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Hypercyclicity of Weighted Composition Operators on L^p-Spaces

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Abstract. Let (X, Σ, μ) be a σ-finite measure space and $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$ $(1 \leq p < \infty)$, defined by $W : f \mapsto u.(f \circ \varphi)$, where $\varphi : X \to X$ is a measurable transformation and u is a weight function on X. In this paper, we study the hypercyclicity of W in terms of u, using the Radon–Nikodym derivatives and the conditional expectations. First, it is shown that if φ is a periodic nonsingular transformation, then W cannot be hypercyclic. The necessary conditions for the hypercyclicity of W are then given in terms of the Radon–Nikodym derivatives provided that φ is non-singular and finitely non-mixing. For the sufficient conditions, we also require that φ is normal. The weakly mixing and topologically mixing concepts are also studied for W. Moreover, under some specific conditions, we establish the subspace hypercyclicity of the adjoint operator W^* with respect to the Hilbert subspace $L^2(\mathcal{A})$. Finally, to illustrate the results, some examples are given.

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1. Introduction and Preliminaries

A bounded linear operator T on a topological vector space Y is *hypercyclic*, if there is a vector $x \in Y$(called a hypercyclic vector) whose orbit i.e., $orb(T, x) := \{T^n x : n = 0, 1, 2, ...\}$ is dense in Y, where T^n stands for the n-th iterate of T and T^0 is the identity map. Moreover, an operator T is *topologically transitive* if for each pair of non-empty open sets (U, V) in Y, there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. By Birkoff's transitivity theorem on a separable F-space setting, notions of topological transitivity and hypercyclicity are equivalent. A continuous operator T is said to be *weakly mixing* if the map $T \times T : Y \times Y \to Y \times Y$ defined by $T \times T(x, y) := (Tx, Ty)$ is topologically transitive. It is worth noting that a linear operator T on a

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separable F-space is weakly mixing if and only if $T \oplus T$ is hypercyclic on $Y \oplus Y$. Moreover, weakly mixing maps are topologically transitive but in the topological setting, the converse is not true. For example, any irrational rotation of the unit circle $\mathbb{T}$ is topologically transitive but it is not weakly mixing. An operator T is *topologically mixing* whenever for each pair of no-empty open sets (U, V) in Y, there exists an $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. As an interesting example, the operators of the form "identity plus a backward shift" are always hypercyclic and even topologically mixing ([3, Chapter 2]).

We say that T satisfies the *hypercyclicity criterion* [17] if there exists an increasing sequence of integers (n_k), two dense sets $D_1, D_2 \subset Y$ and a sequence of maps $S_{n_k} : D_2 \to Y$(not necessarily linear or continuous) such that

- $T^{n_k}(x) \to 0$ for any $x \in D_1$;
- $S_{n_k}(y) \to 0$ for any $y \in D_2$;
- $T^{n_k} S_{n_k}(y) \to y$ for any $y \in D_2$.

For the possible setting, $n_k = k$ and $D_1 = D_2$, it is called the *Kitai's hypercyclicity criterion*. The survey articles [1,5,19,22–24,26] and the books [3,13] are the interesting references in this area.

In what follows, we provide some important notations and definitions that will be later used for the rest of the paper. Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a σ-finite subalgebra of Σ. We use the notation $L^p(\mathcal{A})$ for $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ and henceforth we write μ in place of $\mu_{|\mathcal{A}}$. For $B \in \Sigma$, let $\mathcal{A}_B = \mathcal{A} \cap B$ denote the *relative completion* of the sigma-algebra generated by $\{A \cap B : A \in \mathcal{A}\}$ and denote the *complement* of B by B^c. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear spaces of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The *support* of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. The *characteristic function* of any set A will be denoted by χ_A. Let $S(X)$ be the class of all complex-valued, measurable and simple functions on X such that $\mu(\sigma(f)) < \infty$ for each $f \in S(X)$. Let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, that is, φ is *non-singular*. In this case, we write $\mu \circ \varphi^{-1} \ll \mu$, as usual. Let h denote the *Radon-Nikodym property*, $h := \frac{\mathrm{d}\mu\circ\varphi^{-1}}{\mathrm{d}\mu}$. A measurable transformation φ is said to be *measure preserving* whenever for each $A \in \Sigma$, $\mu(A) = \mu(\varphi^{-1}(A))$ i.e., $h = 1$. The assumption $\mu \circ \varphi^{-1} \ll \mu$ implies that $\mu \circ \varphi^{-n} \ll \mu$ for all $n \in \mathbb{N}$ and $\mu \circ \varphi^{-i} \ll \mu \circ \varphi^{-j}$, for each $i > j$. Set $\Sigma_\infty = \bigcap_{n=1}^{\infty} \varphi^{-n}(\Sigma)$ and suppose that h is Σ_∞-measurable. Then

$$h_n := \frac{\mathrm{d}\mu \circ \varphi^{-n}}{\mathrm{d}\mu} = \frac{\mathrm{d}\mu \circ \varphi^{-n}}{\mathrm{d}\mu \circ \varphi^{-(n-1)}} \cdots \frac{\mathrm{d}\mu \circ \varphi^{-1}}{\mathrm{d}\mu}$$
$$= (h \circ \varphi^{-(n-1)}) \cdots (h \circ \varphi^0) = \prod_{i=0}^{n-1} h \circ \varphi^{-i}.$$

Note that always $h \circ \varphi > 0$ and $h_n = h^n$ whenever $h \circ \varphi = h$. For example, if φ is measure preserving then obviously h is Σ_∞-measurable, $h \circ \varphi = h$ and hence $h_n = h^n$. Throughout this paper, we will use the following *change of variable formula* frequently:

$$\int_{\varphi^{-n}(A)} f \circ \varphi^n \mathrm{d}\mu = \int_A h_n f \mathrm{d}\mu, \qquad A \in \Sigma,\ f \in L^1(\Sigma).$$

The measure μ is said to be *normal* with respect to φ if, $\varphi(\Sigma) \subseteq \Sigma$ and $\mu \circ \varphi \ll \mu$. Hence, it is reasonable to define $h^\sharp = \frac{\mathrm{d}\mu\circ\varphi}{\mathrm{d}\mu}$. Now, consider that

$$h^\sharp = (\frac{\mathrm{d}\mu}{\mathrm{d}\mu \circ \varphi})^{-1} = (\frac{\mathrm{d}\mu \circ \varphi^{-1}}{\mathrm{d}\mu} \circ \varphi)^{-1} = \frac{1}{h \circ \varphi}$$

and so $h^\sharp \circ \varphi^i = (h \circ \varphi^{i+1})^{-1}$ for all $i \geq 1$. Similarly, in this case, we have

$$h^\sharp_n := \frac{\mathrm{d}\mu \circ \varphi^n}{\mathrm{d}\mu} = (h^\sharp \circ \varphi^{(n-1)}) \cdots (h^\sharp \circ \varphi^0) = \prod_{i=0}^{n-1} h^\sharp \circ \varphi^i = \prod_{i=1}^{n} (h \circ \varphi^i)^{-1},$$

$h^\sharp_n \circ \varphi > 0$, $h^\sharp_{n+1} = h^\sharp h^\sharp_n \circ \varphi$ and $\sqrt{h^\sharp_n}(h_n \circ \varphi^n) = \sqrt{h_n \circ \varphi^n}$.

Let $1 \leq p \leq \infty$. For any non-negative Σ-measurable functions f as well as for any $f \in L^p(\Sigma)$, by the Radon–Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A E^{\mathcal{A}}(f)\mathrm{d}\mu = \int_A f \mathrm{d}\mu, \qquad \text{for all} \quad A \in \mathcal{A}.$$

Hence, we obtain a contractive projection $E^{\mathcal{A}}$ from $L^p(\Sigma)$ onto $L^p(\mathcal{A})$ which is called a *conditional expectation operator* associated with the σ-finite subalgebra $\mathcal{A}$. Put $\mathcal{D}(E^{\mathcal{A}}) = \{f \in L^0(\Sigma) : E^{\mathcal{A}}(|f|) \in L^0(\mathcal{A})\}$. Then, $\mathcal{D}(E^{\mathcal{A}})$ the *domain* of $E^{\mathcal{A}}$ contains $\{L^p(\Sigma) : 1 \leq p \leq \infty\} \cup \{f \in L^0(\Sigma) : f \geq 0\}$. Note that for every $f \in \mathcal{D}(E^{\mathcal{A}})$ and every $\mathcal{A}$-measurable function $g \in \mathcal{D}(E^{\mathcal{A}})$, we have $E^{\mathcal{A}}(fg) = gE^{\mathcal{A}}(f)$. For more details, see [15,18,21].

For $n \in \mathbb{N}$, let $\Sigma_n := \varphi^{-n}(\Sigma)$ be a σ-finite subalgebra of Σ. Then $\Sigma_{n+1} = \varphi^{-1}(\Sigma_n) \subseteq \Sigma_n$ and Σ_k is also σ-finite for any $k < n$. Set $E_n = E^{\Sigma_n}$. Since $E_n(f)$ is a Σ_n-measurable function, there is a $g \in L^0(\Sigma)$ such that $E_n(f) = g \circ \varphi^n$. In general, g is not unique. This deficiency can be solved by assuming $\sigma(g) \subseteq \sigma(h_n)$ and as a notation, we then write $g = E_n(f) \circ \varphi^{-n}$([20]). Indeed, note that $g_1 \circ \varphi^n = g_2 \circ \varphi^n = E_n(f) \Leftrightarrow (g_1 - g_2) \circ \varphi^n = 0 \Leftrightarrow \int_{\sigma(h_n)} h_n(g_1 - g_2)\mathrm{d}\mu = 0$. Thus, $g_1 = g_2$ on $\sigma(h_n)$. Now by letting, $g_1 = g_2 = 0$ on $\sigma(h_n)^c$, we obtain that $g_1 = g_2$ on X.

It is known that $h_{n+1} = hE_1(h_n) \circ \varphi^{-1} = h_n E_n(h) \circ \varphi^{-1}$ (see [16]). So $\{\sigma(h_n)\}_n$ is a decreasing sequence. Moreover, if we set $F = (f \circ \varphi^n) \circ \varphi^{-n}$, then $\sigma(F) \subseteq \sigma(h_n)$ and $F \circ \varphi^n = f \circ \varphi^n$. It follows that $h_n F = h_n f$, and so $\chi_{\sigma(h_n)} f = \chi_{\sigma(h_n)} F = F$. Consequently, $(f \circ \varphi^n) \circ \varphi^{-n} = \chi_{\sigma(h_n)} f$.

Let $1 \leq p < \infty$, $u \in L^0(\Sigma)$ and let the measurable transformation $\varphi : X \to X$ be *non-singular*. By a *weighted composition operator* in $L^p(\Sigma)$, we mean a mapping $W = uC_\varphi : L^p(\Sigma) \supseteq \mathcal{D}(W) \to L^p(\Sigma)$ formally defined by

$$Wf(x) = \begin{cases} u(x)f(\varphi(x)) & x \in \sigma(u) \\ 0 & x \notin \sigma(u), \end{cases}$$

for all $f \in \mathcal{D}(W) = \{f \in L^p(\Sigma) : u.(f \circ \varphi) \in L^p(\Sigma)\}$. We use the assumption $(\mu \circ \varphi^{-1})_{|_{\sigma(u)}} \ll \mu$ to see that W is well-defined on $\mathcal{D}(W)$, for more details, see [8]. Now by setting $u = 1$, a *composition operator* C_φ defined by $C_\varphi(f) = f \circ \varphi$ on $L^p(\Sigma)$ is well-defined if and only if the transformation φ is *non-singular*. It is known that $C_\varphi \in \mathcal{B}(L^p(\Sigma))$, the algebra of all bounded linear operators on $L^p(\Sigma)$, if and only if $h \in L^\infty(\Sigma)$.

In this case $\mathcal{D}(C_\varphi) = L^p(\Sigma)$, $\|C_\varphi\|^p = \|h\|_\infty$ and $W = M_u C_\varphi$, where M_u is a *multiplication operator* defined by $M_u(f) = uf$ on $\mathcal{D}(M_u) = \{f \in L^p(\Sigma) : u.f \in L^p(\Sigma)\}$. It is known by the closed graph theorem that $\mathcal{D}(M_u) = L^p(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$, or equivalently, $M_u \in \mathcal{B}(L^p(\Sigma))$. In this case, $\|M_u\| = \|u\|_\infty$ (see [25]). Put $J := hE^{\mathcal{A}}(|u|^p) \circ \varphi^{-1}$. Then $W \in \mathcal{B}(L^p(\Sigma))$ if and only if $J \in L^\infty(\Sigma)$. In this case $\mathcal{D}(W) = L^p(\Sigma)$ and $\|W\|^p = \|J\|_\infty$ (see [16]). The *kernel* and the *range* of W are denoted by $\mathcal{N}(W)$ and $\mathcal{R}(W)$ respectively.

As mentioned before, $h_n = h^n$ whenever $h \circ \varphi = h$. In some cases, this can occur, as happens, for instance, when the composition operator C_φ is an isometry or a normal operator.

In this stage, we confine our attention to the dynamics of known operators, such as weighted shifts, weighted translations and composition operators. In [23], Salas has studied the hypercyclicity of the weighted shifts on $\ell^2(\mathbb{Z})$ widely and technically. The hypercyclicity of *a weighted translation* on $L^p(G)$ (G is a locally compact group) defined by $T_{g,w}f = wf * \delta_g$, $g \in G, f \in L^p(G)$, has been studied in [11] by Chen and Chu. Moreover, Bayart et al. [4], have provided a necessary and sufficient condition for a composition operator C_φ on the L^p-spaces of measurable functions to be topologically transitive or topologically mixing.

To be precise, we mention some of the concrete conditions under which these operators are known to be hypercyclic. In this way, the reader can appreciate how much of the panorama changes for weighted composition operators and how the results in the present paper fit into the general theory.

Theorem 1.1 [23]. *A bilateral weighted shift with positive weight sequence* $\{w_n\}$ *on* $\ell^2(\mathbb{Z})$ *is hypercyclic if and only if given* $\epsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* n *arbitrarily large such that for all* $|j| \leq q$, $\prod_{i=0}^{n-1} w_{i+j} < \epsilon$ *and* $\prod_{i=1}^{n} w_{j-i} > \frac{1}{\epsilon}$.

Theorem 1.2 [11]. *For an aperiodic element* g, *a weighted translation* $T_{g,w}$ *on* $L^p(G)$ *is hypercyclic if and only if for each compact subset* $K \subset G$ *with* $\lambda(K) > 0$*(*λ *is a right Haar measure), there is a sequence of Borel sets* $\{F_n\}$ *in* K *such that* $\lim_{n\to\infty} \lambda(F_n) = \lambda(K)$ *and both sequences* $w_n := \prod_{i=1}^{n} w * \delta^i_{g^{-1}}$ *and* $\tilde{w}_n := (\prod_{i=0}^{n-1} w * \delta^i_g)^{-1}$ *admit respectively subsequences* (w_{k_n}) *and* $(\tilde{w}_{k_n})$ *satisfying* $\lim_{n\to\infty} \|w_{k_n}|_{F_{k_n}}\|_\infty = \lim_{n\to\infty} \|\tilde{w}_{k_n}|_{F_{k_n}}\|_\infty = 0$.

Theorem 1.3 [4]. *A composition operator* C_φ *on* $L^p(\Sigma)$ *is topologically transitive if and only if* $\varphi^{-1}\Sigma = \Sigma$ *and for each* $\epsilon > 0$, $F \in \Sigma$ *with finite measure, there exists a measurable subset* $B \subset F$, $k \geq 1$ *and* $S \in \Sigma$ *such that* $\mu(F \setminus B) < \epsilon$, $\mu(\varphi^{-k}(B)) < \epsilon$, $\varphi^k(B) \subset S$ *and* $\mu(S) < \epsilon$.

These interesting works motivated us to study the hypercyclicity of a weighted composition operator on L^p-spaces of measurable functions.

It is well known that only infinite-dimensional and separable spaces can admit the hypercyclic vectors [3,13]. Upon classical fact, $L^p(X, \Sigma, \mu)$ is separable if and only if (X, Σ, μ) is, i.e., there exists a countable σ-subalgebra $\mathcal{F} \subseteq \Sigma$ such that for each $\epsilon > 0$ and $A \in \Sigma$, we have $\mu(A\Delta B) < \epsilon$ for some $B \in \mathcal{F}$. For more details consult [20].

A measurable transformation $\varphi : X \to X$ is said to be *periodic* if $\varphi^m = I$ for some $m \in \mathbb{N}$, where I is the identity map. It is called *aperiodic*, if it is not periodic. Moreover, it is said to be *finitely non-mixing* if for each subset $F \in \Sigma$ with finite measure, there exists an $N \in \mathbb{N}$ such that $F \cap \varphi^n(F) = \emptyset$ for every $n > N$. It is clear that if φ is finitely non-mixing then for each $x \in X$, $orb(\varphi, x)$ has not finite measure.

Let $W \in \mathcal{B}(L^p(\Sigma))$ and let $\mathcal{M}$ be a nonzero subspace of $L^p(\Sigma)$. Then W is *subspace-hypercyclic* for the subspace $\mathcal{M}$, if there exists $f \in L^p(\Sigma)$ such that $orb(W, f) \cap \mathcal{M}$ is dense in $\mathcal{M}$. The notion of the subspace hypercyclicity has been studied in [19]. In this paper, we investigate the dynamics of a weighted composition operator $W = uC_\varphi : f \mapsto u.(f \circ \varphi)$ on $L^p(\Sigma)$ spaces. It is shown that if φ is a periodic *non-singular* transformation, then W nor C_φ can be hypercyclic. Also, if φ is normal and finitely non-mixing, then using the Radon–Nikodym derivatives and the conditional expectations, conditions that either sufficient or necessary for W to be hypercyclic are given. The weakly mixing and topologically mixing concepts are also studied for W. Further, we show that the adjoint operator W^* is subspace hypercyclic for $L^2(\mathcal{A})$. At the end, to illustrate the obtained results, some examples are given.

2. Hypercyclicity of Weighted Composition Operators on $L^p(\Sigma)$

In this section, we characterize the hypercyclicity of a weighted composition operator W. It is done in two cases, in the first case φ is periodic and in the second case is aperiodic. In the last case, the Kitai's hypercyclicity criterion will be essentially used. Basically, the obtained results are engaged with the Radon–Nikodym derivative and the conditional expectation. The techniques of the proofs are similar to those used in [11,23] nevertheless, the roles of the Radon-Nikodym derivative and the conditional expectation are undeniable.

For this, we adopt some settings and notations. Set $u_n = u.(u\circ\varphi)\cdots(u\circ\varphi^{n-1})$ and $J_n = h_n E_n(|u_n|^p) \circ \varphi^{-n}$. We use the symbols u, h, $h^\sharp$, E and $J = hE(|u|^p) \circ \varphi^{-1}$ instead of u_1, h_1, $h_1^\sharp$, E_1 and J_1, respectively. If $\varphi^m = I$, then $u_{m+n} = u_m u_n$, $(u_n)^m = (u^m)_n = u_{mn}$, $u_m \circ \varphi^n = u_m$ and $\sigma(u_n) = \bigcap_{i=1}^{n-1} \sigma(u \circ \varphi^i)$.

Theorem 2.1. *Let* φ *be a periodic* non-singular *transformation. Then a weighted composition operator* $W \in \mathcal{B}(L^p(\Sigma))$ *is not hypercyclic.*

Proof. Suppose that there exists an $m \in \mathbb{N}$ such that $\varphi^m = I$. Consider the orbit of W at $f \in L^p(\Sigma)$:

$$\begin{aligned} &orb(W, f)\\ &\quad = \{f, Wf, \dots, W^{m-1}f\} \cup \{W^m f, W^{m+1}f, \dots, W^{2m-1}f\}\\ &\quad \cup \cdots \cup \{W^{km}f, W^{km+1}f, \dots, W^{(k+1)m-1}f\} \cup \cdots\\ &\quad = \{f, u.f\circ\varphi, u_2.f\circ\varphi^2, \cdots, u_{m-1}.f\circ\varphi^{m-1}\}\\ &\quad \cup \{u_m f, u_m.(u.f\circ\varphi), u_m.(u_2.f\circ\varphi^2), \dots, u_m.(u_{m-1}.f\circ\varphi^{m-1})\}\\ &\quad \cup \{u_m^2 f, u_m^2.(u.f\circ\varphi), u_m^2.(u_2.f\circ\varphi^2), \dots, u_m^2.(u_{m-1}.f\circ\varphi^{m-1})\} \cup \dots. \end{aligned}$$

First, suppose that $\|u_m\|_\infty \leq 1$. Since $\|W^n\| = \|J_n\|_\infty^{1/p} \leq \|W\|^n = \|J\|_\infty^{n/p}$, then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|W^n f\|_p &\leq \max\{\|f\|_p, \|u.f\circ\varphi\|_p, \|u_2.f\circ\varphi^2\|_p, \cdots, \|u_{m-1}.f\circ\varphi^{m-1}\|_p\}\\ &\leq \|f\|_p \max\{1, \|J\|_\infty^{\frac{1}{p}}, \|J\|_\infty^{\frac{2}{p}}, \cdots, \|J\|_\infty^{\frac{m-1}{p}}\}. \end{aligned}$$

Hence, $orb(W, f)$ is a bounded subset and cannot be dense in $L^p(\Sigma)$. Therefore, W is not hypercyclic.

To prove in the case $\|u_m\|_\infty > 1$, similarly as shown in [11, Lemma 1.1], suppose on contrary that W is hypercyclic. For each $\varepsilon > 0$, there exists a subset $F \in \Sigma$ with $0 < \mu(F) < \infty$, on which $|u_m| > 1$. Hence, one may find $f \in L^p(\Sigma)$ and $n \in \mathbb{N}$, sufficiently large such that

$$\|f - 2\chi_F\|_p < \varepsilon \quad \text{and} \quad \|(W^m)^n f\|_p < \varepsilon.$$

In the last, we have already used the hypercyclicity of W^m and the fact that all hypercyclic vectors are dense in $L^p(\Sigma)$. Set $S = \{t \in F : |f(t)| < 1\}$ and note that $\chi_S \leq \chi_S|f - 2| \leq \chi_S|f - 2\chi_F|$. So it is clear that $\mu(S) < \varepsilon^p$. On the other hand, we have

$$\begin{aligned} \varepsilon^p > \|(W^m)^n f)\|_p^p &= \int_X |u_{mn}.f\circ\varphi^{mn}|^p \mathrm{d}\mu\\ &= \int_X |u_m|^{np}|f|^p \mathrm{d}\mu \geq \int_{F-S} |f|^p \mathrm{d}\mu \geq \mu(\chi_{F-S}). \end{aligned}$$

Therefore, $\mu(F) = \mu(S) + \mu(F - S) < 2\varepsilon^p$, which is a contradiction. □

Remark 2.2. If φ is a periodic *non-singular* transformation, then a composition operator $C_\varphi \in \mathcal{B}(L^p(\Sigma))$ is not hypercyclic either. Since its orbit at $f \in L^p(\Sigma)$ i.e., $orb(C_\varphi, f) = \{f, f\circ\varphi, f\circ\varphi^2, \cdots, f\circ\varphi^{m-1}\}$ forms a bounded subset and cannot be dense in $L^p(\Sigma)$.

Theorem 2.3. *Let* $\varphi : X \to X$ *be a non-singular and finitely non-mixing transformation. If* $W \in \mathcal{B}(L^p(\Sigma))$ *is hypercyclic, then for each* $\varepsilon > 0$ *and subset* $F \in \Sigma$ *with* $0 < \mu(F) < \infty$*, there exists a sequence of measurable sets* $\{V_k\} \subseteq F$ *such that* $\mu(V_k) \to \mu(F)$ *as* $k \to \infty$*, and there is a sequence of integers* $\{n_k\}$ *such that* $\|u_{n_k}^{-1}|_{V_k}\|_\infty < \varepsilon$ *and* $\|\sqrt[p]{h_{n_k}}\ E_{n_k}(u_{n_k})\circ\varphi^{-n_k}|_{V_k}\|_\infty < \varepsilon$*.*

Proof. Since W is hypercyclic, it has dense range and then $\sigma(u) = X$ (see [8, Proposition 17]). So $\sigma(u_k) = \cap_{i=0}^{k-1}\varphi^{-i}(\sigma(u)) = X$ for all $k \in \mathbb{N}$. Let $F \in \Sigma$ be an arbitrary set with $0 < \mu(F) < \infty$ and let ε be an arbitrary positive real number. That a transformation φ is assumed to be finitely non-mixing, ensures the existence of an $N \in \mathbb{N}$ such that $F \cap \varphi^n(F) = \emptyset$ for each $n > N$. Choose ε_1 in such a way that $0 < \varepsilon_1 < \frac{\varepsilon}{1+\varepsilon}$. It is known that the set of all hypercyclic vectors for W and also the set of all simple functions form dense subsets in $L^p(\Sigma)$. Hence, there exists a hypercyclic vector $f \in L^p(\Sigma)$ and $m \in \mathbb{N}$ with $m > N$ such that

$$\|f - \chi_F\|_p < \varepsilon_1^2 \quad \text{and} \quad \|W^m f - \chi_F\|_p < \varepsilon_1^2.$$

Put $P_{\varepsilon_1} = \{t \in F : |f(t) - 1| \geq \varepsilon_1\}$ and $R_{\varepsilon_1} = \{t \in X - F : |f(t)| \geq \varepsilon_1\}$. Then we have

$$\begin{aligned}\varepsilon_1^{2p} > \|f - \chi_F\|_p^p &= \int_X |f - \chi_F|^p \mathrm{d}\mu \\ &\geq \int_{P_{\varepsilon_1}} |f(x) - 1|^p \mathrm{d}\mu(x) + \int_{R_{\varepsilon_1}} |f(x)|^p \mathrm{d}\mu(x) \\ &\geq \varepsilon_1^p(\mu(P_{\varepsilon_1}) + \mu(R_{\varepsilon_1})).\end{aligned}$$

Then, $\max\{\mu(P_{\varepsilon_1}), \mu(R_{\varepsilon_1})\} < \varepsilon_1^p$. Set $S_{m,\varepsilon_1} = \{t \in F : |u_m(t) f \circ \varphi^m(t) - 1| \geq \varepsilon_1\}$ and now consider the following relationships:

$$\begin{aligned}\varepsilon_1^{2p} &> \|W^m f - \chi_F\|_p^p \\ &= \int_X |u_m . f \circ \varphi^m - \chi_F|^p \mathrm{d}\mu \\ &\geq \int_{S_{m,\varepsilon_1}} |u_m(t) f \circ \varphi^m(t) - 1|^p \mathrm{d}\mu(t) \\ &\geq \varepsilon_1^p \mu(S_{m,\varepsilon_1})\end{aligned}$$

to deduce that $\mu(S_{m,\varepsilon_1}) < \varepsilon_1^p$. But for an arbitrary $t \in F$, it is readily seen that $\varphi^m(t) \notin F$ because of $F \cap \varphi^{-m}(F) = \emptyset$. Hence, for each $t \in F - (S_{m,\varepsilon_1} \cup \varphi^{-m}(R_{\varepsilon_1}))$, we have

$$|u_m^{-1}(t)| < \frac{|f \circ \varphi^m(t)|}{1 - \varepsilon_1} < \frac{\varepsilon_1}{1 - \varepsilon_1} < \varepsilon.$$

Now, let $U_{m,\varepsilon_1} = \varphi^{-m}(\{t \in F : \sqrt[p]{h_m(t)}\ |(E_m(u_m) \circ \varphi^{-m})(t) f(t)| \geq \varepsilon_1\})$. The inclusion $U_{m,\varepsilon_1} \subseteq \varphi^{-m}(F) \subseteq X - F$, causes to $0 \leq \int_{U_{m,\varepsilon_1}} E_m(\chi_F)\mathrm{d}\mu = \int_{U_{m,\varepsilon_1}} \chi_F \mathrm{d}\mu = 0$ and hence we may conclude that $E_m(\chi_F) = 0$ on U_{m,ε_1}.

Use the change of variable formula to obtain that

$$\begin{aligned}\varepsilon_1^{2p} &> \|W^m f - \chi_F\|_p^p \\ &= \int_X |u_m.f\circ\varphi^m - \chi_F|^p \mathrm{d}\mu \\ &\geq \int_X |E_m(u_m).f\circ\varphi^m - E_m(\chi_F)|^p \mathrm{d}\mu \\ &\geq \int_{U_{m,\varepsilon_1}} |E_m(u_m).f\circ\varphi^m|^p \mathrm{d}\mu \\ &\geq \int_{\varphi^m(U_{m,\varepsilon_1})} |E_m(u_m)\circ\varphi^{-m} f|^p h_m \mathrm{d}\mu \\ &\geq \varepsilon_1^p \mu(\varphi^m(U_{m,\varepsilon_1})),\end{aligned}$$

which implies in turn that $\mu(\varphi^m(U_{m,\varepsilon_1})) < \varepsilon_1^p$. Note that for each $t \in F - (\varphi^m(U_{m,\varepsilon_1}) \cup P_{\varepsilon_1})$, we have

$$\sqrt[p]{h_m(t)}\ |(E_m(u_m)\circ\varphi^{-m})(t)f(t)| < \frac{\varepsilon_1}{1-\varepsilon_1} < \varepsilon.$$

Finally, put $V_{m,\varepsilon_1} := F - (P_{\varepsilon_1} \cup \varphi^{-m}(R_{m,\varepsilon_1}) \cup S_{m,\varepsilon_1} \cup \varphi^m(U_{m,\varepsilon_1}))$. Then, clearly $\mu(F - V_{m,\varepsilon_1}) < 4\varepsilon_1^p$, $\|u_m^{-1}|_{V_{m,\varepsilon_1}}\|_\infty < \varepsilon$ and $\|\sqrt[p]{h_m}\ E_m(u_m)\circ\varphi^{-m}|_{V_{m,\varepsilon_1}}\|_\infty < \varepsilon$. Proceeding inductively, for each $k \in \mathbb{N}$, there will be a measurable set $V_k \subseteq F$ and an increasing subsequence $\{n_k\}$ such that $\mu(F - V_k) < 4(\frac{1}{k})^p$, $\|u_{n_k}^{-1}|_{V_k}\|_\infty < \varepsilon$ and $\|\sqrt[p]{h_{n_k}}\ E_{n_k}(u_{n_k})\circ\varphi^{-n_k}|_{V_k}\|_\infty < \varepsilon$. □

Theorem 2.4. *Let $W \in \mathcal{B}(L^p(\Sigma))$ with $\sigma(u) = X$, and let φ be a normal and finitely non-mixing transformation provided that $\varphi^{-1}(\Sigma) = \Sigma$. If $\sup_n \|h_n^\sharp\|_\infty < \infty$ and for each $\varepsilon > 0$ and subset $F \in \Sigma$ with $0 < \mu(F) < \infty$, there exists a sequence of measurable sets $\{V_k\} \subseteq F$ such that $\mu(V_k) \to \mu(F)$ as $k \to \infty$, and there is a sequence of integers $\{n_k\}$ such that $\|u_{n_k}^{-1}|_{V_k}\|_\infty < \varepsilon$ and $\|\sqrt[p]{h_{n_k}}\ u_{n_k}\circ\varphi^{-n_k}|_{V_k}\|_\infty < \varepsilon$, then W is hypercyclic.*

Proof. Clearly, $S(X)$ is dense in $L^p(\Sigma)$ and we may take $D_1 = D_2 = S(X)$ in the Kitai's hypercyclicity criterion. For an arbitrary $f \in S(X)$, one can find $\{V_k\} \subseteq \sigma(f)$ such that $\mu(V_k) \to \mu(\sigma(f))$ and moreover, there exists an N_1 such that $\sigma(f) \cap \varphi^n(\sigma(f)) = \emptyset$ for each $n > N_1$. By the hypothesis, $u_n \neq 0$ and $\Sigma_n = \Sigma$ for all $n \in \mathbb{N}$. It follows that $E_n = E = I$, and so for each $g \in \mathcal{D}(E)$, $g\circ\varphi^{-n}$ is well-defined. Now, for each $n_k > N_1$ define the map $W_{n_k} : S(X) \to L^p(\Sigma)$ by $W_{n_k}(f) = \frac{f\circ\varphi^{-n_k}}{u_{n_k}\circ\varphi^{-n_k}}$. In this setting, we have $W^{n_k}(W_{n_k}(f)) = f$.

Just it remains to show that $\|W^{n_k} f\|_p \to 0$ and $\|W_{n_k}(f)\|_p \to 0$ as $k \to \infty$. Let $\varepsilon > 0$ be an arbitrary. According to the hypothesis, there exist $M, N_1 \in \mathbb{N}$, sufficiently large such that $V_{N_1} \subseteq \sigma(f)$ and

$$\mu(\sigma(f) - V_{N_1}) < \frac{\varepsilon}{M\|f\|_\infty^p}.$$

We may assume that $\{\sqrt[p]{h_{n_k}}\ u_{n_k}\circ\varphi^{-n_k}\}$ converges to 0 uniformly on V_{N_1} by Egoroff's theorem. Hence, there exists an N_2 such that for each $n_k > N_2$,

$\| \sqrt[p]{h_{n_k}}\, u_{n_k} \circ \varphi^{-n_k} \|_\infty^p < \frac{\varepsilon}{\|f\|_\infty^p}$ on V_{N_1}. Hence, we may claim that there exists a non-negative real number M such that $\| \sqrt[p]{h_{n_k}}\, u_{n_k} \circ \varphi^{-n_k} \|_\infty^p \leq M < \infty$ on $\sigma(f)$. Recall that $h_{n_k} = h(h \circ \varphi^{-1}) \cdots (h \circ \varphi^{-n_k+1})$ and $u_{n_k} \circ \varphi^{-n_k} = (u \circ \varphi^{-1})(u \circ \varphi^{-2}) \cdots (u \circ \varphi^{-n_k})$. Now, by the change of variable formula, for each $n_k > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
\|W^{n_k} f\|_p^p &= \int_X |u_{n_k} f \circ \varphi^{n_k}|^p \mathrm{d}\mu \\
&= \int_{\sigma(f)} |u_{n_k} \circ \varphi^{-n_k} f|^p h_{n_k} \mathrm{d}\mu \\
&= \int_{\sigma(f)-V_N} |u_{n_k} \circ \varphi^{-n_k} f|^p h_{n_k} \mathrm{d}\mu + \int_{V_N} |u_{n_k} \circ \varphi^{-n_k} f|^p h_{n_k} \mathrm{d}\mu \\
&< \| \sqrt[p]{h_{n_k}}\, u_{n_k} \circ \varphi^{-n_k} \|_\infty^p \|f\|_\infty^p \; \mu(\sigma(f) - V_N) + \frac{\varepsilon}{\|f\|_\infty^p} \|f\|_\infty^p < 2\varepsilon.
\end{aligned}$$

Accordingly, repeat the above method this time for the subsequence $\{u_{n_k}^{-1}\}$ and use the fact that $\sup_n \|h_n^\sharp\|_\infty < \infty$, to derive the following inference:

$$\begin{aligned}
&\lim_{k\to\infty} \|W_{n_k} f\|_p^p \\
&= \lim_{k\to\infty} \int_X |\frac{f \circ \varphi^{-n_k}}{u_{n_k} \circ \varphi^{-n_k}}|^p \mathrm{d}\mu \\
&= \lim_{k\to\infty} \int_{\sigma(f)} |\frac{f}{u_{n_k}}|^p h_{n_k}^\sharp \mathrm{d}\mu \\
&\leq \sup_k \|h_{n_k}^\sharp\|_\infty \left\{ \lim_{k\to\infty} \int_{\sigma(f)-V_N} |\frac{f}{u_{n_k}}|^p \mathrm{d}\mu + \lim_{k\to\infty} \int_{V_N} |\frac{f}{u_{n_k}}|^p \mathrm{d}\mu \right\} = 0.
\end{aligned}$$

Eventually, all third conditions of Kitai's hypercyclicity criterion are satisfied and the proof is completed. □

Proposition 2.5. *Suppose that* $\varphi : X \to X$ *is a normal and finitely non-mixing transformation with* $\varphi^{-1}(\Sigma) = \Sigma$ *and* $\sigma(u) = X$. *Let* $\sup_n \|h_n^\sharp\|_\infty < \infty$ *and* $W \in B(L^p(\Sigma))$. *Then the following conditions are equivalent:*

(i) W *satisfies the hypercyclic criterion.*
(ii) W *is hypercyclic.*
(iii) $W \oplus W$ *is hypercyclic on* $L^p(\Sigma) \oplus L^p(\Sigma)$.
(iv) W *is weakly mixing.*

Proof. Note that an operator satisfies the hypercyclic criterion if and only if it is weakly mixing [6]. Hence, we only prove the implication $(ii) \Rightarrow (iii)$. By Birkoff's transitivity theorem, it is sufficient to show that $W \oplus W$ is topologically transitive. To begin, pick two pairs of non-empty open sets (A_1, B_1) and (A_2, B_2) in $L^p(\Sigma) \oplus L^p(\Sigma)$ arbitrarily. For $j = 1, 2$, choose the functions $f_j, g_j \in S(X)$ with $f_j \in A_j$ and $g_j \in B_j$. Let $F = \sigma(f_1) \cup \sigma(f_2) \cup \sigma(g_1) \cup \sigma(g_2)$. Then $\mu(F) < \infty$. Assume that $\{V_k\} \subseteq F$, $\{u_{n_k}^{-1}\}$ and $\{\sqrt[p]{h_{n_k}}\, u_{n_k} \circ \varphi^{-n_k}\}$ are as provided by Theorem 2.3. By assumption, there exists an $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $F \cap \varphi^n(F) = \emptyset$. Moreover, for each $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$, such that for each $k > N_2$ and $n_k > N_1$,

$\| \sqrt[p]{h_{n_k}}\, u_{n_k} \circ \varphi^{-n_k} \|_\infty^p < \frac{\varepsilon}{\|f_j\|_p^p}$ on V_k. Hence, for $k > N_2$, we apply the change of variable formula to get that

$$\begin{aligned} \|W^{n_k}(f_j \chi_{V_k})\|_p^p &= \int_X |W^{n_k}(f_j \chi_{V_k})|^p \mathrm{d}\mu \\ &= \int_{\varphi^{-n_k}(V_k)} |u_{n_k}(f_j \circ \varphi^{n_k})|^p \, \mathrm{d}\mu \\ &= \int_{V_k} |u_{n_k} \circ \varphi^{-n_k} f_j|^p h_{n_k} \mathrm{d}\mu < \varepsilon. \end{aligned}$$

Now, define a map $D_\varphi(f) = \frac{f \circ \varphi^{-1}}{u \circ \varphi^{-1}}$ on the subspace $S(X)$. Then for each $f \in S(X)$, $W^{n_k} D_\varphi^{n_k}(f) = f$. Again, we may find an $N_3 \in \mathbb{N}$ such that for each $k > N_3$ and $n_k > N_1$, $\|u_{n_k}^{-1}\|_\infty^p < \frac{\varepsilon}{M \|g_j\|_\infty^p}$ on V_k, where $M = \sup_n \|h_n^\sharp\|_\infty < \infty$. On the other hand, for each $k > N_3$, note that

$$\begin{aligned} \|D_\varphi^{n_k}(g_j \chi_{V_k})\|_p^p &= \int_{\varphi^{n_k}(V_k)} \left|\frac{g_j \circ \varphi^{-n_k}}{u_{n_k} \circ \varphi^{-n_k}}\right|^p \mathrm{d}\mu \\ &= \int_{V_k} \left|\frac{g_j}{u_{n_k}}\right|^p \, h_n^\sharp \mathrm{d}\mu < \varepsilon. \end{aligned}$$

For each $k \in \mathbb{N}$, let $f_{j,k}^\natural = f_j \chi_{V_k} + D_\varphi^{n_k}(g_j \chi_{V_k})$. Then we have $f_{j,k}^\natural \in L^p(\Sigma)$,

$$\|f_{j,k}^\natural - f_j\|_p^p \leq \|f_j\|_\infty^p \; \mu(F - V_k) + \|D_\varphi^{n_k}(g_j \chi_{V_k})\|_p^p$$

and

$$\|W^{n_k} f_{j,k}^\natural - g_j\|_p^p \leq \|g_j\|_\infty^p \; \mu(F - V_k) + \|W^{n_k}(f_j \chi_{V_k})\|_p^p.$$

Finally, we obtain that $\lim_{k\to\infty} f_{j,k}^\natural = f_j$, $\lim_{k\to\infty} W^{n_k} f_{j,k}^\natural = g_j$ and $W^{n_k}(A_j) \cap B_j \neq \emptyset$ for some $k \in \mathbb{N}$. So by Birkoff's transitivity theorem, the last means that $W \oplus W$ is hypercyclic on $L^p(\Sigma) \oplus L^p(\Sigma)$. □

Corollary 2.6. *Under the assumptions of Proposition 2.5, the following conditions are equivalent:*

- *(i) W is topologically mixing on $L^p(\Sigma)$.*
- *(ii) For each measurable subset $F \subseteq X$ with $0 < \mu(F) < \infty$, there exists a sequence of measurable sets $\{V_n\} \subseteq F$ such that $\mu(V_n) \to \mu(F)$ as $n \to \infty$ and $\lim_{n\to\infty} \|u_n^{-1}|_{V_n}\|_\infty = \lim_{n\to\infty} \|\sqrt[p]{h_n}(u_n \circ \varphi^{-n})|_{V_n}\|_\infty = 0$.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Theorem 2.4, just using the full sequences instead of subsequences and then apply the similar method used in the proof of the implication $(ii) \Rightarrow (iii)$ in Proposition 2.5. For the implication $(i) \Rightarrow (ii)$, let $\varepsilon > 0$ and $F \in \Sigma$ with $0 < \mu(F) < \infty$ be arbitrary. Consider a non-empty and open subset $U = \{f \in L^p(\Sigma) : \|f - \chi_F\|_p < \varepsilon\}$. Since W is topologically mixing and $\varphi : X \to X$ is finitely non-mixing, one may find $N \in \mathbb{N}$ such that for all $n > N$, $W^n(U) \cap U \neq \emptyset$ and $F \cap \varphi^n(F) = \emptyset$. Hence, for each $n > N$, we can choose a function $f_n \in U$ such that $W^n f_n \in U$. Then $\|f_n - \chi_F\|_p < \varepsilon$ and $\|W^n f_n - \chi_F\|_p < \varepsilon$. Now, for each f_n, the similar arguments used in the proof of Theorem 2.3, can be proceed and the proof is completed. □

Let $W \in \mathcal{B}(L^p(\Sigma))$ and let $\mathcal{M}$ be a non-zero subspace of $L^p(\Sigma)$. Then W is subspace hypercyclic for the subspace $\mathcal{M}$ if there exists $f \in L^p(\Sigma)$ such that $orb(W, f) \cap \mathcal{M}$ is dense in $\mathcal{M}$. In this stage, we are going to characterize the subspace-hypercyclicity of W^*, the adjoint of a weighted composition operator W, for the subspace $L^2(\mathcal{A})$. Hoover, Lambert and Quinn in [16] have shown that the adjoint W^* of $W \in \mathcal{B}(L^2(\Sigma))$ is given by $W^*(f) = hE(\bar{u}f) \circ \varphi^{-1}$, for each $f \in L^2(\Sigma)$. Let μ be normal with respect to φ. Take $h_n^{\mathcal{A}} = \frac{d(\mu\circ\varphi^{-n}|_{\mathcal{A}})}{d(\mu|_{\mathcal{A}})}$ and $h_n^{\sharp\mathcal{A}} = \frac{d(\mu\circ\varphi^{n}|_{\mathcal{A}})}{d(\mu|_{\mathcal{A}})}$, which are non-negative $\mathcal{A}$-measurable Radon-Nikodym derivatives for each $n \in \mathbb{N}$.

Let W^* be a subspace hypercyclic operator for $L^2(\mathcal{A})$. Then $\mathcal{N}(W)^{\perp} = \overline{\mathcal{R}(W^*)} \supseteq \overline{\mathcal{R}(W^*) \cap L^2(\mathcal{A})} = L^2(\mathcal{A})$. So, $\mathcal{N}(W) \subseteq L^2(\mathcal{A})^{\perp}$. It follows that $X\backslash\varphi(X)$ does not contain any $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$. Because otherwise, $\chi_A \in L^2(\mathcal{A})$ and $W(\chi_A) = u.\chi_{\varphi^{-1}(A)} = 0$. Thus, if $\varphi(X) \in \mathcal{A}$, then φ must be onto. Let $n \in \mathbb{N}$, $\sigma(u) = X$ and let $\mathcal{A} \subseteq \Sigma_n$. For $k \leq n$, set $E^{\mathcal{A}}(\bar{u}) \circ \varphi^{-k} = g$. Then $\varphi^{-k}(\sigma(g)) = \sigma(g \circ \varphi^k) = \sigma(E^{\mathcal{A}}(\bar{u})) = X$. If φ is onto, then $\sigma(g) = \varphi^k(\varphi^{-k}(\sigma(g))) = \varphi^k(X) = X$ and hence $\sigma(E^{\mathcal{A}}(\bar{u}_n) \circ \varphi^{-n}) = X$. Moreover, if φ is normal, then for $A \in \mathcal{A}$ with $\mu(A) > 0$, $\int_A h^{\mathcal{A}} \mathrm{d}\mu = \mu(\varphi^{-1}(A)) > 0$. So, in this case, $\sigma(h^{\mathcal{A}}) = \sigma(h_n^{\mathcal{A}}) = X$. In the following proposition, we give a necessary condition for W^* to be subspace hypercyclic. Also, it should be mentioned that if $\mathcal{A}$ is a proper subalgebra of Σ, the the following result cannot be directly obtained from Theorem 2.3.

Proposition 2.7. *Let $\varphi : X \to X$ be a normal and finitely non-mixing transformation such that $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subset \Sigma_\infty = \bigcap_{n=1}^{\infty} \varphi^{-n}(\Sigma)$ and let $W \in B(L^2(\Sigma))$. If W^* is subspace hypercyclic for $L^2(\mathcal{A})$, then for each subset $F \in \mathcal{A}$ with $0 < \mu(F) < \infty$, there exists a sequence of $\mathcal{A}$-measurable sets $\{V_k\} \subseteq F$ such that $\mu(V_k) \to \mu(F)$ as $k \to \infty$, and there is a sequence of integers $\{n_k\}$ such that*

$$\lim_{k\to\infty} \|\sqrt{h_{n_k}^{\mathcal{A}} \circ \varphi^{n_k}} E^{\mathcal{A}}(\bar{u}_{n_k})|_{V_k}\|_\infty = 0.$$

Moreover, if $\sigma(u) = X$ and φ is onto then

$$\lim_{k\to\infty} \|(\sqrt{h_{n_k}^{\mathcal{A}}}\ E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k})^{-1}|_{V_k}\|_\infty = 0.$$

Proof. Choose $f, g \in L^2(\mathcal{A})$ arbitrarily. Since $g \circ \varphi$ is $\mathcal{A}$-measurable and for each $n \in \mathbb{N}$, $E_n E^{\mathcal{A}} = E^{\mathcal{A}} E_n = E^{\mathcal{A}}$, then we have

$$\begin{aligned} \langle g, W^* f\rangle &= \langle W(g), f\rangle = \int_X u g \circ \varphi \bar{f} \mathrm{d}\mu \\ &= \int_X E^{\mathcal{A}}(u\bar{f}) g \circ \varphi \mathrm{d}\mu = \int_X h^{\mathcal{A}}[E^{\mathcal{A}}(u)\bar{f}] \circ \varphi^{-1} g \mathrm{d}\mu \\ &= \int_X h^{\mathcal{A}} \overline{[E^{\mathcal{A}}(\bar{u})f]} \circ \varphi^{-1} g \mathrm{d}\mu = \langle g, h^{\mathcal{A}}[E^{\mathcal{A}}(\bar{u})f] \circ \varphi^{-1}\rangle. \end{aligned}$$

Therefore, $W^* f = h^{\mathcal{A}}[E^{\mathcal{A}}(\bar{u}) \circ \varphi^{-1}] f \circ \varphi^{-1}$ for all $f \in L^2(\mathcal{A})$. Furthermore, for each $n \in \mathbb{N}$ we have

$$W^{*n} f = h_n^{\mathcal{A}}[E^{\mathcal{A}}(\bar{u}_n) \circ \varphi^{-n}] f \circ \varphi^{-n},$$

where $h_n^{\mathcal{A}} = h^{\mathcal{A}}[h^{\mathcal{A}} \circ \varphi^{-1}] \cdots [h^{\mathcal{A}} \circ \varphi^{-n+1}]$. On the other hand, according to the hypothesis that asserts $\varphi^{-n}(F) \cap F = \emptyset$, for some $n \in \mathbb{N}$ and the fact $\sqrt{h_n^{\sharp\mathcal{A}}}\ (h_n^{\mathcal{A}} \circ \varphi^n) = \sqrt{h_n^{\mathcal{A}} \circ \varphi^n}$, we obtain that

$$\begin{aligned}\|W^{*n} f - \chi_F\|_2^2 &= \int_X |h_n^{\mathcal{A}}[E^{\mathcal{A}}(\bar{u}_n) f] \circ \varphi^{-n} - \chi_F|^2 \mathrm{d}\mu \\ &= \int_X |h_n^{\mathcal{A}} \circ \varphi^n [E^{\mathcal{A}}(\bar{u}_n) f] - \chi_{\varphi^{-n}(F)}|^2 h_n^{\sharp\mathcal{A}} \mathrm{d}\mu \\ &\geq \int_{U^{\sharp}_{n,\varepsilon}} |\sqrt{h_n^{\sharp\mathcal{A}}}\ (h_n^{\mathcal{A}} \circ \varphi^n) E^{\mathcal{A}}(u_n) f|^2 \mathrm{d}\mu \\ &= \int_{U^{\sharp}_{n,\varepsilon}} |\sqrt{h_n^{\mathcal{A}} \circ \varphi^n}\ E^{\mathcal{A}}(u_n) f|^2 \mathrm{d}\mu,\end{aligned}$$

where $U^{\sharp}_{n,\varepsilon} = \{t \in F : \sqrt{h_n^{\mathcal{A}}(\varphi^n(t))}\ E^{\mathcal{A}}(\bar{u}_n(t)) f(t)| \geq \varepsilon\}$ is a $\mathcal{A}$-measurable subset for each $\varepsilon > 0$. The rest of the proof is routine and similar to that done in Theorem 2.3, so to avoid a tediously long discussion we do not repeat them again. □

Proposition 2.8. *Let $\varphi : X \to X$ be an onto normal and finitely non-mixing transformation, $\mathcal{A} \subset \Sigma_\infty = \bigcap_{n=1}^{\infty} \varphi^{-n}(\Sigma)$, $\sigma(u) = X$ and let $W \in B(L^2(\Sigma))$. If for each subset $F \in \mathcal{A}$ with $0 < \mu(F) < \infty$, there exists a sequence of $\mathcal{A}$-measurable sets $\{V_k\} \subseteq F$ such that $\mu(V_k) \to \mu(F)$ as $k \to \infty$, and there is a sequence of integers $\{n_k\}$ such that*

$$\lim_{k\to\infty} \|\sqrt{h_{n_k}^{\mathcal{A}} \circ \varphi^{n_k}} E^{\mathcal{A}}(\bar{u}_{n_k})|_{V_k}\|_\infty = 0$$

and

$$\lim_{k\to\infty} \|(\sqrt{h_{n_k}^{\mathcal{A}}}\ E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k})^{-1}|_{V_k}\|_\infty = 0,$$

then W^* is subspace hypercyclic for $L^2(\mathcal{A})$.

Proof. Let $S^{\mathcal{A}}(X)$ be the class of all $\mathcal{A}$-measurable and simple functions on X with finite supports. So, $S^{\mathcal{A}}(X)$ is dense in $L^2(\mathcal{A})$. Define the maps $T_{n_k} : S^{\mathcal{A}}(X) \to L^2(\mathcal{A})$ by

$$T_{n_k}(f) = \frac{f \circ \varphi^{n_k}}{(h_{n_k}^{\mathcal{A}} \circ \varphi^{n_k}) E^{\mathcal{A}}(\bar{u}_{n_k})},$$

where $\{n_k\}$ is chosen by the hypothesis for the $\mathcal{A}$-measurable subset $\sigma(f) \subseteq X$. That φ is an onto map implies $W^{*n_k}(T_{n_k}(f)) = f$ and so we have to only show that $\|W^{*n_k} f\|_2 \to 0$ and $\|T_{n_k}(f)\|_2 \to 0$ as $k \to \infty$. Let $\varepsilon > 0$ be an arbitrary. We know that there exists a non-negative real number M such that $\|\sqrt{h_{n_k}^{\mathcal{A}} \circ \varphi^{n_k}}\ E^{\mathcal{A}}(\bar{u}_{n_k})\|_\infty^2 \leq M < \infty$ on $\sigma(f)$. Moreover, there exists an $N \in \mathbb{N}$, sufficiently large such that $V_N \subseteq \sigma(f)$ and

$$\mu(\sigma(f) - V_N) < \frac{\varepsilon}{2M\|f\|_\infty^2}.$$

We may assume that $\{\sqrt{h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k}}\ E^{\mathcal{A}}(\bar{u}_{n_k})\}$ converges to 0 uniformly on V_N. Hence, for each $n_k > N$,

$$\|\sqrt{h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k}}\ E^{\mathcal{A}}(\bar{u}_{n_k})\|^2_\infty < \frac{\varepsilon}{2\|f\|^2_2}$$

on V_N. Here, for each $n_k > N$ and $f \in S^{\mathcal{A}}(X)$ with the aid of the change of variable formula, we have that

$$\begin{aligned}
\|W^{*n_k} f\|^2_2 &= \int_X |h^{\mathcal{A}}_{n_k} E^{\mathcal{A}}(\bar{u}_{n_k} f) \circ \varphi^{-n_k}|^2 \mathrm{d}\mu \\
&= \int_X (h^{\mathcal{A}}_{n_k})^2 (|E^{\mathcal{A}}(\bar{u}_{n_k})|^2 \circ \varphi^{-n_k}) |f|^2 \circ \varphi^{-n_k} \mathrm{d}\mu \\
&= \int_X (h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k})^2 |E^{\mathcal{A}}(\bar{u}_{n_k})|^2 |f|^2\ h^{\sharp\mathcal{A}}_{n_k}\ \mathrm{d}\mu \\
&= \int_{\sigma(f)} (h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k}) |E^{\mathcal{A}}(\bar{u}_{n_k})|^2 |f|^2 \mathrm{d}\mu \\
&= (\int_{\sigma(f)-V_n} + \int_{V_n}) (h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k} |E^{\mathcal{A}}(\bar{u}_{n_k})|^2 |f|^2) \mathrm{d}\mu \\
&< 2\varepsilon.
\end{aligned}$$

Finally, apply the same method for the subsequence $\{((h^{\mathcal{A}}_{n_k})^{1/2} E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k})^{-1}\}$ to deduce that $\|T_{n_k} f\|^2_2 < 2\varepsilon$ as follows:

$$\begin{aligned}
\|T_{n_k} f\|^2_2 &= \int_X \frac{|f \circ \varphi^{n_k}|^2\ \mathrm{d}\mu}{(h^{\mathcal{A}}_{n_k} \circ \varphi^{n_k})^2 |E^{\mathcal{A}}(\bar{u}_{n_k})|^2} \\
&= \int_X \frac{|f|^2 h^{\mathcal{A}}_{n_k}\ \mathrm{d}\mu}{(h^{\mathcal{A}}_{n_k})^2 |E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k}|^2} \\
&= \int_X |(\sqrt{h^{\mathcal{A}}_{n_k}} E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k})^{-1}|^2\ |f|^2 \mathrm{d}\mu \\
&= (\int_{\sigma(f)-V_n} + \int_{V_n}) |(\sqrt{h^{\mathcal{A}}_{n_k}} E^{\mathcal{A}}(\bar{u}_{n_k}) \circ \varphi^{-n_k})^{-1}|^2\ |f|^2 \mathrm{d}\mu \\
&< 2\varepsilon.
\end{aligned}$$

□

Example 2.9. Let $\mathbb{Z}$ be the integer numbers. Let μ denote the point mass measure on the σ-algebra Σ consisting of all subsets of $\mathbb{Z}$ defined by $\mu(\{n\}) = m_n$. Here, $(m_n)^{+\infty}_{n=-\infty}$ is an increasing and upper bounded sequence of real numbers with $1 \leq m_n \leq M$. For the measurable transformation

$$\varphi(k) = k + 2,\ \ k \in \mathbb{Z},$$

and the weight function

$$u(k) = \begin{cases} 2, & k = 0, 1, 2, 3, \ldots, \\ \frac{1}{2}, & k = -1, -2, \ldots; \end{cases}$$

we show that the corresponding weighted composition operator W is hypercyclic on $L^p(\mathbb{Z})$.

Proof. Concerning a sequence (m_n), one may simply take $M = 2$ and

$$(m_n) = \cdots, \underbrace{1+\frac{1}{9}, 1+\frac{1}{6}, 1+\frac{1}{3}}_{n\leq -1}, \sqrt{2}, \underbrace{2-\frac{1}{2}, 2-\frac{1}{4}, 2-\frac{1}{6}, \cdots}_{n\geq 1}$$

or the counting measure i.e., $m_n = 1$ for each $n \in \mathbb{Z}$ can be considered as well. Clearly, φ is a normal and finitely non-mixing transformation and $\varphi^{-1}(\Sigma) = \Sigma$. Note that for each $k \in \mathbb{Z}$, we shall use the following formulas presently:

$$h(k) = \frac{1}{m_k} \sum_{j\in\varphi^{-1}(k)} m_j$$

and

$$h_n(k) = \frac{1}{m_k} \sum_{j\in\varphi^{-1}(k)} h_{n-1}(j)m_j, \ n \geq 2.$$

Precisely, for each $k \in \mathbb{Z}$, the singleton $\{k\}$ is a μ-atom and so $\int_{\{k\}} h\mathrm{d}\mu = h(k)\mu(\{k\}) = h(k)m_k$. Therefore,

$$h(k)=\frac{1}{m_k}\int_{\{k\}} h\mathrm{d}\mu=\frac{1}{m_k}\int_{\{k\}} \mathrm{d}\mu\circ\varphi^{-1}=\frac{1}{m_k}\int_{\varphi^{-1}(\{k\})} \mathrm{d}\mu=\frac{1}{m_k}\sum_{j\in\varphi^{-1}(\{k\})} m_j.$$

On the other hand, with the aid of the change of variable formula and $h_n = hE(h_{n-1}) \circ \varphi^{-1}$, we see that

$$\begin{aligned} h_n(k) &= \frac{1}{m_k}\int_{\{k\}} h_n\mathrm{d}\mu = \frac{1}{m_k}\int_{\{k\}} hE(h_{n-1})\circ\varphi^{-1}\mathrm{d}\mu \\ &= \frac{1}{m_k}\int_{\{k\}} E(h_{n-1})\circ\varphi^{-1}\mathrm{d}\mu\varphi^{-1} = \frac{1}{m_k}\int_{\varphi^{-1}(\{k\})} E(h_{n-1})\mathrm{d}\mu \\ &= \frac{1}{m_k}\int_{\varphi^{-1}(\{k\})} h_{n-1}\mathrm{d}\mu = \frac{1}{m_k}\sum_{\varphi^{-1}(\{k\})} h_{n-1}(j)m_j. \end{aligned}$$

In our settings, $h(k) = \frac{m_{k-2}}{m_k}$ and $h_n(k) = \frac{m_{k-2n}}{m_k}$, and $h_n^\sharp(k) = \frac{m_{k+2n}}{m_k}, \ n \geq 2$. Subsequently, $\sup_n \|h_n^\sharp\|_\infty \leq M$ and $\|h_n\|_\infty \leq 1$ for each $n \in \mathbb{N}$. Further, for an arbitrary finite subset F of $\mathbb{Z}$, take $V_k = F$ for each $k \in \mathbb{Z}$. Then, one may easily find the suitable subsequences $\{u_{n_k}^{-1}\}$ and $\{\sqrt[p]{h_{n_k}}\ u_{n_k} \circ \varphi^{-n_k}\}$ such that $\|u_{n_k}^{-1}|_{V_k}\|_\infty \to 0$ and $\|\sqrt[p]{h_{n_k}}\ u_{n_k} \circ \varphi^{-n_k}|_{V_k}\|_\infty \to 0$ as $k \to \infty$. Hence by Theorem 2.4, W is hypercyclic. □

Example 2.10. Let $X = \mathbb{R}$, the field of real numbers equipped with the Lebesgue measure μ on the σ-algebra of all Lebesgue measurable functions. Fix a positive $t \in \mathbb{R}$ and consider the transformation $\varphi(x) = x + t, \ x \in \mathbb{R}$. Therefore, $h_n = h_n^\sharp = 1$. Fix $r > 1$ and define the weight function u on $\mathbb{R}$ by

$$u(x) = \begin{cases} \frac{1}{r}, & 1 \leq x, \\ -\frac{x}{2}+1, & -1 < x < 1, \\ r, & x \leq -1. \end{cases}$$

For an arbitrary $F = [a, b]$, take $V_k = [a, b - \frac{1}{k})$. In this circumstance, all conditions of Theorem 2.4 are satisfied and hence W is hypercyclic on $L^p(\mathbb{R})$.

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