{n} \left(\frac{(n-i)}{s} \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u + \frac{(i-1)}{t} \left\| f^t\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u \right), \tag{27}$$

for $i, j = 1, 2, \ldots, n$. Furthermore

$$\prod_{i \neq j} \left\| f\left(|T_{ij}|^r\right) \right\|_u \leq \prod_{i=1}^{n} \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u^{((n-i))/s} \left\| f^t\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u^{((i-1))/t}.$$

Proof: Let $\tilde{T} = \begin{bmatrix} ccT_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{bmatrix}$ be a principle submatrix of T. Since T is accretive-dissipative, it follows that $\tilde{T}$ is accretive-dissipative as well. Now, by applying Theorem 4.4 to the operator $\tilde{T}$ and using the well-known AM-GM inequality, we have

$$\begin{aligned} \left\| f\left(|T_{ij}|^r\right) \right\|_u &\leq \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u^{1/s} \cdot \left\| f^t\left(2^{r/4}\, |T_{jj}|^{r/2}\right) \right\|_u^{1/t} \\ &\leq \frac{1}{s} \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u + \frac{1}{t} \left\| f^t\left(2^{r/4}\, |T_{jj}|^{r/2}\right) \right\|_u, \end{aligned}$$

for $i, j = 1, 2$. Similarly,

$$\left\| f\left(|T_{ji}|^r\right) \right\|_u \leq \frac{1}{s} \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u + \frac{1}{t} \left\| f^t\left(2^{r/4}\, |T_{jj}|^{r/2}\right) \right\|_u.$$

Consequently,

$$\left\| f\left(|T_{ij}|^r\right) \right\|_u + \left\| f\left(|T_{ji}|^r\right) \right\|_u \leq 2 \left(\frac{1}{s} \left\| f^s\left(2^{r/4}\, |T_{ii}|^{r/2}\right) \right\|_u + \frac{1}{t} \left\| f^t\left(2^{r/4}\, |T_{jj}|^{r/2}\right) \right\|_u \right). \tag{28}$$

Here, for the sake of convenience and clarity, we first assume T is an accretive-dissipative 3×3 operator matrices as follows:

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

By applying the inequality (28) for the submatrices $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, $\begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix}$ and $\begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix}$ respectively, we have the following inequalities:

$$\left\| f\left(|T_{12}|^r\right) \right\|_u + \left\| f\left(|T_{21}|^r\right) \right\|_u \leq 2 \left(\frac{1}{s} \left\| f^s\left(2^{r/4}\, |T_{11}|^{r/2}\right) \right\|_u + \frac{1}{t} \left\| f^t\left(2^{r/4}\, |T_{22}|^{r/2}\right) \right\|_u \right),$$

$$\left\| f\left(|T_{13}|^r\right) \right\|_u + \left\| f\left(|T_{31}|^r\right) \right\|_u \leq 2 \left(\frac{1}{s} \left\| f^s\left(2^{r/4}\, |T_{11}|^{r/2}\right) \right\|_u + \frac{1}{t} \left\| f^t\left(2^{r/4}\, |T_{33}|^{r/2}\right) \right\|_u \right),$$

and

$$\left\|f\left(|T_{23}|^r\right)\right\|_u + \left\|f\left(|T_{32}|^r\right)\right\|_u \le 2\left(\frac{1}{s}\left\|f^s\left(2^{r/4}|T_{22}|^{r/2}\right)\right\|_u + \frac{1}{t}\left\|f^t\left(2^{r/4}|T_{33}|^{r/2}\right)\right\|_u\right).$$

Now adding up these inequalities gives

$$\begin{aligned}
&\sum_{i\neq j}\left\|f\left(|T_{ij}|^r\right)\right\|_u \\
&\quad\le \frac{4}{s}\left\|f^s\left(2^{r/4}|T_{11}|^{r/2}\right)\right\|_u + \frac{2}{s}\left\|f^s\left(2^{r/4}|T_{22}|^{r/2}\right)\right\|_u + \frac{2}{t}\left\|f^t\left(2^{r/4}|T_{22}|^{r/2}\right)\right\|_u \\
&\quad+ \frac{4}{t}\left\|f^t\left(2^{r/4}|T_{33}|^{r/2}\right)\right\|_u,
\end{aligned}$$

for $i, j = 1, 2, 3$, satisfying in the first claimed inequality with $n = 3$. Similarly, for a $n \times n$ operator matrix T writing the inequality (28) for all 2×2 submatrices of T in the form $\tilde{T}$, and adding them up yields

$$\begin{aligned}
&\sum_{i\neq j}\left\|f\left(|T_{ij}|^r\right)\right\|_u \\
&\quad\le 2\Bigg(\frac{(n-1)}{s}\left\|f^s\left(2^{r/4}|T_{11}|^{r/2}\right)\right\|_u + \frac{(n-2)}{s}\left\|f^s\left(2^{r/4}|T_{22}|^{r/2}\right)\right\|_u + \cdots \\
&\quad+ \frac{1}{s}\left\|f^s\left(2^{r/4}|T_{(n-1)(n-1)}|^{r/2}\right)\right\|_u \\
&\quad+ \frac{1}{t}\left\|f^t\left(2^{r/4}|T_{22}|^{r/2}\right)\right\|_u + \frac{2}{t}\left\|f^t\left(2^{r/4}|T_{33}|^{r/2}\right)\right\|_u + \cdots \\
&\quad+ \frac{(n-1)}{t}\left\|f^t\left(2^{r/4}|T_{nn}|^{r/2}\right)\right\|_u\Bigg).
\end{aligned}$$

Hence

$$\sum_{i\neq j}\left\|f\left(|T_{ij}|^r\right)\right\|_u \le 2\sum_{i=1}^{n}\left(\frac{(n-i)}{s}\left\|f^s\left(2^{r/4}|T_{ii}|^{r/2}\right)\right\|_u + \frac{(i-1)}{t}\left\|f^t\left(2^{r/4}|T_{ii}|^{r/2}\right)\right\|_u\right),$$

for $i, j = 1, 2, \ldots, n$, as desired. The multiplicative inequality is obtained by using the inequality

$$\left\|f\left(|T_{ij}|^r\right)\right\|_u \le \left\|f^s\left(2^{r/4}|T_{ii}|^{r/2}\right)\right\|_u^{1/s} \cdot \left\|f^t\left(2^{r/4}|T_{jj}|^{r/2}\right)\right\|_u^{1/t}$$

for all 2×2 submatrices of T in the form $\tilde{T}$, in a similar way. ∎

Remark 5.1: Putting $f(t) = t$ and $s = t = 2$ in Theorem 5.1, we immediately obtain

$$\sum_{i\neq j}\left\||T_{ij}|^r\right\|_u \le (n-1)2^{r/2}\sum_{i=1}^{n}\left\||T_{ii}|^r\right\|_u, \quad r > 0$$

$$\prod_{i\neq j}\left\||T_{ij}|^r\right\|_u \le 2^{r(n-1)/2}\prod_{i=1}^{n}\left\||T_{ii}|^r\right\|_u^{((n-1))/2}, \quad r > 0.$$

The first inequality provides a nice improvement of Shatten p-norm results in [11, Theorem 2.4] to all unitarily invariant norms. In addition, a simple comparison shows that the constant $2^{r/2}$ is a better one for all $r > 0$. We emphasize the results in this section are based on the inequality (19) and so letting $r = 1$ and taking p-powers of that inequality leads to [11, Theorem 2.4].

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The authors were supported by Iran National Science Foundation (INSF), Project Number 96009632.

References

[1] Aujla JS, Silva FC. Weak majorization inequalities and convex functions. Linear Algebra Appl. 2003;369:217–233.
[2] Bhatia R. Matrix analysis. New York (NY): Springer-Verlag; 1997.
[3] Bhatia R, Kittaneh F. The singular values of $A + B$ and $A + iB$. Linear Algebra Appl. 2009;431:1502–1508.
[4] Bourin JC. A matrix subadditivity inequality for symmetric norms. Proc Amer Math Soc. 2010;138:495–504.
[5] Bourin JC, Lee EY. Unitary orbits of Hermitian operators with convex or concave functions. Bull Lond Math Soc. 2012;44:1085–1102.
[6] Bourin JC, Shao J. Convex maps on R^n and positive definite matrices. 2019. arXiv:1909.11925 [math.FA]
[7] Bourin JC, Uchiyama M. A matrix subadditivity inequality for $f(A + B)$ and $f(A) + f(B)$. Linear Algebra Appl. 2007;423:512–518.
[8] Gumus IH, Hirzallah O, Kittaneh F. Norm inequalities involving accretive-dissipative 2×2 block matrices. Linear Algebra Appl. 2017;528:76–93.
[9] Hiai F, Petz D. Introduction to matrix analysis and applications. Cham: Springer International Publishing; 2014.
[10] Horn RA, Johnson CR. Topics in matrix analysis. Cambridge: Cambridge University Press; 1991.
[11] Kittaneh F, Sakkijha M. Inequalities for accretive-dissipative matrices. Linear Multilinear Algebra. 2019;67:1–6.
[12] Kosem T. Inequalities between $\|f(A + B)\|$ and $\|f(A) + f(B)\|$. Linear Algebra Appl. 2006;418:153–160.
[13] Lin M. Fischer type determinant inequalities for accretive-dissipative matrices. Linear Algebra Appl. 2013;438:2808–2812.
[14] Lin M, Zhou D. Norm inequalities for accretive-dissipative operator matrices. J Math Anal Appl. 2013;407:436–442.
[15] Simon B. Trace ideals and their applications. Cambridge:Cambridge University Press; 1979.
[16] Zhan X. Singular values of difference of positive semidefinite matrices. SIAM J Matrix Anal Appl. 2000;22:819–823.
[17] Zhan X. Matrix Inequalities. Berlin: Springer-Verlag; 2002. (Lecture Notes in Math).
[18] Zhang F. A matrix decomposition and its applications. Linear Multilinear Algebra. 2015;63:2033–2042.
[19] Zhang Y. Unitarily invariant norm inequalities for accretive-dissipative operator matrices. Math Anal Appl. 2014;412:564–569.