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Substitution conditional type operators on $L^2(\Sigma)$

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Abstract
In this paper point spectrum, compactness, reducibility, generalized inverse and some weak normal classes of substitution conditional type operators acting on $L^2(\Sigma)$ will be investigated.

Keywords Conditional expectation · Reducibility · Generalized inverse · Compact · Normal · Composition operator

Mathematics Subject Classification Primary 47B20; Secondary 47B25

1 Introduction and preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a sub-σ-finite algebra of Σ. We use the notation $L^2(\mathcal{A})$ for $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ and henceforth we write μ in place of $\mu_{|\mathcal{A}}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. We say that S is a localizing set for $\mathcal{A}$ (see [13]) if S is not a null set, and $\mathcal{A}_S = \Sigma_S$, where $\mathcal{A}_S = \{A \cap S : A \in \mathcal{A}\}$. Let $E_{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ be the conditional expectation operator, so that for $f \in L^2(\Sigma)$, $E_{\mathcal{A}}(f)$ is the unique $\mathcal{A}$-measurable function such that $\int_A f d\mu = \int_A E_{\mathcal{A}}(f) d\mu$ for all $A \in \mathcal{A}$. As an operator on $L^2(\Sigma)$, $E = E_{\mathcal{A}}$ is an orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on the properties of E see ([8, 13, 15]). Those properties of E used in our discussion are summarized below. In all cases, we assume that $f, g, fg \in \mathcal{D}(E)$.

- $\chi_A f = f$ whenever $\sigma(f) \subseteq A \in \Sigma$.

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- $E(g) = g$ if and only if g is $\mathcal{A}$-measurable.
- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$.
- If $f \geq g$ then $E(f) \geq E(g)$ and $\sigma(f-g) \subseteq \sigma(E(f-g))$.
- $|E(fg)|^2 \leq E(|f|^2)E(|g|^2)$ (conditonal Cauchy-Schwarz inequality).

For $w, u \in \mathcal{D}(E)$, the mapping $T_1 : L^2(\Sigma) \supseteq \mathcal{D}(T_1) \to L^2(\Sigma)$ given by $T_1(f) = wE(uf)$ for $f \in \mathcal{D}(T_1) = \{f \in L^2(\Sigma) : T_1(f) \in L^2(\Sigma)\}$, is well-defined and linear. Such an operator is called a Lambert conditional type operator induced by the pair (w, u). Let $\varphi : X \to X$ be a measurable transformation on X, that is, $\varphi^{-1}(A) \in \Sigma$ for all $A \in \Sigma$. Define the measure $\mu \circ \varphi^{-1}$ on Σ by $(\mu \circ \varphi^{-1})(A) = \mu(\varphi^{-1}(A))$ for all $A \in \Sigma$. We say that φ is nonsingular, if $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. In this case we write $\mu \circ \varphi^{-1} \ll \mu$, as usual. Let h be the Radon-Nikodym derivative $h = d(\mu \circ \varphi^{-1})/d\mu$ and it is always assumed that h is almost everywhere finite-valued. Equivalently, $\varphi^{-1}(\Sigma)$ is a sub-σ-finite algebra of Σ. By the change of variables formula we observe that for each $f \in L^1(\Sigma)$ and $A \in \Sigma$ we have

$$\int_A f d(\mu \circ \varphi^{-1}) = \int_{\varphi^{-1}(A)} (f \circ \varphi) d\mu = \int_A h f d\mu.$$

Let u and w be in $\mathcal{D}(E)$, the domain of E. The operator $T : L^2(\Sigma) \to L^0(\Sigma)$ that induced by the triple (u, w, φ) is called substitution conditional type operator and defined by $T = M_w E M_u C_\varphi$, where M_w and M_u are multiplication operators and C_φ is a composition operator. If we take $\varphi = id$, the identity map, then $C_\varphi = I$ and T is a Lambert conditional type operator $T_1 = M_w E M_u$. Also, if $\mathcal{A} = \Sigma$, then $E = I$ and T is a weighted composition operator $T_2 = M_{uw} C_\varphi$.

Let $1 < p < \infty$. Moy in [14] showed that if $T \in B(L^p(\Sigma))$, $T(L^\infty(\Sigma)) \subset L^\infty(\Sigma)$ and $T(fT(g)) = T(f)T(g)$ for all $f, g \in L^\infty(\Sigma)$. then $T = E_{\mathcal{A}} M_u$ for some $\mathcal{A} \subseteq \Sigma$. Substitution conditional type operators of the form $T = T_1 C_\varphi$ are closely related to the multipication operators, weighted composition operators, integral and averaging operators and to the operators called conditional expectation type which has been studied in [5, 6, 8, 9, 15]. For example, let $X = [0, 1] \times [0, 1]$, $d\mu = dxdy$, Σ be the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0, 1] : A \text{ is a Lebesgue set in } [0, 1]\}$. Then, for each $f \in L^2(\Sigma)$, $(Ef)(x, y) = \int_0^1 f(x, t)dt$, which is independent of the second coordinate. In this case we have

$$T(f) = w(x, y)E(u.f \circ \varphi)(x, y) = w(x, y) \int_0^1 u(x, t) f(\varphi(x, t)) dt.$$

In [5, 11] we have studied substitution conditional type operator $T = M_w E M_u C_\varphi$ induced by the triple (w, u, φ) on $L^2(\Sigma)$. In the next section, we discuss measure theoretic characterizations for substitution conditional type operators. Boundedness, point spectrum, compactness, reducibility, generalized inverse and some weak normal classes of a these type of operators will be investigated.

2 Characterizations

Let $\mathcal{A} \subseteq \Sigma$ be a sub-σ-algebra and let φ be a nonsingular measurable transformation for which the composition operator C_φ is densely defined. For each $u, w \in L^0(\Sigma)$, it is assumed that $u\mathcal{R}(C_\varphi) \subset \mathcal{D}(E)$. It follows that the substitution conditional type operator $T = M_w E M_u C_\varphi : L^2(\Sigma) \to L^0(\Sigma)$ is well-defined. Now, we provide some lemmas for later use.

Lemma 2.1 *Let $\varphi^{-1}(\mathcal{A})$ be a σ-finite subalgebra of $\mathcal{A}$ and let $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$. Then $E_{\varphi^{-1}(\Sigma)} E_{\mathcal{A}} = E_{\mathcal{A}} E_{\varphi^{-1}(\Sigma)} = E_{\mathcal{A}}$, $E_{\varphi^{-1}(\mathcal{A})} E_{\mathcal{A}} = E_{\mathcal{A}} E_{\varphi^{-1}(\mathcal{A})} = E_{\varphi^{-1}(\mathcal{A})}$. Moreover, $E_{\mathcal{A}}(f \circ \varphi) = f \circ \varphi = E_{\varphi^{-1}(\Sigma)}(f \circ \varphi)$ and $E_{\mathcal{A}}(f) \circ \varphi^{-1}$ is well-defined for all $f \in L^0(\mathcal{A}) \cap \mathcal{D}(E)$.*

From now on, we take $E_\varphi = E_{\varphi^{-1}(\Sigma)}$, $E = E_{\mathcal{A}}$ and assume that $h < \infty$.

Lemma 2.2 *Let $T = M_w E M_u C_\varphi : L^2(\Sigma) \to L^0(\Sigma)$ be a densely defined substitution conditional type operator. Then for each $f \in \mathcal{D}(T^*)$, $T^* f = h E_\varphi\{\bar{u} E(\bar{w} f)\} \circ \varphi^{-1}$.*

Proof Let $f \in \mathcal{D}(T^*)$ and $g \in \mathcal{D}(T)$. Then we have

$$\begin{aligned}
\langle g, T^* f\rangle = \langle Tg, f\rangle &= \int_X wE(u(g \circ \varphi))\bar{f} d\mu = \int_X u(g \circ \varphi) E(w\bar{f}) d\mu \\
&= \int_X E_\varphi\left(uE(w\bar{f})\right) g \circ \varphi d\mu = \int_X hE_\varphi\left\{uE(w\bar{f})\right\} \circ \varphi^{-1} g d\mu \\
&= \left\langle g, hE_\varphi\{\bar{u}E(\bar{w}f)\} \circ \varphi^{-1}\right\rangle.
\end{aligned}$$

Thus, the desired conclusion holds. □

We recall that, by assumption, h is finite-valued. Thus, C_φ is densely defined and hence C_φ^* is well-defined (see [1]). Then by Lemma 2.2, $T^* = C_\varphi^* M_{\bar{u}} E M_{\bar{w}}$.

We write $B(L^2(\Sigma))$ and $K(L^2(\Sigma))$ for the space of bounded and compact linear operators on $L^2(\Sigma)$, respectively. Boundedness of T has been proven in [5]. In the following we improve it as follows.

Theorem 2.3 *Let $T : L^2(\Sigma) \to L^0(\Sigma)$ be a substitution conditional type operator $T = M_w E M_u C_\varphi$. If $J_1 = hE_\varphi\{E(|u|^2)E(|w|^2)\} \circ \varphi^{-1} \in L^\infty(\Sigma)$, then T is bounded and $\|T\|^2 \leq \|J_1\|_\infty$. Moreover, if $h \in L^\infty(\Sigma)$ and $T \in B(L^2(\Sigma))$, then $J_2 = E\{|hE_\varphi(\bar{u}\sqrt{E(|w|^2)}) \circ \varphi^{-1}|^2\} \in L^\infty(\mathcal{A})$ with $\|J_2\|_\infty \leq \|h\|_\infty \|T\|^2$.*

Proof Let $f \in L^2(\Sigma)$. Then by Lemma 2.1 we have

$$\begin{aligned}
\|Tf\|^2 &= \int_X E(|w|^2)|E(u(f \circ \varphi))|^2 d\mu = \int_X |E(u\sqrt{E(|w|^2)}(f \circ \varphi))|^2 d\mu \\
&\leq \int_X E(|u|^2 E(|w|^2)) E(|f|^2 \circ \varphi) d\mu = \int_X E(|u|^2) E(|w|^2)(|f|^2 \circ \varphi) d\mu \\
&= \int_X E_\varphi\{E(|u|^2)E(|w|^2)\}(|f|^2 \circ \varphi) d\mu
\end{aligned}$$

$$= \int_X hE_\varphi\{E(|u|^2)E(|w|^2)\} \circ \varphi^{-1}|f|^2 d\mu$$
$$= \int_X J_1|f|^2 d\mu \leq \|J\|_\infty \int_X |f|^2 d\mu = \|J_1\|_\infty \|f\|^2.$$

Thus, T is bounded and $\|T\|^2 \leq \|J_1\|_\infty$. Conversely, let T is bounded. Set $v = u\sqrt{E(|w|^2)}$. Then $\|Tf\|^2 = \int_X |EM_v C_\varphi f|^2 d\mu = \|EM_v C_\varphi f\|^2$, for each $f \in L^2(\Sigma)$. Consequently, $EM_v C_\varphi$ is bounded and $\|C_\varphi^* M_{\bar{v}} E\| = \|EM_v C_\varphi\| = \|T\|$. Now, for $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ we have

$$\begin{aligned}
\int_A J_2 d\mu &= \int_A h^2|E_\varphi(\bar{v}) \circ \varphi^{-1}|^2 d\mu = \int_A h|E_\varphi(\bar{v})|^2 \circ \varphi^{-1} d(\mu \circ \varphi^{-1}) \\
&= \int_{\varphi^{-1}(A)} (h \circ \varphi)|E_\varphi(\bar{v})|^2 d\mu = \int_X (h \circ \varphi)|E_\varphi(\bar{v}\chi_{\varphi^{-1}(A)})|^2 d\mu \\
&= \int_X h^2|E_\varphi(\bar{v}\chi_{\varphi^{-1}(A)})|^2 \circ \varphi^{-1} d\mu = \int_X |hE_\varphi(\bar{v}E\chi_{\varphi^{-1}(A)}) \circ \varphi^{-1}|^2 d\mu \\
&= \int_X |C_\varphi^* M_{\bar{v}} E(\chi_{\varphi^{-1}(A)})|^2 d\mu = \|C_\varphi^* M_{\bar{v}} E(\chi_{\varphi^{-1}(A)})\|^2 \\
&\leq \|C_\varphi^* M_{\bar{v}} E\|^2 \mu(\varphi^{-1}(A)) = \|T\|^2 \int_A d(\mu \circ \varphi^{-1}) \\
&= \|T\|^2 \int_A h d\mu \leq \|T\|^2 \|h\|_\infty\, \mu(A).
\end{aligned}$$

Thus, $\|J_2\|_\infty = \sup\{\frac{1}{\mu(A)} \int_A J_2 d\mu : 0 < \mu(A) < \infty\} \leq \|T\|^2\|h\|_\infty < \infty$, and hence $\|J_2\|_\infty \leq \|h\|_\infty \|T\|^2$. □

Corollary 2.4 *$T_1 = M_w E M_u \in B(L^2(\Sigma))$ if and only if $E(|u|^2)E(|w|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T_1\|^2 = \|E(|u|^2)E(|w|^2)\|_\infty$.*

Proof Put $\varphi = id$ in Theorem 2.3. Then $E_\varphi = I = C_\varphi = C_\varphi^*$ and $h = 1$. □

Let $v \in L^0(\Sigma)$ and $J = h(E_\varphi(|v|^2) \circ \varphi^{-1})$. We recall that the weighted composition operator $(vC_\varphi)(f) := v.(f \circ \varphi)$ defines a bounded operator on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [3]).

Example 2.5 Let Σ be the Lebesgue subsets of $X = [0, 1] \times [0, 1]$, $d\mu = dxdy$ and let $\varphi : X \to X$ be the baker transformation defined by

$$\varphi(x, y) = \left(2x, \frac{y}{2}\right) \chi_{[0,\frac{1}{2})\times[0,1]} + \left(2x - 1, \frac{y+1}{2}\right) \chi_{[\frac{1}{2},1]\times[0,1]}.$$

Then

$$\varphi^{-1}(x, y) = \left(\frac{x}{2}, 2y\right) \chi_{[0,1]\times[0,\frac{1}{2})} + \left(\frac{x+1}{2}, 2y - 1\right) \chi_{[0,1]\times[\frac{1}{2},1]}.$$

It follows that φ preserves the Lebesgue measure μ on X. Thus, $h = 1 = \|C_\varphi\|$, $E_\varphi = I$ and hence $hE_\varphi(f) \circ \varphi^{-1} = f \circ \varphi^{-1}$ for all $f \in L^2(\Sigma)$. Let $u, w \in L^2(\Sigma)$, $\mathcal{A} = \{\emptyset, X\}$ and let $T = M_w E M_u C_\varphi$. Then $\varphi^{-1}(\mathcal{A}) = \mathcal{A}$ and for each $f \in L^2(\Sigma)$ we have $E(f)(x, y) = \int_0^1 \int_0^1 f(s,t)dsdt$ for all $(x, y) \in X$. Therefor $\{E(f) \circ \varphi^{-1}\}(x, y) = E(f)(x, y)$. It follows that

$$Tf = w\left\{\int_0^1 \int_0^{\frac{1}{2}} u(s,t) f\left(2s, \frac{t}{2}\right) dsdt + \int_0^1 \int_{\frac{1}{2}}^1 u(s,t) f\left(2s-1, \frac{t+1}{2}\right) dsdt\right\}.$$

Moreover, $J_1 = hE_\varphi\{E(|u|^2)E(|w|^2)\} \circ \varphi^{-1} = E(|u|^2)E(|w|^2) = \|u\|_2^2 \, \|w\|_2^2$. Thus, by Theorem 2.3, $T \in B(L^2(\Sigma))$ with $\|T\| = \|u\|_2 \, \|w\|_2 = \|T_1\|$. Also, by Lemma 2.2, we have

$$\begin{aligned} T^* f(x, y) &= \bar{u}(\varphi^{-1}(x, y))E(\bar{w} f)(x, y) \\ &= \left\{\bar{u}\left(\frac{x}{2}, 2y\right)\chi_{[0,1]\times[0,\frac{1}{2})} + \bar{u}\left(\frac{x+1}{2}, 2y-1\right)\chi_{[0,1]\times[\frac{1}{2},1]}\right\} \int_0^1 \int_0^1 \bar{w}(s,t) f(s,t) dsdt. \end{aligned}$$

We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and the range of an operator $T \in B(L^2(\Sigma))$, respectively. In the following we characterize the null-space of T.

Lemma 2.6 *Let $T = M_w E M_u C_\varphi$ be a weighted composition Lambert-type operator on $L^2(\Sigma)$. Then $\mathcal{N}(T) = \{C_\varphi^*(\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A}))\}^\perp$.*

Proof Set $v = \bar{u}\sqrt{E(|w|^2)}$. Since $\|Tf\| = \|EM_v C_\varphi f\|$ for all $f \in L^2(\Sigma)$, then $\mathcal{N}(T) = \mathcal{N}(EM_v C_\varphi)$. Let $g \in L^2(\mathcal{A})$ be an arbitrary. Then we have

$$\begin{aligned} \langle f, C_\varphi^*(\bar{v}g)\rangle = \left\langle f, hE_\varphi(\bar{v}g) \circ \varphi^{-1}\right\rangle &= \int_X f E_\varphi(v\bar{g}) \circ \varphi^{-1} d(\mu \circ \varphi^{-1}) \\ &= \int_X (f \circ \varphi) E_\varphi(v\bar{g}) d\mu = \int_X v(f \circ \varphi)\bar{g} d\mu \\ &= \int_X E(v(f \circ \varphi))\bar{g} d\mu = \langle EM_v C_\varphi(f), g\rangle. \end{aligned}$$

Consequently, $f \in \{C_\varphi^*(\bar{v}L^2(\mathcal{A}))\}^\perp$ if and only if $f \in \mathcal{N}(EM_v C_\varphi) = \mathcal{N}(T)$. □

Theorem 2.7 *Let $\varphi^{-1}(\mathcal{A})$ be σ-finite, $S = \sigma(E(u)E(w))$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$ and let $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$. If $(\varphi^{-1}(\Sigma))_S = \mathcal{A}_S$ and*

$$hE_\varphi\left\{\bar{u}E(|w|^2)E(u)\right\} \circ \varphi^{-1} = wE\left\{u(h \circ \varphi)E_\varphi(\bar{u}E(\bar{w}))\right\}, \tag{2.1}$$

then T is normal on $L^2(\varphi^{-1}(\Sigma))$. Moreover, if T is normal, then (2.1) is holds.

Proof Let $f \in L^2(\mathcal{A})$. Then $f \circ \varphi \in L^0(\varphi^{-1}(\mathcal{A})) \subseteq L^0(\varphi^{-1}(\Sigma))$ and $(f \circ \varphi) \circ \varphi^{-1} = f$ on $\sigma(h)$. It follows that

$$T^*Tf = hE_\varphi\left\{\bar{u}E(|w|^2)E(u)\right\} \circ \varphi^{-1} f,$$

$$TT^*f = wE\left\{u(h\circ\varphi)E_\varphi(\bar{u}E(\bar{w}))\right\}f.$$

Let $(\varphi^{-1}(\Sigma))_S = \mathcal{A}_S$ and (2.1) is true. For $A\in\mathcal{A}$ with $\mu(A)<\infty$, put $g=\chi_{A\cap S}$. Since $EE_\varphi = E$ and $\sigma(hE_\varphi\{\bar{u}E(|w|^2)E(u)\}\circ\varphi^{-1}) = \sigma(wE\{u(h\circ\varphi)E_\varphi(\bar{u}E(\bar{w}))\}) = \sigma(w)\cap\sigma(E(u))\cap\sigma(\bar{u}E(\bar{w}))\subseteq\sigma(E(u)E(w)) = S$, then $T^*Tg = TT^*g$. From $(\varphi^{-1}(\Sigma))_S=\mathcal{A}_S$, we obtain that $T^*Tg = TT^*g$ holds for any $\varphi^{-1}(\Sigma)$-measurable subset A of S, as long as A has finite measure. Consequently, $T^*T\chi_A = TT^*\chi_A$ for all such A, implying T is normal on $L^2(\varphi^{-1}(\Sigma))$.

Now, let $T\in B(L^2(\Sigma))$ is normal. Since $\varphi^{-1}(\mathcal{A})$ be σ-finite, there exists $\{A_n\}_n\subseteq\mathcal{A}$ such that $A_n\subseteq A_{n+1}$, $X=\cup_n(\varphi^{-1}(A_n))$ with $\mu(\varphi^{-1}(A_n))<\infty$. Put $f_n=\chi_{\varphi^{-1}(A_n)}$. Since $T^*Tf_n = TT^*f_n$ and $f_n\nearrow\chi_X$, then we obtain (2.1). □

Corollary 2.8 *([3, 10]) Under assumptions of Theorem 2.7, if $\varphi = id$, then $T_1 = M_wEM_u$ is normal if and only if $T_1 = M_{g\bar{u}}EM_u$ for some $g\in L^0(\mathcal{A})$. Moreover, if $\mathcal{A}=\Sigma$ then the weighted composition operator $T_2 = M_{uw}C_\varphi$ is normal whenever $hE_\varphi(|uw|^2)\circ\varphi^{-1} = uw(h\circ\varphi)E_\varphi(\overline{uw})$.*

Theorem 2.9 *Let $T = M_wEM_uC_\varphi\in B(L^2(\Sigma))$ be self adjoint. Then*
(i) $hE_\varphi(\bar{u}E(\bar{w}))\circ\varphi^{-1} = wE(u)$.
(ii) $wE(u.(wE(u))\circ\varphi) = hE_\varphi(\bar{u}E(|w|^2E(u)))\circ\varphi^{-1}$.
(iii) $\varphi_D^2 = id$, where φ_D is a measurable self adjoint map on $D = \sigma(wE(u(wE(u))\circ\varphi)$ is a localising set for $\mathcal{A}$.

Proof Let $f\in L^2(\Sigma)$. Since $T^*=T$, then

$$hE_\varphi\{\bar{u}E(\bar{w}f)\}\circ\varphi^{-1} = wE(u.f\circ\varphi). \tag{2.2}$$

Since $\mathcal{A}$ is σ-finite, choose an increasing sequence of measurable set $\{C_n\}$, each of finite measurable, whose union is all of X. Setting $f=\chi_{C_n}$ in (2.2) and letting $n\to\infty$, we obtain $hE_\varphi\{\bar{u}E(\bar{w})\}\circ\varphi^{-1} = wE(u)$. Now, since $T^2 = T^*T$ then for each $f\in L^2(\mathcal{A})$ we have

$$w(u(wE(w))\circ\varphi)f\circ\varphi^2 = hE_\varphi\{\bar{u}E(|w|^2)E(u)\}\circ\varphi^{-1}f. \tag{2.3}$$

Put again $f=\chi_{C_n}$ as above. Then we obtain (ii). Dividing both side of (2.3) by $wE(u(wE(w))\circ\varphi)$ and using (ii), we have $\chi_Df\circ\varphi^2 = \chi_Df$. Now, let $f=\chi_C$, when $C\in\mathcal{A}$. Then $\chi_D\chi_C\circ\varphi^2 = \chi_{D\cap C}$, and so $\chi_{D\cap\varphi^{-2}(C)} = \chi_{D\cap C}$. Since $\varphi^{-2}(\mathcal{A})\subseteq\mathcal{A}$, then $\varphi_D^2 = I$ on $(D,\mathcal{A}_D,\mu_{|\mathcal{A}_D}) = (D,\Sigma_D,\mu_{|\Sigma_D})$, because D is a localising set for $\mathcal{A}$. □

If we take $\varphi = id$ in Theorem 2.9, we get that (a) $\bar{u}E(\bar{w}) = wE(u)$ and (b) $wE(uw) = \bar{u}E(|w|^2)$. Put $A=\sigma(E(w))$, $B=\sigma(E(|w|^2))$ and $C=\sigma(E(uw))$. Then

$$E(w)\bar{u} \overset{(b)}{=\!=} \frac{E(w)wE(uw)}{E(|w|^2)}\chi_B = \frac{E(wE(uw))}{E(|w|^2)}w\chi_B$$

$$\overset{(b)}{=} \frac{E(\bar{u}E(|w|^2))}{E(|w|^2)} w\chi_B = E(\bar{u})w\chi_B = wE(\bar{u}),$$

$$wE(|u|^2) = wE(u\bar{u}) \overset{(a)}{=} wE(\frac{uwE(u)}{E(\bar{w})}\chi_A)$$

$$= \frac{wE(uw)E(u)}{E(\bar{w})}\chi_A \overset{(a)}{=} \frac{\bar{u}E(\bar{w})E(uw)}{E(\bar{w})}\chi_A = \bar{u}E(uw)\chi_A.$$

Let $T_1 = M_w E M_u \in B(L^2(\Sigma))$ be self-adjoint. It is known that ([10, Theorem 2.21]) ess range $E(uw)$ =spec$(T_1) \subseteq \mathbb{R}$, the spectrum of T_1, and so $E(uw)$ is a real valued function. Then, using (b), we get that $T_1 = M_{g\bar{u}} E M_u$ where $g = \frac{E(|w|^2)}{E(uw)}\chi_C \in L^0(\mathcal{A})$ with $\bar{g} = g$ (see [10, Theorem 2.32(a)]).

Now, take $\mathcal{A} = \Sigma$ and $v = uw$ in Theorem 2.9. Setting $T_2^* = T_2 = M_v C_\varphi$ yields $v = hE_\varphi(\bar{v}) \circ \varphi^{-1}$, $v(v \circ \varphi) = hE_\varphi(|v|^2) \circ \varphi^{-1}$ and $\varphi_D^2 = id$, where $D = \sigma(v(v \circ \varphi)) = \sigma(hE_\varphi(|v|^2) \circ \varphi^{-1})$ (see [2]).

We recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$, such that for each $B \in \mathcal{A}$, if $B \subseteq A$, then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure space $(X, \mathcal{A}, \mu)$ with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu)$ can be partitioned uniquely as $X = (\cup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and $B \in \mathcal{A}$, being disjoint from each A_n, is non-atomic (see [17]).

It is known that every $\mathcal{A}$-measurable function f is constant on each $\mathcal{A}$-atom A, namely $f_{|A} = c$. Since $f(x)\chi_A(x) = c\chi_A(x) = (f\chi_A)(x)$ for all $x \in X$, we take $f(A) = c$ when no confusion can arise. The cardinal number of a set $A \in \Sigma$ is represented as $\#(A)$. Set $\varphi_0(x) = x$ and put $\varphi_k(x) = \varphi(\varphi_{k-1}(x))$, for all $x \in X$. An $\mathcal{A}$-atom A in $(X, \mathcal{A}, \mu)$ is called a fixed atom of φ of order $n \in \mathbb{N}$ if $\varphi_n(A) = A$ and $\varphi_k(A) \neq A$ for $1 \leq k \leq n-1$.

For Lambert conditional operators and compact weighted composition operators on L^p some properties of their ra were described by Herron [8] and Takagi [16]. In the following, we show that the point spectrum $\Pi_0(T)$ of a substitution conditional type operator T on $L^2(\Sigma)$ contains some special numbers.

Theorem 2.10 *Let $(X, \mathcal{A}, \mu)$ be partitioned as $X = (\cup_{n\in\mathbb{N}} A_n) \cup B$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, $W = E(u(w \circ \varphi))$ and let $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$. Then $\Lambda \subseteq spec(T)$ where $\Lambda = \{\lambda \in \mathbb{C} : \lambda^n = W(A)W(\varphi(A)) \cdots W(\varphi_{n-1}(A))$ for some fixed atom A of φ of order $n\}$.*

Proof To prove the theorem, we adopt the method used by Kamowitz [12]. Let $W(\varphi_n(A)) = 0$ for some $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We claim that T is not onto and so $\lambda = 0 \in \sigma_p(T)$. Suppose not, then T^{n+2} is onto. This yields a function $f \in L^2(\Sigma)$ with $T^{n+2}f = w\{W(W \circ \varphi) \cdots (W \circ \varphi_n)\}E(uf \circ \varphi) \circ \varphi_{n-1} = \chi_A$. Thus, $\sigma(T^{n+2}f) \subseteq \sigma(W \circ \varphi_n)$. Since $\sigma(W \circ \varphi_n) \cap A = \emptyset$, then $(T^{n+2}f)_{|A} = 0$ whereas $\chi_A(A) = 1$. This shows that T is not onto.

Assume $W(\varphi_k(A)) \neq 0$ for all $k \in \mathbb{N}_0$ and $\lambda^n = W(A)W(\varphi(A)) \cdots W(\varphi_{n-1}(A))$. If $n = 1$, then $\lambda = W(A)$ and $\varphi(A) = A$. Put $g = w\chi_A$. We show that there exists no $f \in L^2(\Sigma)$ such that $\lambda f - Tf = g$. Indeed,

$$\lambda f - Tf = g \xrightarrow{C_\varphi} \lambda f \circ \varphi - (w \circ \varphi)E(u.f \circ \varphi) \circ \varphi = g \circ \varphi$$
$$\xrightarrow{\times u} \lambda uf \circ \varphi - u(w \circ \varphi)E(u.f \circ \varphi) \circ \varphi = ug \circ \varphi$$
$$\xrightarrow{E} \lambda E(uf \circ \varphi) - E(u(w \circ \varphi))E(u.f \circ \varphi) \circ \varphi = E(ug \circ \varphi).$$

Since $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, then $0 = (\lambda - W(A))E(uf \circ \varphi)(A) = W(A)\chi_A(\varphi(A)) = W(A) \neq 0$. This shows that $\lambda I - T$ is not onto and hence $\lambda \in spec(T)$.

Now, let $n \geq 2$, $\lambda^n = W(A)W(\varphi(A)) \cdots W(\varphi_{n-1}(A))$ and $\varphi_n(A) = A$. Put $F = E(uf \circ \varphi)$ and $G = E(ug \circ \varphi)$ where $f \in L^2(\Sigma)$ and $g = w\chi_A$. It follows that $F(\varphi_n(A)) = F(A)$, $G(\varphi_{n-1}(A)) = E(u(w \circ \varphi))(\varphi_{n-1}(A))\chi_A(\varphi_n(A)) = W(\varphi_{n-1}(A))$ and $G(\varphi_k(A)) = 0$ for $0 \leq k \leq n-2$. Again we claim that there exists no $f \in L^2(\Sigma)$ which satisfies $\lambda f - Tf = g$. For, if such a function f exists, then by induction and using the same method in case $n = 1$, we have

$$\lambda^n F - W(W \circ \varphi) \cdots (W \circ \varphi_{n-1})F \circ \varphi_n = \lambda^{n-1}G + \lambda^{n-2}WG \circ \varphi + \cdots$$
$$+ W(W \circ \varphi) \cdots (W \circ \varphi_{n-2})G \circ \varphi_{n-1}.$$

It follows that

$$\left\{\lambda^n - W(A) \cdots W(\varphi_{n-1}(A))\right\}F(A) = W(A) \cdots W(\varphi_{n-1}(A)). \tag{2.4}$$

Since $W(\varphi_k(A)) \neq 0$ for all $k \in \mathbb{N}_0$, the right hand side of (2.4) is non-zero, whereas the left hand side of (2.4) is zero. This contradiction shows that $\lambda I - T$ is not onto and thus $\lambda \in spec(T)$. ◻

Corollary 2.11 *Under assumptions of Theorem 2.10, if* $T = M_w E M_u C_\varphi \in K(L^2(\Sigma))$ *then* $\Lambda \cup \{0\} \subseteq \Pi_0(M_w E M_u) \cup \{0\}$.

Corollary 2.12 *Let* $\{M_w E M_u, M_u C_\varphi\} \subset K(L^2(\Sigma))$. $\Lambda_1 = \{\lambda \in \mathbb{C} : \lambda^n = u(A) \cdots u(\varphi_{n-1}(A))$, *for some fixed* Σ*-atom* A *of* φ *of order* $n\}$ *and* $\Lambda_2 = \{\lambda \in \mathbb{C} : \lambda = E(uw)(A)$, *for some* $\mathcal{A}$*-atom* $A\}$. *Then* $\Lambda_1 \cup \{0\} \subseteq \Pi_0(M_u C_\varphi) \cup \{0\}$ *and* $\Lambda_2 \cup \{0\} \subseteq \Pi_0(M_w E M_u) \cup \{0\}$.

Recall that a linear operator T on a Hilbert space $\mathcal{H}$ is said to be compact if for each bounded sequence $\{f_n\}_n \subseteq \mathcal{H}$, there is a subsequence of $\{Tf_n\}_n$ that is convergent. In the following theorem we give a sufficient and necessary conditions for the compactness of $T = M_w E M_u C_\varphi$ on $L^2(\Sigma)$.

Theorem 2.13 *Let* $(X, \mathcal{A}, \mu)$ *be partitioned as* $X = (\cup_{n\in\mathbb{N}} A_n) \cup B$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ *and let* $T = M_w E M_u C_\varphi$ *be a substitution conditional type operator on* $L^2(\Sigma)$. *If* T *is compact, then for each* $\varepsilon > 0$, $\mu(B \cap K_\varepsilon) = 0$ *and* $\#\{n \in \mathbb{N} : K_\varepsilon \supseteq A_n \in \mathcal{A}\} < \infty$, *where* $K_\varepsilon := \{x \in X : h(x)E_\varphi\{E(|w|^2)|E(u)|^2\} \circ \varphi^{-1}(x) \geq \varepsilon\}$. *Conversely,* T *is compact whenever for each* $\varepsilon > 0$, $\mu(B \cap G_\varepsilon) = 0$ *and* $\#\{n \in \mathbb{N} : G_\varepsilon \supseteq A_n \in \mathcal{A}\} < \infty$, *where* $G_\varepsilon := \{x \in X : h(x)E_\varphi\{E(|w|^2)|E(|u|^2)\} \circ \varphi^{-1}(x) \geq \varepsilon\}$.

Proof Suppose T is a compact operator. We show that for each $\varepsilon > 0$ the set K_ε consists of finitely many $\mathcal{A}$ atoms. Assume the contrary. Then for some $\varepsilon > 0$ the set K_ε either contains a subset of nonatomic part $B \in \mathcal{A}$ or has infinitely many $\mathcal{A}$-atoms. In both cases we can find a sequence $\{A_n\}_n$ of pairwise disjoint $\mathcal{A}$-measurable sets with $0 < \mu(A_n) < \infty$. Define $f_n = \frac{\chi_{A_n}}{\sqrt{\mu(A_n)}}$. Then for each $n \in \mathbb{N}$, $\{f_n, f_n \circ \varphi\} \subset L^2(\mathcal{A}) \cup L^2(\varphi^{-1}(\mathcal{A})) = L^2(\mathcal{A})$ with $\|f_n\|_2 = 1$ and

$$\begin{aligned}\|Tf_n\|_2^2 &= \int_X |wE(u(f_n \circ \varphi))|^2 d\mu = \int_X |w|^2 |E(u)|^2 |f_n|^2 \circ \varphi d\mu \\ &= \int_X E(|w|^2)|E(u)|^2 |f_n|^2 \circ \varphi d\mu = \int_X hE_\varphi\{E(|w|^2)|E(u)|^2\} \circ \varphi^{-1} |f_n|^2 d\mu \\ &= \frac{1}{\mu(A_n)} \int_{A_n} hE_\varphi\{E(|w|^2)|E(u)|^2\} \circ \varphi^{-1} d\mu \geq \varepsilon.\end{aligned}$$

For $n \neq m$, $\mu(\sigma(Tf_n) \cap \sigma(Tf_m)) = 0$ and hence $\|Tf_n - Tf_m\|_2^2 = \|Tf_n\|^2 + \|Tf_m\|^2 \geq 2\varepsilon$. Thus, the sequence $\{Tf_n\}$ does not contain any convergent subsequence, and so T is not compact.

Conversely, suppose for each $\varepsilon > 0$, $G_\varepsilon \cap \{A_n\}_n = \{A_\varepsilon^1, \cdots, A_\varepsilon^k\}$. Put $B_\varepsilon = A_\varepsilon^1 \cup \cdots \cup A_\varepsilon^k$, $v = u\chi_{B_\varepsilon}$ and take $T_\varepsilon = M_w E M_v C_\varphi$. Then for each $f \in L^2(\Sigma)$ we have

$$T_\varepsilon f = wE(u\chi_{B_\varepsilon}(f \circ \varphi)) = (Tf)\chi_{B_\varepsilon} = w \sum_{i=1}^{k} \Big(E(u(f \circ \varphi))(A_\varepsilon^i)\Big) \chi_{A_\varepsilon^i}.$$

Thus, T_ε is a finite rank operator. Also, since $u = v$ on B_ε, then

$$\int_{B_\varepsilon} |(T - T_\varepsilon)f|^2 d\mu = \int_{B_\varepsilon} |M_w E M_{(u-v)} C_\varphi f|^2 d\mu = 0.$$

It follows that

$$\begin{aligned}\|(T - T_\varepsilon)f\|_2^2 &= \int_{X \setminus B_\varepsilon} |Tf - T_\varepsilon f|^2 d\mu = \int_{X \setminus B_\varepsilon} |Tf|^2 d\mu \\ &= \int_{X \setminus B_\varepsilon} |w|^2 |E(u(f \circ \varphi))|^2 d\mu \\ &= \int_{X \setminus B_\varepsilon} E(|w|^2) E(u(f \circ \varphi))|^2 d\mu \\ &\leq \int_{X \setminus B_\varepsilon} E(|w|^2) E(|u|^2) E(|f|^2 \circ \varphi) d\mu \\ &= \int_{X \setminus B_\varepsilon} E(|w|^2) E(|u|^2)(|f|^2 \circ \varphi) d\mu \\ &= \int_{X \setminus B_\varepsilon} hE_\varphi \Big\{E(|w|^2) E(|u|^2)\Big\} \circ \varphi^{-1} |f|^2 d\mu\end{aligned}$$

$$\leq \varepsilon \int_X |f|^2 d\mu = \varepsilon \|f\|_2^2.$$

Consequently, T is compact on $L^2(\Sigma)$. □

Corollary 2.14 *Let $T : L^2(\Sigma) \to L^0(\Sigma)$ be a substitution conditional type operator $T = M_w E M_u C_\varphi$. Then the followings hold.*

(i) If $T_1 = M_w E M_u$ is a compact operator on $L^2(\Sigma)$, then for each $\varepsilon > 0$ the set $\{E(|w|^2)|E(u)|^2 \geq \varepsilon\}$ consists of finitely many $\mathcal{A}$-atoms. Conversely, if for each $\varepsilon > 0$ the set $\{E(|w|^2)E(|u|^2) \geq \varepsilon\}$ consists of finitely many $\mathcal{A}$-atoms, then T is compact on $L^2(\Sigma)$.

(ii) The weighted composition operator $T_2 = M_{uw} C_\varphi$ is compact on $L^2(\Sigma)$ if and only if for each $\varepsilon > 0$ the set $\{hE_\varphi(|uw|^2) \circ \varphi^{-1} \geq \varepsilon\}$ consists of finitely many Σ-atoms.

Proof Take $\varphi = id$ and $\mathcal{A} = \Sigma$ in Theorem 2.13, respectively. □

Let $T \in B(\mathcal{H})$. We recall that the unique operator $S \in B(\mathcal{H})$ satisfying

$$(1)\ TST = T, \quad (2)\ STS = S, \quad (3)\ (TS)^* = TS, \quad (4)\ (ST)^* = ST$$

is called the Moore-Penrose inverse of T and is denoted by $T^\dagger$. Let $T\{i, ..., j\}$ denote the set of all operators S satisfying condition (k) for all labels k in the list $\{i, \cdots, j\}$. In this case $S \in T\{i, ..., j\}$ is a $\{i, ..., j\}$-inverse of T and is denoted by $T^{(i,...,j)}$. Note that $T^{(1,2,3,4)} = T^\dagger$. For other important properties of $T^\dagger$ (see [4, 7]).

Lemma 2.15 *Let $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$, $h \in L^0(\mathcal{A})$, $K = h\{E(uE_\varphi(\bar{u}))E(|w|^2)\} \circ \varphi^{-1}$ and $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$. Then K is bounded away from zero on $\sigma(K)$ whenever T has closed range.*

Proof Suppose T has closed range, but K is not bounded away from zero on $\sigma(K)$. Then for fixed $\varepsilon > 0$, there exists $\{A_n\}_n \in \mathcal{A}$ with $A_n \subseteq A_{n+1} \subseteq \sigma(K)$ and $0 < \mu(A_n) < \infty$ such that $|K|\chi_{A_n} < 1/\sqrt{n}$. Put $f_n = \sqrt{h} E_\varphi(\bar{u}\sqrt{E(|w|^2)}) \circ \varphi^{-1} \chi_{A_n}$. Then by Theorem 2.3 we have

$$\begin{aligned}\|f_n\|_2^2 &= \int_X |E_\varphi(\bar{u}\sqrt{E(|w|^2)})|^2 (\chi_{A_n} \circ \varphi) d\mu \leq \int_X E(|u|^2)E(|w|^2)(\chi_{A_n} \circ \varphi) d\mu \\ &= \int_{A_n} hE_\varphi\{E(|u|^2)E(|w|^2)\} \circ \varphi^{-1} d\mu \leq \|J_1\|_\infty \mu(A_1) < \infty.\end{aligned}$$

Now, let $g \in \mathcal{N}(T)$. Then we get that

$$\begin{aligned}|\langle g, f_n \rangle|^2 &= |\int_X \frac{\chi_{\sigma(h)}}{\sqrt{h}} g E_\varphi(u\sqrt{E(|w|^2)}) \circ \varphi^{-1} \chi_{A_n} d(\mu \circ \varphi^{-1})|^2 \\ &= |\int_X \frac{g \circ \varphi}{\sqrt{h \circ \varphi}} (u\sqrt{E(|w|^2)})(\chi_{A_n} \circ \varphi) d\mu|^2\end{aligned}$$

$$= |\int_X E(u(g\circ\varphi))\sqrt{\frac{E(|w|^2)}{h\circ\varphi}}(\chi_{A_n}\circ\varphi)d\mu|^2$$
$$\leq \int_X |E(u(g\circ\varphi))|^2\frac{E(|w|^2)}{h\circ\varphi}(\chi_{A_n}\circ\varphi)d\mu$$
$$= \int_X |wE(u(g\circ\varphi))|^2\frac{1}{h\circ\varphi}(\chi_{A_n}\circ\varphi)d\mu$$
$$= \int_X |Tg|^2\frac{\chi_{A_n}\circ\varphi}{h\circ\varphi}du = 0.$$

It follows that $f_n \in L^2(\Sigma)\cap\mathcal{N}(T)^{\perp}$ and satisfies

$$\|Tf_n\|^2 = \int_{\varphi^{-1}(A_n)} |w|^2|E(u(\sqrt{h}\circ\varphi)E_{\varphi}(\bar{u}))\sqrt{E(|w|^2)}\,|^2 d\mu$$
$$= \int_{\varphi^{-1}(A_n)} (h\circ\varphi)|E(uE_{\varphi}(\bar{u}))E(|w|^2)|^2 d\mu$$
$$= \int_{A_n} h^2|E(uE_{\varphi}(\bar{u}))E(|w|^2)|^2\circ\varphi^{-1}d\mu$$
$$= \int_{A_n} |K|^2\chi_{A_n}d\mu \leq \frac{1}{n}\mu(A_1)\to 0, \quad \text{as } n\to\infty.$$

But this is a contradiction. □

In Theorem 2.15, if we take $\varphi = id$ or $\mathcal{A} = \Sigma$, then we have the following corollary.

Corollary 2.16 *(a) If $T_1 = M_w E M_u \in B(L^2(\Sigma))$ has closed range, then $K_1 = E(|u|^2)E(|w|^2)$ is bounded away from zero on $\sigma(K_1)$.*

(b) Let $v = uw$ and $\varphi^{-1}(\Sigma) = \Sigma$. Then $K_2 = h(|v|^2\circ\varphi^{-1})$ is bounded away from zero on $\sigma(K_2)$ whenever the weighted composition operator $T_2 = M_v C_{\varphi}$ has closed range.

Lemma 2.17 *Let $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$, $A = \sigma(hE\{uE_{\varphi}(\bar{u})E(|w|^2)\}\circ\varphi^{-1})$, $h \in L^0(\mathcal{A})$ and for each $f \in L^2(\Sigma)$,*

$$Sf = \frac{\chi_A}{hE\{uE_{\varphi}(\bar{u})E(|w|^2)\}\circ\varphi^{-1}}hE_{\varphi}\{\bar{u}E(\bar{w}f)\}\circ\varphi^{-1}.$$

Then $S \in B(L^2(\Sigma))$ whenever $T = M_w E M_u C_{\varphi} \in B(L^2(\Sigma))$ has closed range.

Proof Let $f \in L^2(\Sigma)$. By Lemma 2.15, $|K|^2 = h^2|\{E(uE_{\varphi}(\bar{u}))E(|w|^2)\}|^2\circ\varphi^{-1} \geq \alpha$ on $\sigma(K) = \sigma(A)$ for some $\alpha > 0$. Using Theorem 2.3 and the conditional Cauchy inequality we have

$$\|Sf\|_2^2 = \int_X \frac{\chi_A h^2|E_{\varphi}\{\bar{u}E(\bar{w}f)\}|^2\circ\varphi^{-1}}{h^2|E\{uE_{\varphi}(\bar{u})E(|w|^2)\}|^2\circ\varphi^{-1}}d\mu$$

$$
\begin{aligned}
&\leq \frac{1}{\alpha}\int_X h\chi_A|E_\varphi\{\bar{u}E(\bar{w}f)\}|^2 d(\mu\circ\varphi^{-1})\\
&= \frac{1}{\alpha}\int_X (h\circ\varphi)(\chi_A\circ\varphi)|E_\varphi\{\bar{u}E(\bar{w}f)\}|^2 d\mu\\
&\leq \frac{1}{\alpha}\int_X (h\circ\varphi)(\chi_A\circ\varphi)E(|u|^2)E(|w|^2)E(|f|^2)d\mu\\
&= \frac{1}{\alpha}\int_X (h\circ\varphi)(\chi_A\circ\varphi)E(|u|^2)E(|w|^2)|f|^2 d\mu\\
&= \frac{1}{\alpha}\int_X (h\circ\varphi)(\chi_A\circ\varphi)E_\varphi\left\{E(|u|^2)E(|w|^2)\right\}E_\varphi(|f|^2)d\mu\\
&\leq \frac{1}{\alpha}\int_X \left(hE_\varphi\left\{E(|u|^2)E(|w|^2)\right\}\circ\varphi^{-1}\right)\left(hE_\varphi(|f|^2)\circ\varphi^{-1}\right)d\mu\\
&\leq \frac{\|J_1\|_\infty}{\alpha}\int_X hE_\varphi(|f|^2)\circ\varphi^{-1}d\mu = \frac{\|J_1\|_\infty}{\alpha}\|f\|_2^2.
\end{aligned}
$$

□

Let $\sigma(uE_\varphi(\bar{u})) \supseteq \sigma(u)$. As the assumptions of Lemma 2.17 let $h \in L^0(\mathcal{A})$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$ and $A = \sigma(hE\{uE_\varphi(\bar{u})E(|w|^2)\}\circ\varphi^{-1})$. Since $\sigma(h\circ\varphi) = X$, then $\varphi^{-1}(A) = \sigma((h\circ\varphi)E\{uE_\varphi(\bar{u})E(|w|^2)\}) = \sigma(E\{uE_\varphi(\bar{u})E(|w|^2)\})$ and so $\chi_A\circ\varphi = \chi_{\varphi^{-1}(A)} = \chi_{\sigma(uE_\varphi(\bar{u}))}\chi_{\sigma(E(|w|^2))}$. Thus, for each $f \in L^2(\Sigma)$ we have

$$
\begin{aligned}
(\chi_A\circ\varphi)Tf &= w\chi_{\sigma(E(|w|^2))}E\left(\chi_{\sigma(uE_\varphi(\bar{u}))}u(f\circ\varphi)\right)\\
&= wE(u\circ\varphi) = Tf.
\end{aligned}
$$

It follows that

$$
STf = \frac{\chi_A}{E\{uE_\varphi(\bar{u})E(|w|^2)\}\circ\varphi^{-1}}E_\varphi\left\{\bar{u}E(|w|^2)E(u(f\circ\varphi))\right\}\circ\varphi^{-1}
$$

and

$$
\begin{aligned}
TSTf &= wE\left\{\frac{u(\chi_A\circ\varphi)}{E\{uE_\varphi(\bar{u})E(|w|^2)\}}E_\varphi(\bar{u})E(|w|^2)E(u(f\circ\varphi))\right\}\\
&= (\chi_A\circ\varphi)wE(f\circ\varphi) = (\chi_A\circ\varphi)Tf = Tf.
\end{aligned}
$$

By Similar computations, we have

$$
\begin{aligned}
STS &= S\left(wE\left(\frac{u(\chi_A\circ\varphi)}{E\{uE_\varphi(\bar{u})\}E(|w|^2)}E_\varphi\{\bar{u}E(\bar{w}f)\}\right)\right) = S\left(w(\chi_A\circ\varphi)\frac{E(\bar{w}f)}{E(|w|^2)}\right)\\
&= \frac{\chi_A}{E\{uE_\varphi(\bar{u})E(|w|^2)\}\circ\varphi^{-1}}E_\varphi\left\{\bar{u}E\left(w\bar{w}(\chi_A\circ\varphi)\frac{E(\bar{w}f)}{E(|w|^2)}\right)\right\}\circ\varphi^{-1}\\
&= \frac{\chi_A}{E\{uE_\varphi(\bar{u})E(|w|^2)\}\circ\varphi^{-1}}E_\varphi\{\bar{u}E(\bar{w}f)\}\circ\varphi^{-1} = Sf.
\end{aligned}
$$

These observations establish the following result.

Theorem 2.18 *Let $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$, $A = \sigma(hE\{uE_\varphi(\bar{u})E(|w|^2)\} \circ \varphi^{-1})$, $h \in L^0(\mathcal{A})$ and let $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$ has closed range and*

$$Sf = \frac{\chi_A}{E\{uE_\varphi(\bar{u})E(|w|^2)\} \circ \varphi^{-1}} E_\varphi\{\bar{u}E(\bar{w}f)\} \circ \varphi^{-1}, \quad f \in L^2(\Sigma).$$

Then $T^{(2)} = S$. Moreover, if $\sigma(uE_\varphi(\bar{u})) \supseteq \sigma(u)$ then $T^{(1,2)} = S$.

Corollary 2.19 *Let $T_1 = M_w E M_u$ and $T_2 = M_v C_\varphi$, where $v = uw$. If $T_i \in B(L^2(\Sigma))$ has closed range, then for each $f \in L^2(\Sigma)$,*

$$T_1^\dagger(f) = \frac{\chi_{\sigma(E(|w|^2)}}{E(|u|^2)E(|w|^2)} \bar{u}E(\bar{w}f);$$
$$T_2^\dagger(f) = \frac{\chi_{A_2}}{|v|^2 \circ \varphi^{-1}} (f \circ \varphi^{-1})$$

where $A_2 = \sigma(h(|v|^2 \circ \varphi^{-1}))$ and $E_\varphi = I$.

Let $f \in L^2(\Sigma)$ and $hE_\varphi(f) \circ \varphi^{-1} = 0$. Then $(h \circ \varphi)\{E_\varphi(f) \circ \varphi^{-1}\} \circ \varphi = (h \circ \varphi)E_\varphi(f) = 0$. Since $h \circ \varphi > 0$, then $E_\varphi(f) = 0$. Conversely, suppose $E_\varphi(f) = 0$. Then for each $B \in \Sigma$ we have

$$\int_B hE_\varphi(f) \circ \varphi^{-1} d\mu = \int_{\varphi^{-1}(B)} E_\varphi(f) d\mu = 0.$$

and so $hE_\varphi(f) \circ \varphi^{-1} = 0$. Let $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ be a σ-finite subalgebra of $\mathcal{A}$ and let $E_\varphi(f) = 0$ for all $f \in L^2(\Sigma)$. Then there is $\{B_n\} \subseteq \varphi^{-1}(\mathcal{A}) \subseteq \varphi^{-1}(\Sigma)$ with $B_n \subseteq B_{n+1}$, $0 < \mu(B_n) < \infty$ and $X = \cup_n B_n$. Take $f_n = \chi_{B_n}$. Then $E_\varphi(u) = \lim E_\varphi(u) f_n = \lim E_\varphi(uf_n) = 0$.

Now, let $\varphi^{-1}(\Sigma)$ is a σ-finite. Then $h = d(\mu \circ \varphi^{-1})/d\mu$ is finite-valued and hence C_φ is densely defined. It follows that (see [1])

$$\begin{aligned}\overline{\mathcal{R}(C_\varphi)} &= \mathcal{R}(E_\varphi) = L^2(\varphi^{-1}(\Sigma)) \\ &= \left\{ f \circ \varphi : \int_X |f|^2 d(\mu \circ \varphi^{-1}) < \infty \right\} = \{f \circ \varphi : f \in L^2(hd\mu)\}.\end{aligned}$$

Hence we have the following lemma.

Lemma 2.20 *Let $u \in \mathcal{D}(E_\varphi)$ and $\varphi^{-1}(\mathcal{A})$ be a σ-finite subalgebra of $\varphi^{-1}(\Sigma)$. Then the following assertions hold.*

(a) $hE_\varphi(f) \circ \varphi^{-1} = 0$ if and only if $E_\varphi(f) = 0$, for all $f \in L^2(\mathcal{A})$.

(b) If $E_\varphi(uf) = 0$ for all $f \in L^2(\mathcal{A})$, then $E_\varphi(u) = 0$.

(c) $\overline{\mathcal{R}(C_{\varphi|L^2(\mathcal{A})})} = \overline{\mathcal{R}(C_\varphi E_\varphi)} = \overline{C_\varphi(L^2(\mathcal{A}))} = L^2(\varphi^{-1}(\mathcal{A})) = \{f \circ \varphi : \int_X |f|^2 d(\mu \circ \varphi^{-1}) < \infty\} = \{f \circ \varphi : f \in L^2(hd\mu)\}$.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. If $T \in B(\mathcal{H})$, then T can be written 2×2 matrix with operator entries $[T_{ij}]_{1 \le i,j \le 2}$, where $T_{11} \in B(\mathcal{M})$, $T_{12} \in B(\mathcal{M}^{\perp}, \mathcal{M})$, $T_{21} \in B(\mathcal{M}, \mathcal{M}^{\perp})$ and $T_{22} \in B(\mathcal{M}^{\perp})$. $\mathcal{M}$ is said to be a reducing subspace for T if $\mathcal{M}$ is invariant subspace for T and T^*, or equivalently, $T_{12} = T_{12}^* = 0$.

Relative to the direct sum decomposition $L^2(\Sigma) = \mathcal{R}(E) \oplus \mathcal{N}_2(E)$, any element f of $L^2(\Sigma)$ can be written uniquely as $f = f_1 + f_2$ where $f_1 = E(f) \in L^2(\mathcal{A})$ and $f_2 = f - E(f) \in \mathcal{N}_2(E) = \{f \in L^2(\Sigma) : E(f) = 0\}$. Now, let $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$. Then the matrix representation of T and T^* with respect to the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}_2(E)$ are

$$T = \begin{bmatrix} M_{w_1u_1}C_\varphi & EM_{w_1u_2}C_\varphi \\ M_{w_2u_1}C_\varphi & M_{w_2}EM_{u_2}C_\varphi \end{bmatrix} \text{ and } T^* = \begin{bmatrix} C_\varphi^* M_{\overline{w_1u_1}} & EC_\varphi^* M_{\overline{w_2u_1}} \\ C_\varphi^* M_{\overline{w_1u_2}} & C_\varphi^* M_{\bar{u}_2} E M_{\bar{w}_2} \end{bmatrix}.$$

Consequently, $L^2(\mathcal{A})$ is a reducing subspace for T if and only if $M_{w_2u_1}C_\varphi : L^2(\mathcal{A}) \to \mathcal{N}_2(E)$ and $C_\varphi^* M_{\overline{w_1u_2}} : \mathcal{N}_2(E) \to L^2(\mathcal{A})$ are 0. Let $\varphi^{-1}(\mathcal{A}) = \mathcal{A} \subseteq \varphi^{-1}(\Sigma)$. Then by Lemma 2.20, $\overline{\mathcal{R}(C_{\varphi|L^2(\mathcal{A})})} = L^2(\mathcal{A})$. In this case we have

$$\begin{aligned} M_{w_2u_1}C_\varphi = 0 &\Longleftrightarrow w_2u_1(f \circ \varphi) = 0, \quad \forall f \in L^2(\mathcal{A}) \\ &\Longleftrightarrow w_2u_1 = 0 \qquad\qquad (\text{by Lemma} 2.20) \\ &\Longleftrightarrow (w - w_1)u_1 = wu_1 - w_1u_1 = 0 \\ &\Longleftrightarrow w\chi_{\sigma(u_1)} = w_1\chi_{\sigma(u_1)} \in L^0(\mathcal{A}) \end{aligned}$$

and

$$\begin{aligned} C_\varphi^* M_{\overline{w_1u_2}} = 0 &\Longleftrightarrow hE_\varphi(\overline{w_1u_2}f) \circ \varphi^{-1} = 0, \quad \forall f \in L^2(\mathcal{A}) \\ &\Longleftrightarrow E_\varphi(\overline{w_1u_2}f) = 0 \qquad (\text{by Lemma} 2.20) \\ &\Longleftrightarrow E_\varphi(w_1(u - u_1)) = E_\varphi(w_1u) - E_\varphi(w_1u_1) = 0 \\ &\Longleftrightarrow E_\varphi(u)w_1 = w_1u_1 \qquad (\text{since } \mathcal{A} \subseteq \varphi^{-1}(\Sigma)) \\ &\Longleftrightarrow E_\varphi(u)\chi_{\sigma(w_1)} = u_1\chi_{\sigma(w_1)} \in L^0(\mathcal{A}). \end{aligned}$$

These observations establish the following result.

Theorem 2.21 *Let $\varphi^{-1}(\mathcal{A}) = \mathcal{A}$ be a σ-finite algebra of Σ and let $T = M_w E M_u C_\varphi \in B(L^2(\Sigma))$. Then $L^2(\mathcal{A})$ is a reducing subspace of T if and only if*

$$\left\{w\chi_{\sigma(u_1)}, E_\varphi(u)\chi_{\sigma(w_1)}\right\} \subseteq L^0(\mathcal{A}).$$

In Theorem 2.21 if we take $\varphi = id$, then we have the following corollary.

Corollary 2.22 *Let $T_1 = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$. Then $\mathcal{R}(E) = L^2(\mathcal{A})$ is a reducing subspace of T_1 if and only if $w\chi_{\sigma(E(u))}$ and $u\chi_{\sigma(E(w))}$ are $\mathcal{A}$-measurable functions.*

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Data availability The data that support the findings of this study are available from the corresponding author, [M. R. Jabbarzadeh], upon reasonable request.

Declarations

Competing interests The authors declare no competing interests.

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