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Complex Analysis and Operator Theory

Weighted Pullback Transforms on Riemann Surfaces

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Abstract

In this paper, using properties of the conditional expectation operators we give an explicit formula for the adjoint of a bounded weighted pullback transform uC_φ with analytic symbol φ and measurable weight u on the measurable differential form spaces for Riemann surfaces. Also, some properties of these transforms are discussed.

Keywords Riemann surfaces · Differential forms · Weighted composition operators · Multiplication operators

Mathematics Subject Classification Primary 47B38; Secondary 30F30

1 Introduction and Preliminaries

A two dimensional manifold M is a connected Haussdorf topological space such that every x in M has a neighborhood homeomorphic to an open disc in the plane. If M is a two dimensional manifold, a complex chart on M is a homeomorphism $\alpha : U_\alpha \to \alpha(U_\alpha)$ of an open subset $U_\alpha \subset M$ onto an open subset $\alpha(U_\alpha) \subset \mathbb{C}$. Two charts α and β are analytically compatible if transition map

$$\tau_{\alpha\beta} = \beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$$

is biholomorphic. A complex atlas on M is a collection of analytically equivalent compatible charts $\mathcal{A} = \{(\alpha, U_\alpha)\}$ whose domains cover M, i.e. $M = \cup_\alpha U_\alpha$. Two complex atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are analytically equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a complex atlas. An analytic structure on a two dimensional manifold M is an equivalence class of

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analytically equivalent atlases. A Riemann surface is a two dimensional manifold with an analytic structure. A 0-form on M is a complex valued function on M. A 1-form ω on M is an ordered assignment of two functions f^α and g^α to each local coordinate chart (α, U_α) on M such that the expression $f^\alpha d\alpha + g^\alpha d\bar{\alpha}$ is invariant under coordinate changes. A 2-form Ω on M is an assignment of a function f^α to each local coordinate α such that the expression $f^\alpha d\alpha \wedge d\bar{\alpha}$ is invariant under coordinate changes. For $k \in \{0, 1, 2\}$, we let $\Lambda^k(M)$ denote the vector space of k-forms. Note that $\Lambda^k(M) = 0$, for all $k \geq 3$.

Since M locally looks like an open subset of $\mathbb{C}$, it is clear that measurability can be lifted up from $\mathbb{C}$ to M using local charts. Let $(\mathbb{C}, \mathcal{M}_{\mathbb{C}}, A)$ be a Lebesque measure space. A subset $B \subseteq M$ is said to be Lebesque measurable, if for every $b \in B$ there is a local chart (α, U_α) with $b \in U_\alpha$ such that $\alpha(B \cap U_\alpha) \in \mathcal{M}_{\mathbb{C}}$. This approach is independent of coordinate system. Put $\Sigma_M = \{B \subseteq M : B \text{ is Lebesque measurable}\}$. It is easy to see that Σ_M is a σ-algebra over M and contains the Borel σ-algebra $\mathcal{B}(M)$. Let $M = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Using the trivial chart (id, M), $\Sigma_{\mathbb{D}} = \mathcal{M}_{\mathbb{D}} = \{B \cap \mathbb{D} : B \in \mathcal{M}_{\mathbb{C}}\}$ is a σ-algebra restricted to $\mathbb{D}$. Note that the change of variable formula is in disagreement with the change of chart formula, so the Lebesque measure and hence the Lebesque integral on $\cup_\alpha U_\alpha$ can not be lifted to M in general. However, we can also speak about measures on Σ_M of Lebesque sets on M (e.g. see [10]). We say that $f : M \to \mathbb{C}$ is measurable if and only if $f^{-1}(\mathcal{M}_{\mathbb{C}}) \subseteq \Sigma_M$. So, $f : M \to \mathbb{C}$ is measurable if and only if $f^\alpha = f_{|U_\alpha} : (U_\alpha, \Sigma_{U_\alpha}) \to \mathbb{C}$ is measurable, for all α. Equivalently, f is measurable if and only if $f^\alpha \circ \alpha^{-1} : (\alpha(U_\alpha), \mathcal{M}_{\alpha(U_\alpha)}) \to \mathbb{C}$ is measurable, for all α. In particular, $A \in \Sigma_M$ has measure zero if for every local chart (α, U_α) of M, the set $\alpha(A \cap U_\alpha)$ has measure zero. Since the change of coordinates maps between charts are diffeomorphisms, then the null sets remain null under coordinate change. A measurable 1-form with respect to local chart (α, U_α) is an expression ω of the form $\omega = f^\alpha d\alpha + g^\alpha d\bar{\alpha}$, where $f^\alpha, g^\alpha : (U_\alpha, \Sigma_{U_\alpha}) \to \mathbb{C}$ are measurable.

A weighted pullback transform on measurable differential form spaces is an operator induced by pullback with a analytic transformation of the underlying Riemann surfaces, followed by a multiplication (see section 2 for precise definitions). The pullback transforms (composition operators) on Riemann surfaces were first studied by Mihaila [11]; she obtained some results on pullback transforms on Riemann surfaces and posed some problems on these operators. Then Cao [3, 4] characterized invertibility and Fredholmness of pullback and Toeplitz transforms on measurable and analytic differential forms for Riemann surfaces. Boundedness criteria and the adjoint of a weighted pullback transform has been given in [10]. In the next section, we provide another one which is complete and written in terms of conditional expectation operators different than that used in [10]. Also, some properties of these transforms are discussed.

2 Main Results

First we review some basic results on pullback transforms and state some general assumptions. Let us start by recalling the definitions and fixing the notation in case

$M = N = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Take $L^2(\mathbb{D}, \mathcal{M}_{\mathbb{D}}, A) = L^2(\mathbb{D})$ and set $\Lambda_2^1(\mathbb{D}) = \{\omega = f dz + g d\bar{z} : f, g \in L^2(\mathbb{D})\}$. Then $\Lambda_2^1(\mathbb{D})$ is a vector space of measurable 1-forms on the Riemann surface $\mathbb{D}$. For $z = x + iy$ and $\omega = f dz + g d\bar{z} \in \Lambda_2^1(\mathbb{D})$, we have $\overline{dz} = d\bar{z} = dx - idy$ and $^*\omega := -if dz + ig d\bar{z}$. So, $\bar{\omega} = \bar{g} dz + \bar{f} d\bar{z}$ and $^*\bar{\omega} = -i\bar{g}dz + i\bar{f}d\bar{z}$. Set $dx \wedge dy = dxdy = dA$. Then $dz \wedge d\bar{z} = -2idxdy$ and hence $\omega \wedge {}^*\overline{\omega} = i(|f|^2 + |g|^2) dz \wedge d\bar{z} = 2(|f|^2 + |g|^2) dxdy$. For $\omega_i = f_i dz + g_i d\bar{z} \in \Lambda_2^1(\mathbb{D})$, set $\langle \omega_1, \omega_2 \rangle = \int_{\mathbb{D}} \omega_1 \wedge^* \overline{\omega_2} = \int_{\mathbb{D}} 2(f_1 \bar{f_2} + g_1 \bar{g}_2) dxdy$. Then $\Lambda_2^1(\mathbb{D})$ is an inner product space with the induced norm given by $\|\omega\|_{\mathbb{D}}^2 = 2\|f\|^2_{L^2(\mathbb{D})} + 2\|g\|^2_{L^2(\mathbb{D})}$. Now let $\{\omega_n\} = \{f_n dz + g_n d\bar{z}\}$ be a Cauchy sequence in $\Lambda_2^1(\mathbb{D})$. Then

$$\max\{\|f_n - f_m\|^2_{L^2(\mathbb{D})}, \|g_n - g_m\|^2_{L^2(\mathbb{D})}\} \le \int_{\mathbb{D}} 2(|f_n - f_m|^2 + |g_n - g_m|^2) dxdy \to 0.$$

Hence there are $f, g \in L^2(\mathbb{D})$ such that $\max\{\|f_n - f\|^2_{L^2(\mathbb{D})}, \|g_n - g\|^2_{L^2(\mathbb{D})}\} \to 0$. Set $\omega = f dz + g d\bar{z}$. Then $\omega \in \Lambda_2^1(\mathbb{D})$ and $\|\omega_n - \omega\|_{\mathbb{D}} \to 0$. Thus, $(\Lambda_2^1(\mathbb{D}), \| \|_{\mathbb{D}})$ is a Hilbert space (see [6]). We remark that $\Lambda_2^1(\mathbb{D}) \cong L^2(\mathbb{D}) \times L^2(\mathbb{D})$ with the natural norm $\|(f, g)\|^2 = 2\|f\|^2_{L^2(\mathbb{D})} + 2\|g\|^2_{L^2(\mathbb{D})}$. The Bergman space $L_a^2(\mathbb{D})$ is the set of analytic functions on $\mathbb{D}$, square integrable with respect to Lebesque area measure A, i.e. $L_a^2(\mathbb{D}) = L^2(\mathbb{D}) \cap H(\mathbb{D})$, where $H(\mathbb{D})$ denote the class of functions analytic in the unit disc $\mathbb{D}$. It is a closed subspace of $L^2(\mathbb{D})$ and hence is a Hilbert space with inner product $\langle f, g \rangle = 1/\pi \int_{\mathbb{D}} f(z)\bar{g}(z) dA(z)$ (see [15]). Since for each $f \in L_a^2(\mathbb{D})$, $\|f dz\|_{\mathbb{D}} = 2\pi \|f\|_{L_a^2(\mathbb{D})}$, so $\Lambda^1_{2,a}(\mathbb{D}) := \{f dz : f \in L_a^2(\mathbb{D})\} \cong L_a^2(\mathbb{D})$ (see [11]) and hence is an Hilbert space.

Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic and nonconstant function. For $w \in \varphi(\mathbb{D})$, let $c(w, \varphi)$ denote the countable collection of zeros of $\varphi(z) - w$ including multiplicities, i.e. $c(w, \varphi) = \{\xi \in \mathbb{D} : \varphi(\xi) = w\}$. Let W and g be two non-negative measurable functions defined on $\mathbb{D}$. By the area formula [5, Theorem 2.32], we have

$$\int_{\mathbb{D}} g(\varphi(z)) W(z) |\varphi'(z)|^2 dA(z) = \int_{\varphi(\mathbb{D})} g(w) N_\varphi(W)(w) dA(w), \tag{2.1}$$

where $N_\varphi(f) : \mathbb{D} \to \mathbb{C} \cup \{\infty\}$ is the generalized counting function defined by $N_\varphi(f)(w) = \{\Sigma f(\xi) : \xi \in c(w, \varphi)\}$ for all $f \in L^0(\mathbb{D})$, the space of all finite-valued measurable functions on $\mathbb{D}$. If $c(w, \varphi) = \emptyset$, then we take $N_\varphi(f)(w) = 0$. Then the support of $N_\varphi(f)$ is $\varphi(\mathbb{D})$ and so $\chi_{\varphi(\mathbb{D})} N_\varphi(f) = N_\varphi(f)$. For $f \in L^0(\mathbb{D})$, set $g = |f|^2$ in (2.1). Then

$$\int_{\mathbb{D}} W |f \circ \varphi|^2 |\varphi'|^2 dA = \int_{\mathbb{D}} N_\varphi(W) |f|^2 dA.$$

Set $N_\varphi(1) = N_\varphi = \#\{z \in \mathbb{D} : \varphi(z) = w\}$ where the number of z above is counted with appropriate multiplicity. If we take $W = 1$, then we have

$$\int_{\mathbb{D}} |f \circ \varphi|^2 |\varphi'|^2 dA = \int_{\mathbb{D}} N_\varphi |f|^2 dA. \tag{2.2}$$

Suppose $\psi : \mathbb{D} \to \mathbb{D}$ is an analytic and invertible. Then $c(\psi^{-1}(w), \varphi) = c(w, \psi \circ \varphi)$ for all $\omega \in \varphi(\mathbb{D})$, and so $(N_\varphi(f))_\psi := N_\varphi(f) \circ \psi^{-1} = N_{\psi\circ\varphi}(f)$ for all non-negative function on $\mathbb{D}$. Moreover, $N_\varphi(f \circ \varphi) = f N_\varphi$. In particular, let $f(w, a_0)$ be a Green function on $\mathbb{D}$ where a_0 is some fixed point. Then $N_\varphi(f(.\, a_0))$ is called the Nevanlinna counting function on $\mathbb{D}\backslash\{\varphi(a_0)\}$ (see [14]).

The space $L^\infty(\mathbb{D})$ is the set of all essentially bounded functions on $\mathbb{D}$. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, the pullback transform $C_\varphi : \Lambda^1_2(\mathbb{D}) \to \Lambda^1_2(\mathbb{D})$ defined as $C_\varphi(\omega) = \varphi^*(\omega)$, where $\varphi^*(\omega) = (f \circ \varphi)d\varphi + (g \circ \varphi)d\bar{\varphi}$ is the pullback of the form $\omega = f dz + g d\bar{z} \in \Lambda^1_2(\mathbb{D})$. Since $d\varphi = \varphi' dz$ and $d\bar{\varphi} = \overline{d\varphi} = \overline{\varphi'} d\bar{z}$, then $C_\varphi(\omega) = (f \circ \varphi)\varphi' dz + (g \circ \varphi)\overline{\varphi'} d\bar{z}$. Using 2.2 we obtain that

$$\|C_\varphi(\omega)\|^2_{\mathbb{D}} = 2\int_{\mathbb{D}} (|f \circ \varphi|^2 + |g \circ \varphi|^2)|\varphi'|^2 dA = 2\int_{\mathbb{D}} (|f|^2 + |g|^2) N_\varphi dA. \quad (2.3)$$

Thus, for some $k > 0$,

$$\begin{aligned} C_\varphi \in B(\Lambda^1_2(\mathbb{D})) &\Longleftrightarrow \|C_\varphi(\omega)\|_{\mathbb{D}} \le k\|\omega\|_{\mathbb{D}}, \quad \forall \omega \in \Lambda^1_2(\mathbb{D}) \\ &\Longleftrightarrow k^2\|\omega\|^2_{\mathbb{D}} - \|C_\varphi(\omega)\|^2_{\mathbb{D}} \ge 0 \\ &\Longleftrightarrow 2\int_{\mathbb{D}} (|f|^2 + |g|^2)(k^2 - N_\varphi) dA \ge 0 \\ &\Longleftrightarrow N_\varphi \le k^2 \Longleftrightarrow N_\varphi \in L^\infty(\mathbb{D}). \end{aligned}$$

In this case $\|C_\varphi(\omega)\|_{\mathbb{D}} \le \|N_\varphi\|^{1/2}_\infty \|\omega\|_{\mathbb{D}}$, and hence $\|C_\varphi\|_{\mathbb{D}} \le \|N_\varphi\|^{1/2}_\infty$. Also,

$$\begin{aligned} \langle C^*_\varphi C_\varphi(\omega), \omega\rangle &= \langle C_\varphi(\omega), C_\varphi(\omega)\rangle = \|C_\varphi(\omega)\|^2_{\mathbb{D}} = 2\int_{\mathbb{D}} (|f|^2 + |g|^2) N_\varphi dA \\ &= \int_{\mathbb{D}} (\sqrt{N_\varphi}\omega) \wedge^* (\overline{\sqrt{N_\varphi}\omega}) = \langle \sqrt{N_\varphi}\omega, \sqrt{N_\varphi}\omega\rangle = \langle N_\varphi \omega, \omega\rangle. \end{aligned}$$

Since $C^*_\varphi C_\varphi$ is self-adjoint, then $C^*_\varphi C_\varphi(\omega) = N_\varphi \omega$ for all $\omega \in \Lambda^1_2(\mathbb{D})$. Using (2.3), $\|C_\varphi(\omega)\|_{\mathbb{D}} = \|\omega\|_{\mathbb{D}}$ if and only if $N_\varphi = 1$ on $\mathbb{D}$. But $N_\varphi = 1$ if and only if φ is a bijection. The support of $f \in L^0(\mathbb{D})$ is defined by $\sigma(f) = \{x \in \mathbb{D} : f(x) \neq 0\}$. In our case, $\sigma(f dz + g d\bar{z}) = \sigma(f) \cup \sigma(g)$. It is worth nothing that $\sigma(N_\varphi) = \varphi(\mathbb{D})$ and $\sigma(N_\varphi) = \mathbb{D}$ if and only if φ is onto. Let $K = \mathbb{D}\backslash\varphi(\mathbb{D})$. Using (2.3), $C_\varphi(\omega) = 0$ if and only if $\sigma(\omega) \subseteq K$ for all $\omega \in \Lambda^1_2(\mathbb{D})$. It follows that $\mathcal{N}(C_\varphi) \cong \Lambda^1_2(K)$ and hence $\mathcal{R}(C^*_\varphi) = \Lambda^1_2(K)^\perp \cong \Lambda^1_2(\varphi(\mathbb{D}))$. So, C_φ is one-to-one if and only if $A(K) = 0$ and C_φ is partial isometry if and only of $N_\varphi = 1$ on $\varphi(\mathbb{D})$. Note that $\|C_\varphi(f dz)\|^2_{\mathbb{D}} = 2\int_{\mathbb{D}} |f|^2 N_\varphi dA$ for all $f \in L^2_a(\mathbb{D})$. So, C_φ is bounded if and only if N_φ is bounded. In particular, if we take $\varphi(z) = z^n$, then $\|C_\varphi\|^2 = \|C^*_\varphi C_\varphi\| = \|M_n\| = n$ and hence $\|C_\varphi\| = \sqrt{n}$ (see [11, p.26]).

Now we show that the measure $A \circ \varphi^{-1}$ defined by $A \circ \varphi^{-1}(K) = A(\varphi^{-1}(K))$, for all $K \in \mathcal{M}_{\mathbb{D}}$, is absolutely continuous with respect to A. For this, let $A(K) = 0$ but $A(\varphi^{-1}(K)) \neq 0$ for some $K \in \mathcal{M}_{\mathbb{D}}$. Since φ is a non-constant analytic self-map on $\mathbb{D}$, so there exists a collection of disjoint open sets $\{V_i\}$ such that $A(\mathbb{D}\backslash \cup V_i) = 0$ and

$\varphi_{|V_i}$ is one-to-one. So, $A(V_i \cap \varphi^{-1}(K)) \neq 0$, for some $i \in \mathbb{N}$. Set $F = V_i \cap \varphi^{-1}(K)$. Then $F = \varphi^{-1}(F')$ for some $F' \subseteq K$. Note that $\varphi_{|F}$ is one-to-one and so $|\varphi'| > 0$ on F. Thus, $\int_{V_i} \chi_F |\varphi'|^2 dA = \int_F |\varphi'|^2 dA > 0$. Since $\varphi(F) \subseteq F' \subseteq K$ and $A(K) = 0$, then $A(\varphi(V_i) \cap \varphi(F)) \leq A(K) = 0$. Then by the area formula we obtain that

$$0 < \int_{V_i \cap \varphi^{-1}(F')} |\varphi'|^2 dA = \int_{V_i} \chi_F |\varphi'|^2 dA$$
$$= \int_{\varphi(V_i)} \chi_F \circ \varphi^{-1} dA = \int_{\varphi(V_i) \cap \varphi(F)} dA = 0.$$

But this is a contradiction. These observations establish the following result.

Proposition 2.1 *Let $C_\varphi : \Lambda^1_2(\mathbb{D}) \to \Lambda^1_2(\mathbb{D})$ be the pullback transform induced by a non-constant analytic self-map φ on the unit disc $\mathbb{D}$. Then the following statements hold:*

(a) [11, Theorem 2.1] *C_φ is bounded if and only if $N_\varphi \in L^\infty(\mathbb{D})$, and in this case $\|C_\varphi\|^2_{\mathbb{D}} \leq \|N_\varphi\|_\infty$.*

(b) [11, Corollary 2.1] *$C^*_\varphi C_\varphi = M_{N_\varphi}$, the multiplication operator induced by N_φ.*

(c) *$\mathcal{N}(C_\varphi) \cong \Lambda^1_2(K)$, where $K = \mathbb{D} \backslash \varphi(\mathbb{D})$.*

(d) *C_φ is an isometry if and only if $N_\varphi = 1$ on $\mathbb{D}$.*

(e) *C_φ is a partial isometry if and only if $N_\varphi = 1$ on $\varphi(\mathbb{D})$.*

(f) *$A \circ \varphi^{-1}$ is absolutely continuous with respect to A on $\mathbb{D}$.*

Let M and N be Riemann surfaces. A continuous map $\varphi : M \to N$ is said to be analytic if for any chart α on M and for any chart β on N with $\varphi(U_\alpha) \subset U_\beta$, the function $\varphi_{\alpha\beta} = \beta \circ \varphi \circ \alpha^{-1} : \alpha(U_\alpha) \to \beta(U_\beta)$ is analytic. Throughout the paper $\varphi : M \to N$ will be an analytic map, $\mathcal{A} = \{(\alpha, U_\alpha)\}$, $\mathcal{B} = \{(\beta, U_\beta)\}$, $M = \cup_\alpha U_\alpha$ and $N = \cup_\beta U_\beta$. Let $\mathcal{M}_{\beta(U_\beta)}$ be the Lebesque σ-algebra in $\beta(U_\beta)$, f^β, g^β, $F^\beta \in L^0(U_\beta)$, $\omega \in \Lambda^1(N)$, $\Omega \in \Lambda^2(N)$, $\omega^\beta = \omega_{|U_\beta} = f^\beta d\beta + g^\beta d\bar{\beta}$, $\Omega^\beta = \Omega_{|U_\beta} = F^\beta d\beta \wedge d\bar{\beta}$. Take

$$L^p(\beta(U_\beta)) = L^p(\beta(U_\beta), \mathcal{M}_{\beta(U_\beta)}, A_{|\mathcal{M}_{\beta(U_\beta)}});$$
$$f^\beta_\beta = f^\beta \circ \beta^{-1} \in L^0(\beta(U_\beta));$$
$${}^*\omega^\beta = -i f^\beta d\beta + i g^\beta d\bar{\beta}$$

and $\varphi^*(\omega^\beta) = (f^\beta \circ \varphi) d\beta \circ \varphi + (g^\beta \circ \varphi) d\bar{\beta} \circ \varphi$. Then

$$\omega^\beta \wedge^* \overline{\omega^\beta} = i(|f^\beta|^2 + |g^\beta|^2) d\beta \wedge d\bar{\beta};$$
$$\omega^\beta_\beta = (f^\beta \circ \beta^{-1}) d\beta \circ \beta^{-1} + (g^\beta \circ \beta^{-1}) d\bar{\beta} \circ \beta^{-1} = f^\beta_\beta dz + g^\beta_\beta d\bar{z};$$
$$\Omega^\beta_\beta = (F^\beta \circ \beta^{-1}) d(\beta \wedge d\bar{\beta}) \circ \beta^{-1} = F^\beta_\beta dz \wedge d\bar{z};$$
$$\int_{U_\beta} \Omega^\beta = \int_{\beta(U_\beta)} \Omega^\beta_\beta = \int_{\beta(U_\beta)} F^\beta_\beta dz \wedge d\bar{z} = \int_{\beta(U_\beta)} -2i F^\beta_\beta dA.$$

A triangle on M is a Jordan domain together with a homeomorphism onto a triangle in $\mathbb{C}$. A two dimensional manifold is called triangulable if there are countable triangles $\{\triangle_\alpha\}$ on M such that $\cup\triangle_\alpha = M$, for $\alpha \neq \beta$, $\text{int}\triangle_\alpha \cap \text{int}\triangle_\beta = \emptyset$ and for each $p \in M$ there is a neighborhoods V of p such that that set $\{\alpha : \triangle_\alpha \cap V \neq \emptyset\}$ is finite. By subdividing a triangulation it is always possible to have each triangle contained in the domain of a chart [11, p. 24]. It is known that a connected surface is triangulable if and only if it admits a countable base. In particular, every Riemann surface is triangulable (see [13]). Since there might exist several charts containing a given triangle, using the axiom of choice, we pick one of them and then we restrict it to the interior of the triangle. So, for $\triangle_\alpha \subset U'_\alpha$, $U_\alpha := \text{int}\triangle_\alpha \cap U'_\alpha$ is the restriction of U'_α to the interior of $\triangle_\alpha$. For brevity, we consider the following standing assumption.

$\triangle$-property: We say that triangulations $\{\triangle_\alpha\}$ of M and $\{\triangle_\beta\}$ of N have Δ-property if each triangle $\triangle_\alpha$ is contained in the domain of some chart on M and each triangle $\triangle_\beta$ is contained in the domain of some chart on N and for every α, there is a β such that $\varphi(\triangle_\alpha) \subseteq \triangle_\beta$, $U_\alpha = \text{int}\triangle_\alpha$, $U_\beta = \text{int}\triangle_\beta$ and so $\{\alpha : \alpha \in \mathcal{A}\} = \{(\alpha, \beta) : \alpha \in \mathcal{A}, \varphi(\triangle_\alpha) \subseteq \triangle_\beta\}$.

The space $\Lambda^1_2(N)$ of measurable 1-forms on the Riemann surface N defined as $\Lambda^1_2(N) = \{\omega \in \Lambda^1(N) : \omega^\beta = f^\beta d\beta + g^\beta d\bar{\beta},\ f^\beta_\beta, g^\beta_\beta \in L^2(\beta(U_\beta)),\ \text{for all}\ \beta \in \mathcal{B}\}$. Consider triangulation $\{\triangle_\beta\}_{\beta\in\mathcal{B}}$ of N with Δ-property. Let $\omega_1, \omega_2 \in \Lambda^1_2(N)$ and $\omega^\beta_i = f^\beta_i d\beta + g^\beta_i d\bar{\beta}$, for $\beta \in \mathcal{B}$. Set $\langle\omega_1, \omega_2\rangle_N = \int_N \omega_1 \wedge^* \overline{\omega_2}$. Then we have

$$\begin{aligned}
\langle\omega_1, \omega_2\rangle_N &= \int_{\cup\triangle_\beta} \omega_1 \wedge^* \overline{\omega_2} = \sum_{\beta\in\mathcal{B}} \int_{\triangle_\beta} \omega^\beta_1 \wedge^* \overline{\omega^\beta_2} \\
&= \sum_{\beta\in\mathcal{B}} \int_{\triangle_\beta} (f^\beta_1\, \overline{f^\beta_2} + g^\beta_1\, \overline{g^\beta_2}) i d\beta \wedge d\bar{\beta} \\
&= \sum_{\beta\in\mathcal{B}} \int_{\beta(\triangle_\beta)} \left\{(f_1)^\beta_\beta\, \overline{(f_2)^\beta_\beta} + (g_1)^\beta_\beta\, \overline{(g_2)^\beta_\beta}\right\} i dz \wedge d\bar{z} \\
&= \sum_{\beta\in\mathcal{B}} 2 \int_{\beta(\triangle_\beta)} \left\{(f_1)^\beta_\beta\, \overline{(f_2)^\beta_\beta} + (g_1)^\beta_\beta\, \overline{(g_2)^\beta_\beta}\right\} dA.
\end{aligned}$$

The space $\Lambda^1_2(N)$ which satisfy the following

$$\|\omega\|^2_N = \int_N \omega \wedge^* \overline{\omega} = \sum_{\beta\in\mathcal{B}} 2 \int_{\beta(\triangle_\beta)} \left\{|f^\beta_\beta|^2 + |g^\beta_\beta|^2\right\} dA < \infty$$

is a Hilbert space with inner product $\langle\omega_1, \omega_2\rangle = \int_N \omega_1 \wedge^* \overline{\omega_2} = \sum_\beta \langle\omega^\beta_1, \omega^\beta_2\rangle_{\triangle_\beta}$ (see [6]). Since $\{(f_i)^\beta_\beta, (g_i)^\beta_\beta\} \subset L^2(\beta(U_\beta))$, then $\langle\omega^\beta_1, \omega^\beta_2\rangle_{\triangle_\beta} < \infty$ for all $\beta \in \mathcal{B}$. So, if $\mathcal{B}$ is finite, then $\|\omega\|_N < \infty$ for all $\omega \in \Lambda^1_2(N)$. So we have the following result.

Proposition 2.2 *Let $\{\triangle_\beta\}_{\beta\in\mathcal{B}}$ be a triangulation of N with $\triangle$-property. Then $\Lambda^1_2(N) \cong \bigoplus_\beta \Lambda^1_2(\beta(\triangle_\beta))$, $\|\omega\|^2_N = \sum_\beta \|\omega^\beta\|^2_{\triangle_\beta} = \sum_\beta \|\omega^\beta_\beta\|^2_{\beta(\triangle_\beta)}$ and*

$$\|\omega^\beta_\beta\|^2_{\beta(\triangle_\beta)} = \int_{\beta(\triangle_\beta)} 2(|f^\beta_\beta|^2 + |g^\beta_\beta|^2)dA,$$

for all $\omega \in \Lambda^1_2(N)$.

Now, let $\omega \in \Lambda^1_2(N)$. Using $\triangle$-property, we take $(\omega \circ \varphi)_\alpha = \omega^\beta \circ \varphi \circ \alpha^{-1}$. Since

$$\begin{aligned}
(f^\beta \circ \varphi)_\alpha &= f^\beta \circ \varphi \circ \alpha^{-1} = (f^\beta \circ \beta^{-1}) \circ (\beta \circ \varphi \circ \alpha^{-1}) = f^\beta_\beta \circ \varphi_{\alpha\beta};\\
(g^\beta \circ \varphi)_\alpha &= g^\beta \circ \varphi \circ \alpha^{-1} = g^\beta_\beta \circ \varphi_{\alpha\beta};\\
d(\beta \circ \varphi)_\alpha &= d(\beta \circ \varphi \circ \alpha^{-1}) = d\varphi_{\alpha\beta} = \varphi'_{\alpha\beta}dz;\\
d(\bar\beta \circ \varphi)_\alpha &= \overline{d(\beta \circ \varphi)_\alpha} = \overline{\varphi'_{\alpha\beta}dz} = \overline{\varphi'_{\alpha\beta}}d\bar z,
\end{aligned}$$

then

$$\begin{aligned}
[C_\varphi(\omega)]^\alpha_\alpha = [\omega \circ \varphi]^\alpha_\alpha = (\omega^\beta \circ \varphi)_\alpha &= [(f^\beta d\beta) \circ \varphi + (g^\beta d\bar\beta) \circ \varphi]_\alpha\\
&= (f^\beta \circ \varphi)_\alpha d(\beta \circ \varphi)_\alpha + (g^\beta \circ \varphi)_\alpha d(\bar\beta \circ \varphi)_\alpha\\
&= (f^\beta_\beta \circ \varphi_{\alpha\beta})\varphi'_{\alpha\beta}dz + (g^\beta_\beta \circ \varphi_{\alpha\beta})\overline{\varphi'_{\alpha\beta}}d\bar z.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|C_\varphi(\omega)\|^2_M &= \int_{\cup\triangle_\alpha} C_\varphi(\omega) \wedge^* \overline{C_\varphi(\omega)} = \sum_{\alpha\in\mathcal{A}} \int_{\triangle_\alpha} (\omega \circ \varphi)^\alpha \wedge^* \overline{(\omega \circ \varphi)^\alpha}\\
&= \sum_{\alpha\in\mathcal{A}} \int_{\alpha(\triangle_\alpha)} [\omega \circ \varphi]^\alpha_\alpha \wedge^* \overline{[\omega \circ \varphi]^\alpha_\alpha}\\
&= \sum_{\{(\alpha,\beta):\alpha\in\mathcal{A},\ \varphi(\triangle_\alpha)\subseteq\triangle_\beta\}} 2\int_{\alpha(\triangle_\alpha)} \left\{|f^\beta_\beta \circ \varphi_{\alpha\beta}|^2 + |g^\beta_\beta \circ \varphi_{\alpha\beta}|^2\right\} |\varphi'_{\alpha\beta}|^2 dA\\
&= \sum_{\beta\in\mathcal{B}} \sum_{\alpha\in\mathcal{A}_\beta} 2\int_{\alpha(\triangle_\alpha)} \left\{(|f^\beta_\beta|^2 + |g^\beta_\beta|^2) \circ \varphi_{\alpha\beta}\right\} |\varphi'_{\alpha\beta}|^2 dA,
\end{aligned}$$

where $\mathcal{A}_\beta = \{\alpha \in \mathcal{A} : \varphi(\triangle_\alpha) \subseteq \triangle_\beta\}$.

We now introduce conditional expectations as another application of the Radon–Nikodym theorem. Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{C} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{C}, \mu_{|\mathcal{C}})$ is abbreviated to $L^2(\mathcal{C})$ where $\mu_{|\mathcal{C}}$ is the restriction of μ to $\mathcal{C}$. For each non-negative $f \in L^0(\Sigma)$, the linear space of all complex-valued Σ-measurable functions on X, or $f \in L^2(\Sigma)$, by the Radon–Nikodym theorem, there exists a unique $\mathcal{C}$-measurable function $E^{\mathcal{C}}(f) =$

$E(f \mid \mathcal{C})$ such that $\int_A f d\mu = \int_A E^{\mathcal{C}}(f) d\mu$, where A is any $\mathcal{C}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{C} \subseteq \Sigma$, the mapping $E^{\mathcal{C}} : L^2(\Sigma) \to L^2(\mathcal{C})$ uniquely defined by the assignment $f \mapsto E^{\mathcal{C}}(f)$, is called the conditional expectation operator with respect to $\mathcal{C}$. The mapping $E^{\mathcal{C}}$ is a linear orthogonal projection onto $L^2(\mathcal{C})$. Note that $\mathcal{D}(E^{\mathcal{C}})$, the domain of $E^{\mathcal{C}}$, contains $\cup_{p\geq 1} L^p(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on the properties of $E^{\mathcal{C}}$ see [7, 12]. Conditional expectation operator plays a crucial role in our considerations. Those properties of $E^{\mathcal{C}}$ used in our discussion are summarized below. In all cases we assume that $f, g, fg \in \mathcal{D}(E^{\mathcal{C}})$ and $p \geq 1$.

- If g is $\mathcal{C}$-measurable then $E^{\mathcal{C}}(fg) = E^{\mathcal{C}}(f)g$.
- $|E^{\mathcal{C}}(f)|^p \leq E^{\mathcal{C}}(|f|^p)$.
- If $f \geq 0$ then $E^{\mathcal{C}}(f) \geq 0$.
- $|E^{\mathcal{C}}(fg)|^2 \leq (E^{\mathcal{C}}(|f|^2))(E^{\mathcal{C}}(|g|^2))$.

Let G_1 and G_2 be an open and connected sets in $\mathbb{C}$ and let $\varphi : G_1 \to G_2$ be a non-constant analytic function. Still proceeding as in the proof of Proposition 2.1(f), one establishes that $A \circ \varphi^{-1}$ is absolutely continuous with respect to A, i.e., $A(\varphi^{-1}(K)) = 0$ for all $K \in \mathcal{M}_{G_2}$ with $A(K) = 0$. Let $h_\varphi = A \circ \varphi^{-1}/dA$ be the Radon–Nikodym derivative. Consider the σ-finite algebra $\mathcal{C}(\varphi) = \varphi^{-1}(\mathcal{M}_{G_2})$ of G_1 and take $E^{\mathcal{C}(\varphi)} = E(. \mid \mathcal{C}(\varphi)) = E_\varphi$. It is known that for each non-negative G_1-measurable function f or for each $f \in L^2(G_1)$, there exists a G_2-measurable function g such that $E_\varphi(f) = g \circ \varphi$. Moreover, g is uniquely determined in $\sigma(h_\varphi)$, the support of h_φ. Therefore, even though φ is not invertible, the expression $g = E_\varphi(f) \circ \varphi^{-1}$ is well defined, whenever $\sigma(g) \subseteq \sigma(h_\varphi)$ (see [1]). Recall that for $0 \leq f \in L^0(G_2)$ and $0 \leq W \in L^0(G_1)$ we have

$$\int_{G_1} W(z) f(\varphi(z)) |\varphi'(z)|^2 dA(z) = \int_{\varphi(G_1)} \left\{ \sum_{z \in c(w,\varphi)} W(z) \right\} f(w) dA(w).$$

Set $G_0 = \{z \in G_1 : \varphi'(z) \neq 0\}$ Then G_0 is countable and so $G_0 = G_1$ a.e. $[A]$. For $0 \leq g \in L^0(G_1)$, put $W(z) = \chi_{G_0} g(z) |\varphi'(z)|^{-2}$. Then we have that

$$\int_{G_1} g(z) f(\varphi(z)) dA(z) = \int_{\varphi(G_1)} \left\{ \sum_{z \in c(w,\varphi) \cap G_0} \frac{g(z)}{|\varphi'(z)|^2} \right\} f(w) dA(w). \quad (2.4)$$

On the other hand, by the change of variable formula in the measure theory setting, we have ([8])

$$\begin{aligned} \int_{G_1} g(f \circ \varphi) dA = \int_{G_1} E_\varphi(g)(f \circ \varphi) dA &= \int_{\varphi(G_1)} \{E_\varphi(g) \circ \varphi^{-1}\} f dA \circ \varphi^{-1} \\ &= \int_{\varphi(G_1)} \left\{ h_\varphi E_\varphi(g) \circ \varphi^{-1} \right\} f dA. \end{aligned}$$

Now, for each $A \in \mathcal{M}_{G_1}$, take $f = \chi_A$ and set $J_\varphi[g] = h_\varphi E_\varphi(g) \circ \varphi^{-1}$. Using (2.4) we get that

$$\int_A \left\{ J_\varphi[g](w) - \sum_{z \in c(w,\varphi) \cap G_0} \frac{g(z)}{|\varphi'(z)|^2} \right\} \chi_{\varphi(G_1)} dA(w) = 0.$$

It follows that

$$J_\varphi[g](w) = \sum_{z \in c(w,\varphi) \cap G_0} \frac{g(z)}{|\varphi'(z)|^2}, \ (w \in \varphi(G_1)).$$

If $N_\varphi(.)$ is bounded on $\varphi(G_1)$, then $J_\varphi[g]$ is finite-valued. Note that $J_\varphi[1] = h$ and $c(w, \varphi) = \emptyset$ for $w \in G_2 \backslash \varphi(G_1)$. Also, if $B \subseteq G_2 \backslash \varphi(G_1)$ is in $\mathcal{M}_{G_2}$, then $\varphi^{-1}(B) \cap G_1 = \emptyset$ and hence $\int_B h dA = \int_B dA \circ \varphi^{-1} = A(\varphi^{-1}(B) \cap G_1) = 0$. Thus, $\sigma(J_\varphi[g]) \subseteq \sigma(h) \subseteq \varphi^{-1}(G_1)$. These observations establish the following result.

Theorem 2.3 *Let G_1 and G_2 be an open and connected sets in $\mathbb{C}$, $\varphi : G_1 \to G_2$ be a non-constant analytic function and let $G_0 = \{z \in G_1 : \varphi'(z) \neq 0\}$. Then for each $0 \leq g \in L^0(G_1)$ we have*

$$J_\varphi[g](w) = \begin{cases} \sum\limits_{z \in c(w,\varphi) \cap G_0} \frac{g(z)}{|\varphi'(z)|^2} & w \in \varphi(G_1) \\ 0 & w \notin G_2 \backslash \varphi(G_1), \end{cases}$$

where $J_\varphi[g] = h_\varphi E_\varphi(g) \circ \varphi^{-1}$. In particular, $J_\varphi[|\varphi'|^2] = N_\varphi(\chi_{G_0})$ and

$$h(w) = \begin{cases} \sum\limits_{z \in c(w,\varphi) \cap G_0} \frac{1}{|\varphi'(z)|^2} & w \in \varphi(G_1) \\ 0 & w \notin G_2 \backslash \varphi(G_1). \end{cases}$$

Let $E_\varphi(L^2_a(\mathbb{D})) \subseteq L^2_a(\mathbb{D})$. Then by [2, Theorem 2], non-negativity of $f \in L^2_a(\mathbb{D})$ is not required as mentioned in Theorem 2.3 for $J_\varphi[f]$.

Example 2.4 Let $G_1 = \mathbb{D}$, $\{\alpha, \beta, \gamma\} \subset \mathbb{R}$, $\varphi(z) = \alpha z^2 + \beta z + \gamma$ and let $G_2 = \varphi(\mathbb{D})$. Then for each $w \in \varphi(\mathbb{D})$, $c(w^n, \varphi) = \{w, -\frac{\beta + \alpha w}{\alpha}\}$ and $c(w, \varphi) = \{w_1, w_2\} = \{\frac{-\beta - \sqrt{\beta^2 - 4\alpha(\gamma - w)}}{2\alpha}, \frac{-\beta + \sqrt{\beta^2 - 4\alpha(\gamma - w)}}{2\alpha}\}$. Then by [2, Theorem 2] and Theorem 2.3 we obtain

$$E_\varphi(f)(w) = \frac{1}{2} f(w) + \frac{1}{2} f\left(-\frac{\beta + \alpha w}{\alpha}\right);$$
$$h(w) = \frac{2}{|\beta^2 - 4\alpha(\gamma - w)|^2}$$

and $(E_\varphi(f)\circ\varphi^{-1})(w)=\frac{1}{2}\{f(w_1)+f(w_2)\}$. Consequently,

$$J_\varphi[f](w)=\frac{1}{|\beta^2-4\alpha(\gamma-w)|^2}\{f(w_1)+f(w_2)\},\quad f\in L_a^2(\mathbb{D}),\ w\in\varphi(\mathbb{D}).$$

Boundedness of pullback transforms on between differential form spaces for Riemann surfaces has been characterised in [11, Theorem 2.2]. In [10, Theorem 2.2], we studied bounded operators of the form $\omega\mapsto u(\omega\circ\varphi)$ for $\omega\in\Lambda_2^1(N)$. In the following, the boundedness of weighted pullback transforms acting between two different measurable differential form spaces are characterized using some properties of conditional expectation operators.

Theorem 2.5 *Let M and N be Riemann surfaces, $u\in\Lambda^0(M)$ and let $\varphi:M\to N$ be an analytic map with $\triangle$-property. Then the weighted pullback transform $uC_\varphi:\Lambda_2^1(N)\to\Lambda_2^1(M)$ is bounded if and only if $N_\varphi(|u^2|)$ is essentially bounded. In this case $\|uC_\varphi\|_M^2\le\|N_\varphi(|u|^2)\|_\infty$.*

Proof Let $\omega\in\Lambda_2^1(N)$. Then for each $\alpha\in\mathcal{A}$ we have

$$\begin{aligned}[uC_\varphi(\omega)]_\alpha^\alpha&=(u^\alpha(\omega^\beta\circ\varphi))_\alpha=u_\alpha^\alpha(\omega^\beta\circ\varphi)_\alpha\\&=u_\alpha^\alpha(f_\beta^\beta\circ\varphi_{\alpha\beta})\varphi'_{\alpha\beta}dz+u_\alpha^\alpha(g_\beta^\beta\circ\varphi_{\alpha\beta})\overline{\varphi'_{\alpha\beta}}d\bar{z}\end{aligned}$$

Let $\mathcal{A}_\beta=\{\alpha\in\mathcal{A}:\varphi(\triangle_\alpha)\subseteq\triangle_\beta\}$. Then by the change of variable formula we have

$$\begin{aligned}\|uC_\varphi(\omega)\|_M^2&=\sum_{\alpha\in\mathcal{A}}\int_{\alpha(\triangle_\alpha)}[u(\omega\circ\varphi)]_\alpha^\alpha\wedge^*\overline{[u(\omega\circ\varphi)]_\alpha^\alpha}\\&=2\sum_{\beta\in\mathcal{B}}\sum_{\alpha\in\mathcal{A}_\beta}\int_{\alpha(\triangle_\alpha)}\left\{|f_\beta^\beta|^2+|g_\beta^\beta|^2\right\}(\varphi_{\alpha\beta}(z))|u_\alpha^\alpha(z)\varphi'_{\alpha\beta}(z)|^2dA(z)\\&=2\sum_{\beta\in\mathcal{B}}\sum_{\alpha\in\mathcal{A}_\beta}\int_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))}(|f_\beta^\beta(w)|^2+|g_\beta^\beta(w)|^2)J_{\alpha\beta}[|u_\alpha^\alpha\varphi'_{\alpha\beta}|^2](w)dA(w)\\&=2\sum_{\beta\in\mathcal{B}}\sum_{\alpha\in\mathcal{A}_\beta}\int_{\beta(\triangle_\beta)}(|f_\beta^\beta(w)|^2+|g_\beta^\beta(w)|^2)\chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))}(w)\\&\quad\times J_{\alpha\beta}[|u_\alpha^\alpha\varphi'_{\alpha\beta}|^2](w)dA(w),\end{aligned}$$

where

$$\begin{aligned}J_{\alpha\beta}[|u_\alpha^\alpha\varphi'_{\alpha\beta}|^2](w)&=h_{\alpha\beta}(w)\left\{E_{\alpha\beta}(|u_\alpha^\alpha\varphi'_{\alpha\beta}|^2)\circ\varphi_{\alpha\beta}^{-1}\right\}(w);\\E_{\alpha\beta}&=E(.\mid\varphi_{\alpha\beta}^{-1}(\mathcal{M}_{\beta(\triangle_\beta)}));\\h_{\alpha\beta}&=\frac{dA\circ\varphi_{\alpha\beta}^{-1}}{dA}.\end{aligned}$$

Put $c(w, \varphi_{\alpha\beta}) = \{z \in \alpha(\triangle_\alpha) : \varphi'_{\alpha\beta}(z) \neq 0, \varphi_{\alpha\beta}(z) = w\}$. Using Theorem 2.3, we have

$$J_{\alpha\beta}[|u^\alpha_\alpha \, \varphi'_{\alpha\beta}|^2](w) = \sum_{z \in c(w,\varphi_{\alpha\beta})} \frac{|u^\alpha_\alpha(z)|^2 |\varphi'_{\alpha\beta}(z)|^2}{|\varphi'_{\alpha\beta}(z)|^2} = \sum_{z \in c(w,\varphi_{\alpha\beta})} |u^\alpha_\alpha(z)|^2$$
$$= N_{\varphi_{\alpha\beta}}(|u^\alpha_\alpha|^2)(w) = \chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} N_{\varphi_{\alpha\beta}}(|u^\alpha_\alpha|^2)(w).$$

Thus

$$\|uC_\varphi(\omega)\|^2_M = 2 \sum_{\beta \in \mathcal{B}} \int_{\beta(\triangle_\beta)} (|f^\beta_\beta(w)|^2 + |g^\beta_\beta(w)|^2) \left\{ \sum_{\alpha \in \mathcal{A}_\beta} N_{\varphi_{\alpha\beta}}(|u^\alpha_\alpha|^2)(w) \right\} dA(w)$$

in which

$$\sum_{\alpha \in \mathcal{A}_\beta} N_{\varphi_{\alpha\beta}}(|u^\alpha_\alpha|^2)(w) = \sum_{\alpha \in \mathcal{A}_\beta} \left\{ \sum |u^\alpha_\alpha(z)|^2 : z \in \alpha(\triangle_\alpha), \varphi_{\alpha\beta}(z) = w \right\}$$
$$= \sum_{\alpha \in \mathcal{A}_\beta} \left\{ \sum |u^\alpha(\alpha^{-1}(z))|^2 : \alpha^{-1}(z) \in \triangle_\alpha, \varphi(\alpha^{-1}(z)) = \beta^{-1}(w) \right\}$$
$$= \left\{ \sum |u(x)|^2 : x \in M, \varphi(x) = \beta^{-1}(w) \right\} = N_\varphi(|u|^2)(\beta^{-1}(w)) = (N_\varphi(|u|^2))_\beta(w).$$

Consequently,

$$\|uC_\varphi(\omega)\|^2_M = 2 \sum_{\beta \in \mathcal{B}} \int_{\beta(\triangle_\beta)} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2)(N_\varphi(|u|^2))_\beta dA.$$

Now, if $N_\varphi(|u|^2) \in L^\infty(M)$ then

$$\|uC_\varphi(\omega)\|^2_M \leq \|(N_\varphi(|u|^2))_\beta\|_\infty \left\{ \sum_{\beta \in \mathcal{B}} 2 \int_{\beta(\triangle_\beta)} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2) dA \right\}$$
$$\leq \|N_\varphi(|u|^2)\|_\infty \|\omega\|^2_N,$$

and so $\|uC_\varphi\|^2_M \leq \inf_{\beta \in \mathcal{B}} \|(N_\varphi(|u|^2))_\beta\|_\infty \leq \|N_\varphi(|u|^2)\|_\infty$. Conversely, suppose uC_φ is bounded. If $\mathcal{A}$ and $\mathcal{B}$ is finite, then for each $\beta \in \mathcal{B}$, $(N_\varphi(|u|^2))_\beta$ is essentially bounded and hence

$$\|N_\varphi(|u|^2)\|_\infty = \max_{\beta \in \mathcal{B}} \|(N_\varphi(|u|^2))_\beta\|_\infty < \infty.$$

Now, let $\mathcal{A}$ and $\mathcal{B}$ be countably infinite sets. If $N_\varphi(|u|^2) \notin L^\infty(M)$, then there exists $\{\beta_n\} \subset \mathcal{B}$ such that $(N_\varphi(|u|^2))_{\beta_n} \geq 2^n$. For each n, choose $U_n \subseteq \beta_n(\triangle_{\beta_n})$ with

$0 < A(U_n) < \infty$. Let $\omega \in \Lambda_2^1(N)$ be represented by

$$\omega^{\beta_n} = \frac{(\chi_{U_n} \circ \beta_n) d\beta_n}{\sqrt{N_\varphi(|u|^2) A(U_n)}}, \quad n \geq 1.$$

Then

$$\begin{aligned} \|\omega\|_N^2 &= 2 \sum_{n=1}^{\infty} \int_{\beta_n(\triangle_{\beta_n})} \frac{\chi_{U_n} dA}{(N_\varphi(|u|^2))_{\beta_n} A(U_n)} \\ &\leq 2 \sum_{n=1}^{\infty} \frac{1}{2^n A(U_n)} \int_{U_n} dA = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2, \end{aligned}$$

and

$$\|uC_\varphi(\omega)\|_M^2 = 2 \sum_{n=1}^{\infty} \int_{\beta_n(\triangle_{\beta_n})} \frac{\chi_{U_n} (N_\varphi(|u|^2))_{\beta_n} dA}{(N_\varphi(|u|^2))_{\beta_n} A(U_n)} = 2 \sum_{n=1}^{\infty} 1 = \infty.$$

But this is a contradiction. This completes the proof. □

Corollary 2.6 (a) [11, Theorem 2.2] *The pullback transform $C_\varphi : \Lambda_2^1(N) \to \Lambda_2^1(M)$ is bounded if and only if the counting function N_φ is bounded.*
(b) *If $M = N = \mathbb{D}$, then $J_\varphi(|u\varphi'|^2) = h_\varphi E_\varphi(|u\varphi'|^2) \circ \varphi^{-1} = N_\varphi(\chi_{G_0}|u|^2)$.*

Let (β, U_β) be any local chart in N and let Σ_β be the σ-algebra generated by $\{\beta^{-1}(K) \cap U_\beta : K \in \mathcal{M}_\mathbb{C}\}$. Define $\mu_\beta(B) = A(\beta(B))$ for all $B \in \Sigma_\beta$. Thus, $(U_\beta, \Sigma_\beta, \mu_\beta)$ is a non-atomic measure space.

Let $\omega \in \mathcal{N}(uC_\varphi)$ and $N_\varphi(|u|^2) > 0$ on N. Then for all $\beta \in \mathcal{B}$, $(N_\varphi(|u|^2))_\beta > 0$ on $\beta(\triangle_\beta)$ and

$$2 \sum_{\beta \in \mathcal{B}} \int_{\beta(\triangle_\beta) \cap \sigma(\omega^\beta)} (|f_\beta^\beta|^2 + |g_\beta^\beta|^2)(N_\varphi(|u|^2))_\beta dA = \|uC_\varphi(\omega)\|_M^2 = 0.$$

It follows that $\mu_\beta(\beta(\triangle_\beta) \cap \sigma(\omega^\beta)) = 0$, and so $\omega^\beta = 0$ for all $\beta \in \mathcal{B}$. Thus, $\omega = 0$. Now, suppose for some $\beta \in \mathcal{B}$ and $B \in \Sigma_\beta$ with $0 < \mu_\beta(B) = A(\beta(B)) < \infty$, $\chi_B N_\varphi(|u|^2) = 0$. Set $\omega_0 = \chi_B d\beta$ Then $\omega_0 \neq 0$ and $\|uC_\varphi(\omega_0)\|_M = 0$. Using this and Proposition 2.1 we have the following corollary.

Corollary 2.7 *Let $uC_\varphi \in B(\Lambda_2^1(N)\ \Lambda_2^1(M))$. Then the followings hold.*
(a) *Then uC_φ is injective if and only if $N_\varphi(|u|^2) > 0$ on N.*
(b) *uC_φ is an isometry if and only if $N_\varphi(|u|^2) = 1$ on N.*
(c) *uC_φ is a partial isometry if and only if $N_\varphi(|u|^2) = 1$ on $\varphi(N)$.*

Now, we try to give an explicit formula for the adjoint of these type operators by the language of conditional expectation operators.

Let $\varphi : M \to N$ be an analytic map with $\triangle$-property and let $\omega \in \Lambda_2^1(N)$ and $\eta \in \Lambda_2^1(M)$ be represented by $\omega^\beta = f^\beta d\beta + g^\beta d\bar{\beta}$ and $\eta^\alpha = k^\alpha d\alpha + l^\alpha d\bar{\alpha}$, for each $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$. Then $\eta_\alpha^\alpha = k_\alpha^\alpha dz + l_\alpha^\alpha d\bar{z}$ and hence ${}^*\overline{\eta_\alpha^\alpha} = -i\overline{l_\alpha^\alpha} dz + i\overline{k_\alpha^\alpha} d\bar{z}$. Then we have

$$
\begin{aligned}
\langle uC_\varphi(\omega), \eta\rangle_M &= \int_{\cup\triangle_\alpha} uC_\varphi(\omega) \wedge^* \overline{\eta} = \sum_{\alpha\in\mathcal{A}} \int_{\alpha(\triangle_\alpha)} [u(\omega\circ\varphi)]_\alpha^\alpha \wedge^* \overline{\eta_\alpha^\alpha} \\
&= \sum_{\beta\in\mathcal{B}} \sum_{\alpha\in\mathcal{A}_\beta} 2\int_{\alpha(\triangle_\alpha)} \left\{ (u_\alpha^\alpha\, \overline{k_\alpha^\alpha} (f_\beta^\beta \circ \varphi_{\alpha\beta})\varphi'_{\alpha\beta} + u_\alpha^\alpha\, \overline{l_\alpha^\alpha} (g_\beta^\beta \circ \varphi_{\alpha\beta})\overline{\varphi'_{\alpha\beta}} \right\} dA \\
&= \sum_{\beta\in\mathcal{B}} \sum_{\alpha\in\mathcal{A}_\beta} 2\int_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} f_\beta^\beta \left\{ h_{\alpha\beta} E_{\alpha\beta} (\overline{k_\alpha^\alpha}\, u_\alpha^\alpha\, \varphi'_{\alpha\beta}) \circ \varphi_{\alpha\beta}^{-1} \right\} dA \\
&\quad + \sum_{\beta\in\mathcal{B}} \sum_{\alpha\in\mathcal{A}_\beta} 2\int_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} g_\beta^\beta \left\{ h_{\alpha\beta} E_{\alpha\beta} (\overline{l_\alpha^\alpha}\, u_\alpha^\alpha\, \overline{\varphi'_{\alpha\beta}}) \circ \varphi_{\alpha\beta}^{-1} \right\} dA \\
&= 2\sum_{\beta\in\mathcal{B}} \int_{\beta(\triangle_\beta)} f_\beta^\beta \left\{ \sum_{\alpha\in\mathcal{A}_\beta} \chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} \left[h_{\alpha\beta} E_{\alpha\beta} (\overline{k_\alpha^\alpha}\, u_\alpha^\alpha\, \varphi'_{\alpha\beta}) \circ \varphi_{\alpha\beta}^{-1} \right] \right\} dA \\
&\quad + 2\sum_{\beta\in\mathcal{B}} \int_{\beta(\triangle_\beta)} g_\beta^\beta \left\{ \sum_{\alpha\in\mathcal{A}_\beta} \chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} \left[h_{\alpha\beta} E_{\alpha\beta} (\overline{l_\alpha^\alpha}\, u_\alpha^\alpha\, \overline{\varphi'_{\alpha\beta}}) \circ \varphi_{\alpha\beta}^{-1} \right] \right\} dA
\end{aligned}
$$

Take

$$
K^\beta = \left\{ \sum_{\alpha\in\mathcal{A}_\beta} \chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} \left[h_{\alpha\beta} E_{\alpha\beta} (k_\alpha^\alpha\, \overline{u_\alpha^\alpha\, \varphi'_{\alpha\beta}}) \circ \varphi_{\alpha\beta}^{-1} \right] \right\} \circ \beta; \tag{2.5}
$$

$$
L^\beta = \left\{ \sum_{\alpha\in\mathcal{A}_\beta} \chi_{\varphi_{\alpha\beta}(\alpha(\triangle_\alpha))} \left[h_{\alpha\beta} E_{\alpha\beta} (l_\alpha^\alpha\, \overline{u_\alpha^\alpha}\, \varphi'_{\alpha\beta}) \circ \varphi_{\alpha\beta}^{-1} \right] \right\} \circ \beta. \tag{2.6}
$$

Then

$$
\begin{aligned}
\langle \omega, (uC_\varphi)^*(\eta)\rangle_N &= 2\sum_{\beta\in\mathcal{B}} \int_{\beta(\triangle_\beta)} \left\{ f_\beta^\beta \overline{K_\beta^\beta} + g_\beta^\beta \overline{L_\beta^\beta} \right\} dA \\
&= \langle \omega, \sum_{\beta\in\mathcal{B}} (K^\beta d\beta + L^\beta d\bar{\beta})\chi_{\triangle_\beta} \rangle_N.
\end{aligned}
$$

Consequently, $[(uC_\varphi)^*(\eta)]^\beta = K^\beta d\beta + L^\beta d\bar{\beta}$. So we have the following result.

Theorem 2.8 *Let M and N be Riemann surfaces, $u \in \Lambda^0(M)$ and let $\varphi : M \to N$ be an analytic map with $\triangle$-property. If $uC_\varphi : \Lambda_2^1(N) \to \Lambda_2^1(M)$ is bounded, then for each $\eta^\alpha = k^\alpha d\alpha + l^\alpha d\bar{\alpha}$ in (α, U_α), the adjoint of uC_φ is given by the formula $[(uC_\varphi)^*(\eta)]^\beta = K^\beta d\beta + L^\beta d\bar{\beta}$, where K^β and L^β are given as (2.5) and (2.6).*

Corollary 2.9 *Let $M = N = \mathbb{D}$ and $\omega = f dz + g d\bar{z} \in \Lambda^1_2(\mathbb{D})$. Then*

$$(uC_\varphi)^*(\omega) = \left[h_\varphi E_\varphi(\overline{u\varphi'} f) \circ \varphi^{-1}\right] dz + \left[h_\varphi E_\varphi(\overline{u}\varphi' g) \circ \varphi^{-1}\right] d\bar{z}$$

Let $uC_\varphi \in B(\Lambda^1_2(\mathbb{D}))$ and $\omega = f dz + g d\bar{z} \in \Lambda^1_2(\mathbb{D})$. Then by Corollaries 2.6 and 2.9 we get that

$$\begin{aligned}(uC_\varphi)^*(u(\omega \circ \varphi)) &= (uC_\varphi)^*(u(f \circ \varphi)\varphi' dz + u(g \circ \varphi)\varphi' d\bar{z}) \\ &= \left[h_\varphi E_\varphi(|u|^2|\varphi'|^2 f \circ \varphi) \circ \varphi^{-1}\right] dz + \left[h_\varphi E_\varphi(|u|^2|\varphi'|^2 f \circ \varphi) \circ \varphi^{-1}\right] d\bar{z} \\ &= [h_\varphi E_\varphi(|u\varphi'|^2) \circ \varphi^{-1}](f dz + g d\bar{z}) = J_\varphi(|u\varphi'|^2)\omega = N_\varphi(|u|^2)\omega \ \ a.e.[A].\end{aligned}$$

Consequently, $(uC_\varphi)^*(uC_\varphi) = M_{N_\varphi(|u|^2)}$, $|u|^2$ is viewed as a function defined A-almost everywhere on $D_0 = \{z \in \mathbb{D} : \varphi'(z) \neq 0\}$. Also,

$$(uC_\varphi)(uC_\varphi)^*(\omega) = u\varphi'(h_\varphi \circ \varphi)E_\varphi(\overline{u\varphi'} f)dz + u\overline{\varphi'}(h_\varphi \circ \varphi)E_\varphi(\overline{u}\varphi' g)d\bar{z}.$$

Example 2.10 Let $M = N = \mathbb{D}$ and $\varphi(z) = z^n$. Then for $w \in \mathbb{D}$, $c(w, \varphi) = \{e^{\theta_1} w, \ldots, e^{\theta_{n-1}} w, w\}$ and $c(w^n, \varphi) = \{z_1, \ldots, z_n\}$ where $z_k = \sqrt[n]{|w|}$ and $\theta_k = e^{\frac{2k\pi i}{n}}$. Since $A(\mathbb{D} \setminus \mathbb{D}_0) = 0$, then by Theorem 2.3 we get that

$$h(w) = \sum_{k=1}^{n} \frac{1}{|\varphi'(z_k)|^2} = \frac{1}{n|w|^{\frac{2(n-1)}{n}}}$$

and for each $0 \leq f \in L^0(\Sigma)$ we have (also see [2, 9])

$$\begin{aligned}E_\varphi(f)(w) &= \frac{1}{n} \sum_{z \in c(w^n, \varphi)} f(z) = \frac{1}{n} \sum_{k=1}^{n} f(e^{\theta_k} w);\\ (E_\varphi(f) \circ \varphi^{-1})(w) &= \frac{1}{n} \sum_{z \in c(w, \varphi)} f(z) = \frac{1}{n} \sum_{k=1}^{n} f(z_k);\\ J_\varphi[f](w) = h(w)(E_\varphi(f) \circ \varphi^{-1})(w) &= \frac{1}{n^2|w|^{\frac{2(n-1)}{n}}} \sum_{k=1}^{n} f(z_k).\end{aligned}$$

Thus so for $u \in \Lambda^0(M)$, $N_\varphi(|u|^2) = J_\varphi[|u\varphi'|^2](w) = \sum_{k=1}^{n} |u(z_k)|^2$. In particular, if $u(z) = z$ then $J_\varphi[|z\varphi'|^2](w) = n|w|^{\frac{2}{n}}$. Also, if $\overline{u\varphi'} f$ and $\bar{u}\varphi' g$ are non-negative, then by Corollary 2.9 we have

$$\begin{aligned}(uC_\varphi)^*(\omega) &= J_\varphi[\overline{u\varphi'} f](w)dw + J_\varphi[\bar{u}\varphi' g](w)d\bar{w}\\ &= \frac{1}{n|w|^{\frac{n-1}{n}}} \sum_{k=1}^{n} \bar{u}(z_k) \left\{e^{\theta_k} f(z_k)dw + e^{-\theta_k} g(z_k)d\bar{w}\right\}.\end{aligned}$$

Proposition 2.11 *Let $u \in \Lambda^0(M)$, let $\varphi : M \to N$ be an analytic map with $\triangle$-property and $uC_\varphi \in B(\Lambda^1_2(N)\, \Lambda^1_2(M))$. Then*
(a) $\dim \mathcal{N}(uC_\varphi) = 0$ *or* ∞.
(b) $\dim \mathcal{N}((uC_\varphi)^*) = 0$ *or* ∞.

Proof (a) Let $0 \neq \omega \in \mathcal{N}(uC_\varphi)$ be represented by $\omega^\beta = f^\beta d\beta + g^\beta d\bar{\beta}$ in any local chart (β, U_β). Then $\mu_\beta(\sigma(f^\beta) \cup \sigma(g^\beta)) = \mu_\beta(\sigma(\omega^\beta)) = A(\beta(\sigma(\omega^\beta)) > 0$. Choose a sequence $\{K_n\}$ of pairwise disjoint $\mathcal{M}_{\mathbb{C}}$-measurable sets in $\beta(\sigma(\omega^\beta))$ with $0 < A(K_n) < \infty$. Let $\omega^\beta_n = \omega^\beta \chi_{\beta^{-1}(K_n)}$ for $n \in \mathbb{N}$. Then $\omega_n \neq 0$ and for all $n \neq m$,

$$\langle \omega_n, \omega_m \rangle_N = \sum_{\beta \in \mathcal{B}} \langle \omega^\beta_n, \omega^\beta_m \rangle_{\triangle_\beta} = \sum_{\beta \in \mathcal{B}} \int_{\beta(\triangle_\beta)} 2(|f^\beta_\beta|^2 + |g^\beta_\beta|^2) \chi_{E_n \cap E_m} dA = 0$$

and

$$\begin{aligned} \|uC_\varphi(\omega_n)\|^2_M &= \sum_{\beta \in \mathcal{B}} \sum_{\alpha \in \mathcal{A}_\beta} 2 \int_{\varphi^{-1}_{\alpha\beta}(E_n)} \left\{ (|f^\beta_\beta|^2 + |g^\beta_\beta|^2) \circ \varphi_{\alpha\beta} \right\} |u^\alpha_\alpha \varphi'_{\alpha\beta}|^2 dA \\ &\leq \sum_{\beta \in \mathcal{B}} \sum_{\alpha \in \mathcal{A}_\beta} 2 \int_{\alpha(\triangle_\alpha)} \left\{ (|f^\beta_\beta|^2 + |g^\beta_\beta|^2) \circ \varphi_{\alpha\beta} \right\} |u^\alpha_\alpha \varphi'_{\alpha\beta}|^2 dA = \|uC_\varphi(\omega)\|^2_M = 0. \end{aligned}$$

Consequently, $\dim \mathcal{N}(uC_\varphi) = \infty$.

(b) Let $0 \neq \eta \in \mathcal{N}((uC_\varphi)^*)$ be represented by $\eta^\alpha = k^\alpha d\alpha + l^\alpha d\bar{\alpha}$ in any local chart (α, U_α). Then by Theorem 2.8 we have

$$\langle \omega, (uC_\varphi)^*(\eta) \rangle_N = 2 \sum_{\beta \in \mathcal{B}} \int_{\beta(\triangle_\beta)} \left\{ f^\beta_\beta \overline{K^\beta_\beta} + g^\beta_\beta \overline{L^\beta_\beta} \right\} dA = 0$$

for all $\omega \in \Lambda^1_2(N)$. Put $p^\alpha_\alpha = \max\{|k^\alpha_\alpha|^2, |l^\alpha_\alpha|^2\}$. Then

$$\int_{\alpha(\triangle_\alpha)} E_{\alpha\beta}(p^\alpha_\alpha) dA = \int_{\alpha(\triangle_\alpha)} p^\alpha_\alpha dA > 0.$$

So for some $\delta > 0$, $\varphi^{-1}_{\alpha\beta}(\mathcal{M}_{\beta(\triangle_\beta)})$-measurable set $F = \{z \in \alpha(\triangle_\alpha) : E_{\alpha\beta}(p^\alpha_\alpha)(z) \geq \delta\}$ has positive measure. There is $\mathcal{M}_{\beta(\triangle_\beta)}$-measurable set $G \subseteq \beta(\triangle_\beta)$ such that $F = \varphi^{-1}_{\alpha\beta}(G)$. It follows that there exists a sequence $\{G_n\} \subseteq \mathcal{M}_{\beta(\triangle_\beta)}$ of pairwise disjoint sets in G such that $0 < A(\varphi^{-1}_{\alpha\beta}(G_n)) < \infty$. Take $\eta_n = \eta \chi_{\alpha^{-1}(\varphi^{-1}_{\alpha\beta}(G_n))}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} \|\eta_n\|^2_M &= 2 \sum_{\alpha \in \mathcal{A}} \int_{\alpha(\triangle_\alpha)} \left(|l^\alpha_\alpha|^2 + |k^\alpha_\alpha|^2 \right) \chi_{\varphi^{-1}_{\alpha\beta}(G_n)} dA \geq 2 \sum_{\alpha \in \mathcal{A}} \int_{\varphi^{-1}_{\alpha\beta}(G_n)} p^\alpha_\alpha dA \\ &= 2 \sum_{\alpha \in \mathcal{A}} \int_{\varphi^{-1}_{\alpha\beta}(G_n)} E_{\alpha\beta}(p^\alpha_\alpha) dA \geq 2\delta A(\varphi^{-1}_{\alpha\beta}(G_n)) > 0, \end{aligned}$$

$\langle \eta_n, \eta_m \rangle_M = 0$ for all $n \neq m$ and $\|(uC_\varphi)^*(\eta_n)\|_N^2 \leq \|(uC_\varphi)^*(\eta)\|_N^2 = 0.$ □

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Declarations

Competing interests The authors declare no competing interests.

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