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Lambert conditional operators on C^*-algebras

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Abstract
In this paper, for each $a \in \mathcal{A}$ we introduce an algebra $\mathcal{K}_a \subseteq \mathcal{K}$ of bounded Lambert conditional operators on a unital C^*-algebra $\mathcal{A}$, which is defined in terms of the left multiplication operators and conditional expectations. The commutant of $\mathcal{K}$ is studied, as well as the question of when $\mathcal{K}$ is closed in the norm operator topology.

Keywords C^*-algebra · Conditional expectation · Commutant

Mathematics Subject Classification Primary 46L05 · 47B47

1 Introduction and preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. From now on, $B(\mathcal{A})$ will denote the Banach algebra of all bounded and linear maps defined on $\mathcal{A}$ and with values in $\mathcal{A}$. In addition, if $T \in B(\mathcal{A})$, then $\mathcal{N}(T)$ and $\mathcal{R}(T)$ will stand for the null space and the range of T, respectively. Note also that $I \in B(\mathcal{A})$ will denote the identity operator on $\mathcal{A}$. A linear mapping $T : \mathcal{A} \to \mathcal{B}$ is said to be positive, if $T(a) \geq 0$ whenever $a \geq 0$. Recall that an element a in a C^*-algebra $\mathcal{A}$ is called positive, and we write $a \geq 0$, if $a = a^*$ and the spectrum of a lies on the nonnegative real axis. A linear mapping $E : \mathcal{A} \to \mathcal{B}$ is called a projection if $E(b) = b$ for every $b \in \mathcal{B}$. In this case $E^2 = E$ and $\|E\| \geq 1$. Tomiyama in [18] proved that if E is a projection of norm 1 from $\mathcal{A}$ onto $\mathcal{B}$, then E is positive, $E(a^*)E(a) \leq E(a^*a)$ and $\mathcal{B}$-linear. A $\mathcal{B}$-linear projection $E : \mathcal{A} \to \mathcal{B}$ which is also a positive mapping, is called

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a conditional expectation. A positive linear mapping $T : \mathcal{A} \to \mathcal{B}$ is called faithful if $T(a^*a) = 0$ implies $a = 0$.

For each $a \in \mathcal{A}$, we denote by L_a the left multiplication operator on $\mathcal{A}$. Our interest in operators of the form L_aEL_b stems from the fact that such products tend to appear often in the study of those operators related to the conditional expectation E. These types of operators appear in [6], where it is shown that every contractive projection on certain L^1-spaces can be decomposed into an operator of the form L_aEL_b and a nilpotent operator. In [8] and [9] operators that are representable as products involving multiplications and conditional expectations are studied. Also, in [14], Moy has characterized all operators on L^p of the form EL_b and L_aEL_b. Some classical properties of the Lambert conditional operator L_aEL_b on L^2-spaces are characterized in [7, 10–12]. In general, the theory of conditional measures and conditional expectations is extremely rich and varied (see e.g. [1] for references).

Put $\mathcal{K} = \{L_aEL_b : a, b \in \mathcal{A}\}$. Then $\mathcal{K}$ is a subset of $B(\mathcal{A})$. Set $\mathcal{K}_a = \{L_aEL_b : b \in \mathcal{A}\}$ and $\mathcal{K}^b = \{L_aEL_b : a \in \mathcal{A}\}$. Then $\mathcal{K}_a$ and $\mathcal{K}^b$ are subalgebras of $B(\mathcal{A})$. In the next section we prove that $\mathcal{K}_a$ and $\mathcal{K}^b$ are closed in the norm operator topology and then, under some assumptions, we determine the commutant of $\mathcal{K}$.

2 Characterizations

We recall that a positive linear mapping $E : \mathcal{A} \to \mathcal{B}$ is said to be a conditional expectation when $E(\mathcal{A}) = \mathcal{B}$, $E(b) = b$ and $E(bac) = bE(a)c$ if $a \in \mathcal{A}$ and $b, c \in \mathcal{B}$. It follows that $E^2 = E$, $\|E\| = 1$, $E(1_\mathcal{A}) = 1_\mathcal{B}$, $E(x^*) = E(x)^*$ and $E(xE(a)) = E(E(x)a) = E(x)E(a)$ for all $x, a \in \mathcal{A}$. Note that the existence of a conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is a rich subject, and is often a very tricky matter (see e.g. [1, 2, 17]). Also, conditional expectations may not be unique. Every conditional expectation E from $\mathcal{A}$ onto $\mathcal{B}$ is a Schwarz mapping (see [15]), i.e., $E(x^*)E(x) \le E(x^*x)$ and $\|E(x^*a^*ax)\| \le \|a^*a\| \, \|E(x^*x)\|$ (see [16]) for all $x, a \in \mathcal{A}$. In Schwarz's inequality, equality holds if and only if E is multiplicative (see [3]). Henceforth we take $1_\mathcal{A} = 1$, $1_\mathcal{B} = e$ and $\mathcal{N}(E) = \mathcal{N}$. Note that the closed subspace $\mathcal{N}$ is a $\mathcal{B}$-module and $\mathcal{N}^2 \nsubseteq \mathcal{N}$, in general.

Definition 2.1 Let $S(\mathcal{A}|\mathcal{N}) = \{x \in \mathcal{A} : \mathcal{A}x \subseteq \mathcal{N}\}$. $\mathcal{N}$ is called a subspace of type zero if $S(\mathcal{A}|\mathcal{N}) = \{0\}$ and is of type one if $S(\mathcal{A}|\mathcal{N}) = \mathcal{N}$.

As an immediate consequence of Definition 2.1, $S(\mathcal{A}|\mathcal{N})$ is a closed left ideal of $\mathcal{A}$, $\{0, 1-e\} \subseteq S(\mathcal{A}|\mathcal{N}) \subseteq \mathcal{N}$ and $(\mathcal{A}|\mathcal{N})$ contains every left ideal of $\mathcal{A}$ which is contained in $\mathcal{N}$. Because $\mathcal{N}^* = \mathcal{N}$, $\mathcal{N}$ is a two-sided ideal whenever $\mathcal{N}$ is of type one. By using Peirce decomposition, we have $\mathcal{N} = e\mathcal{A}(1-e) \oplus (1-e)\mathcal{A}e \oplus (1-e)\mathcal{A}(1-e)$. The left annihilator of $e\mathcal{A}$ is $\mathcal{A}(1-e)$ and $e\mathcal{A}(1-e) \oplus (1-e)\mathcal{A}(1-e) \subseteq \mathcal{A}(1-e) \subseteq S(\mathcal{A}|\mathcal{N})$. So, if we take $\mathcal{B} = e\mathcal{A}e$ and define $E(a) = eae$ for all $a \in \mathcal{A}$, then $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}$. In Definition 2.1, if we take $S_1(\mathcal{A}|\mathcal{N}) = \{x \in \mathcal{A} : x\mathcal{A} \subseteq \mathcal{N}\}$, then $S_1(\mathcal{A}|\mathcal{N})$ is a closed right ideal of $\mathcal{A}$, $\{0, 1-e\} \subseteq S_1(\mathcal{A}|\mathcal{N}) \subseteq \mathcal{N}$ and $(1-e)\mathcal{A}e \oplus (1-e)\mathcal{A}(1-e) \subseteq (1-e)\mathcal{A} \subseteq S_1(\mathcal{A}|\mathcal{N})$.

Proposition 2.2 *The followings are equivalent:*

(i) *E is multiplicative.*
(ii) *$\mathcal{N}$ is an ideal.*
(iii) $S(\mathcal{A}|\mathcal{N}) = \mathcal{N}$.
(iv) $\|E(a^*a)\| = \|E(a)\|^2$, *for all* $a \in \mathcal{A}$.

Proof (i)$\Rightarrow$ (ii) Let $a \in \mathcal{A}$ and $x \in \mathcal{N}$. Then $E(ax) = E(a)E(x) = 0 = E(x)E(a) = E(xa)$. So $\mathcal{A}\mathcal{N} \subseteq \mathcal{N}$ and $\mathcal{N}\mathcal{A} \subseteq \mathcal{N}$.

(ii)$\Rightarrow$ (iii) Let $x \in \mathcal{N}$. Since $\mathcal{N}$ is an ideal, then $ax \in \mathcal{N}$ for all $a \in \mathcal{A}$. Thus, $x \in S(\mathcal{A}|\mathcal{N}) \subseteq \mathcal{N}$.

(iii)$\Rightarrow$ (v) Let $b \in \mathcal{A}$. Then $b - E(b) \in \mathcal{N} = S(\mathcal{A}|\mathcal{N})$, and so $a(b - E(b)) \in \mathcal{N}$ for all $a \in \mathcal{A}$. Hence $0 = E(a(b - E(b))) = E(ab) - E(a)E(b)$. In particular, $\|E(a^*a)\| = \|E(a)\|^2$, for all $a \in \mathcal{A}$.

(iv)$\Rightarrow$ (i) Let $a, b \in \mathcal{A}$. Then $\|E(a(b - E(b))\|^2 = \|E((b - E(b))^*a^*a(b - E(b))\| \leq \|a^*a\| \, \|E((b - E(b))^*(b - E(b))\| = \|a^*a\| \, \|E(b - E(b))\|^2 = 0$. Thus, $E(ab) = E(a)E(b)$. □

Let $S_0 = S_0(\mathcal{A}|\mathcal{B}) = \{x \in \mathcal{A} : \mathcal{A}ex \subseteq \mathcal{B}\}$ and $S_1 = \{x \in \mathcal{A} : \mathcal{A}x \subseteq \mathcal{B}\}$. $\mathcal{B}$ is a subalgebra of type zero if $S_0 = \{0\}$ and is of restricted type zero if $e \neq 1$ and $S_1 = \{0\}$. It follows that $S_1 \subseteq \mathcal{B} \cap S_0$ is a closed two-sided ideal of $\mathcal{A}$ and is the annihilator of $\mathcal{N}$ (see [4]).

Example 2.3 Let (X, Σ, μ) be a σ-finite measure space. For any sub-σ-finite algebra $\Sigma_1 \subseteq \Sigma$ and $1 \leq p \leq \infty$, the L^p-space $L^p(X, \Sigma_1, \mu|_{\Sigma_1})$ is abbreviated by $L^p(\Sigma_1)$. In this case one may identify $L^p(\Sigma_1)$ isometrically with a subspace of $L^p(\Sigma)$. Let $\mathcal{A} = L^\infty(\Sigma)$, $\mathcal{B} = L^\infty(\Sigma_1)$ and the mapping $E = E(.|\Sigma_1)$ be a classical conditional expectation operator from abelian C^*-algebra $\mathcal{A}$ onto $\mathcal{B}$. So for $f \in \mathcal{A}$, $\overline{E(f)} = E(\bar{f})$, $E(f)E(\bar{f}) = |E(f)|^2 \leq E(|f|^2)$ and if $E(f\bar{f}) = 0$, then $\int_X |f|^2 d\mu = \int_X E(|f|^2) d\mu = 0$, and hence $f = 0$. Thus, the probabilistic conditional expectations are always faithful. Also, $E = E^2 = E^*$ and so $\|E\| = 1$. Here any C^*-subalgebra $\mathcal{B}$ of $L^\infty(\Sigma)$ is of this form. In fact $\mathcal{B} = L^\infty(\Sigma_1)$ where $\Sigma_1 = \{A \in \Sigma : \chi_A \in \mathcal{B}\}$. Let $\varphi : X \to X$ be a non-singular measurable transformation, i.e. $\mu \circ \varphi^{-1} \ll \mu$. Here the non-singularity of φ guarantees that the operator $f \to f \circ \varphi$ is well defined. For each $n \in \mathbb{N}$, let $h_n = d\mu \circ \varphi^{-n}/d\mu$ be the Radon-Nikodym derivative. We also assume that h_n for each $n \in \mathbb{N}$ is finite-valued, or equivalently $\varphi^{-n}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra, $\mathcal{A}_n = L^\infty(\varphi^{-n}(\Sigma))$ and $E_n = E(.|\varphi^{-n}(\Sigma))$. Then $\{E_n : \mathcal{A} \to \mathcal{A}_n\}_n$ is a sequence of conditional expectations and for each $m \geq n$, $E_m E_n = E_n E_m = E_m = E_m^n E_n$ where E_m^n is a conditional expectation from $\mathcal{A}_n$ onto $\mathcal{A}_m$. Now, let $X = [-1, 1]$, $2d\mu = dx$, Σ be the Lebesgue sets and let Σ_1 be the sub-σ-algebra of Σ consisting of sets symmetric about the origin. Then for $f \in \mathcal{A}$, $E_0(f)$ is the even part of f and those in the kernel of $E_0 = E_0(.|\Sigma_1)$ are the odd functions. Define $E_1(f)(x) = f(|x|)$ (see [4]) and $E_2(f)(x) = f(-|x|)$. Then E_1, E_2 are distinct non-probabilistic conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$, $\mathcal{N}(E_1) = \{f \in \mathcal{A} : \chi_{[0,1]} f = 0\}$ and $\mathcal{N}(E_2) = \{f \in \mathcal{A} : \chi_{[-1,0]} f = 0\}$. For $i, j \in \{1, 2\}$, $E_0 E_i = E_i$, $2E_i E_0 = E_1 + E_2$ and $E_i E_j = E_j$. The null space of E_1 and E_2 are ideals in $\mathcal{A}$ but $\mathcal{N}^2(E_0) \subseteq \mathcal{B}$ and hence $\mathcal{N}(E_0)$ is not an ideal. In fact $S_0 = S_1 = \{0\}$, $S(\mathcal{A}, \mathcal{N}(E_0)) = \{0\}$ and for $i \in \{1, 2\}$, $S(\mathcal{A}, \mathcal{N}(E_i)) = \mathcal{N}(E_i)$. So, $\mathcal{B}$ and $\mathcal{N}(E_0)$ are of type zero and $\mathcal{N}(E_1)$ and

$\mathcal{N}(E_2)$ are of type one. Note that E_1 and E_2 are far from faithful and these conditional expectations play no role in the definition of S_0 and S_1.

Remark 2.4 Conditional expectations play an important role in classical probability theory (see [5]). Some properties of probabilistic conditional expectations cannot hold in our situation. For example, probabilistic conditional expectation of the idempotent element χ_A is zero if and only if $\chi_A = 0$. In fact, the support of $E(\chi_A)$ contains the support of χ_A (see [13]). However, if we define $E : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ as $E\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, then E is a conditional expectation with $S(M_2(\mathbb{C})|\mathcal{N}) = \{0\}$ and for the idempotent element $p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $E(p) = 0$.

For each $a \in \mathcal{A}$, we denote by R_a the left and right multiplication operators on $\mathcal{A}$. Define the linear operator $T_{a,b} : \mathcal{A} \to \mathcal{A}$ by $T_{a,b}(x) = aE(bx)$, where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation operator. It is clear that $T_{a,b} = L_a E L_b$ is linear and bounded and $\|aE(b)\| \le \|T_{a,b}\| \le \|a\|\,\|b\|$. Put $\mathcal{K} = \mathcal{K}(\mathcal{A}|\ \mathcal{B}; E) = \{T_{a,b} : a, b \in \mathcal{A}\}$. Then $\alpha T_{a,b} = T_{\alpha a,b} = T_{a,\alpha b}$, $T_{a,b}T_{c,d} = T_{aE(bc),d} = T_{a,E(bc)d}$, $T_{a,b} + T_{a,d} = T_{a,b+d}$ and $T_{a,b} + T_{c,b} = T_{a+c,b}$ for all $\{a, b, c, d\} \subset \mathcal{A}$ and $\alpha \in \mathbb{C}$. Thus, $\mathcal{K}$ is closed under the scalar multiplication and product operators, but not closed under addition. Note that for a fixed $a \in \mathcal{A}$, the mapping $K_a : \mathcal{A} \to \mathcal{K}_a$ given by $K_a(b) = T_{a,b}$ is linear with $\|ae\| \le \|K_a\| \le \|a\|$. Recall that $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}(E)$. So any $a \in \mathcal{A}$ can be written uniquely as $a = a_1 + a_2$, where $a_1 = E(a) \in \mathcal{B}$ and $a_2 = a - E(a) \in \mathcal{N}(E)$. Then $T_{a,b}(x) = a_1b_1x_1 + a_2b_1x_1 + a_1E(b_2x_2) + a_2E(b_2x_2)$ for all $x \in \mathcal{A}$. If $\mathcal{N}^2 \subseteq \mathcal{N}$, then $T_{a,b}(x) = (a_1b_1 + a_2b_1)x_1$.

Proposition 2.5 *For $a \in \mathcal{A}$, $K_a(S(\mathcal{A}|\mathcal{N}))\mathcal{K} = K_a(S_1(\mathcal{A}|\mathcal{N}))\mathcal{K} = \{0\}$ whenever $\mathcal{N}$ is of type one.*

Proof If $\mathcal{N}$ is of type one, then by Proposition 2.2, $\mathcal{N} = S(\mathcal{A}|\mathcal{N}) = S_1(\mathcal{A}|\mathcal{N})$ is a two-sided ideal. So for each $\{a, c, d\} \subset \mathcal{A}$ and $x \in \mathcal{N}$, $T_{a,x}T_{c,d} = T_{aE(xc),d} = T_{0,d} = 0$. It follows that for each $x \in \mathcal{N}$, $T_{a,x}$ is a left annihilator of $\mathcal{K}$. □

In the following, under some conditions, we obtain connections between $\mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_1)$ and $\mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_2) = \{S_{a,b} = L_a E_2 L_b : b \in \mathcal{A}\}$, where $E_1, E_2 : \mathcal{A} \to \mathcal{B}$ are distinct conditional expectations.

Proposition 2.6 *Let E_1, E_2 be multiplicative conditional expectations of $\mathcal{A}$ onto $\mathcal{B}$. For a fixed $a \in \mathcal{A}$, let $E_1(a) = E_2(a)$, $T_{a,x} \in \mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_1)$ and $S_{a,x} \in \mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_2)$ be defined by $T_{a,x}(y) = aE_1(xy)$ and $S_{a,x}(y) = aE_2(xy)$ for all $x \in \mathcal{A}$. Then there is an invertible operator G on $\mathcal{A}$ such that $G^{-1}T_{a,x}G = S_{a,G^{-1}(x)}$ and the mapping $\Gamma : T_{a,x} \to G^{-1}T_{a,x}G$ is an algebra isomorphism of $\mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_1)$ onto $\mathcal{K}_a(\mathcal{A}|\mathcal{B}; E_2)$ which is a homeomorphism.*

Proof Take $G = E_2 + 1 - E_1$. Then G is an invertible operator on $\mathcal{A}$ with $G^{-1} = E_1 + 1 - E_2$ (see [4]). Let $x, y \in \mathcal{A}$, then

$$G^{-1}T_{a,x}G(y) = G^{-1}(aE_1(x(E_2(y) + y - E_1(y))$$

$$
\begin{aligned}
&= (E_1 + 1 - E_2)(aE_1(x)E_2(y) + aE_1(xy) - aE_1(x)E_1(y)) \\
&= aE_1(x)E_2(y) + aE_1(xy) - aE_1(x)E_1(y) = aE_1(x)E_2(y)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
S_{a,G^{-1}x}(y) &= aE_2((E_1 + I - E_2)(x)y) = aE_2(E_1(x)y - E_2(x)y + xy) \\
&= aE_1(x)E_2(y) + aE_2(x)E_2(y) - aE_2(xy) = aE_1(x)E_2(y).
\end{aligned}
$$

Therefore $G^{-1}T_{a,x}G = S_{a,G^{-1}x}$ and for all $x_1, x_2 \in \mathcal{A}$ we have

$$
\begin{aligned}
\Gamma(T_{a,x_1}T_{a,x_2}) &= \Gamma(T_{a,E_1(x_1a)x_2}) = S_{a,G^{-1}(E_1(x_1a)x_2)} \\
&= S_{a,G^{-1}x_1}S_{a,G^{-1}x_2} = \Gamma(T_{a,x_1})\Gamma(T_{a,x_2}).
\end{aligned}
$$

So, Γ is a continuous algebra isomorphism and $\Gamma^{-1}(S_{a,G^{-1}x}) = GS_{a,G^{-1}x}G^{-1}$ is also continuous with respect to any of the operator topologies. □

Proposition 2.7 *Let $E : \mathcal{A} \to \mathcal{B}$ be a conditional expectation. Then $\mathcal{N}e + \mathcal{B} = \vee_{a,b\in\mathcal{A}}\mathcal{R}(T_{a,b})$, where $\vee$ denotes the algebraic span. Moreover, $\cap_{a\in\mathcal{B}}\mathcal{N}(T_{a,e}) = \mathcal{N}$.*

Proof Recall that for each $x \in \mathcal{A}$, $x_1 = E(x)$ and $x_2 = x - E(x)$. Then for each $a, b \in \mathcal{A}$ we have

$$
\begin{aligned}
T_{a,b}(x) &= (a_1 + a_2)E((b_1 + b_2)(x_1 + x_2)) \\
&= (a_1 + a_2)E(b_1x_1 + b_1x_2 + b_2x_1 + b_2x_2) \\
&= (a_1 + a_2)(b_1x_1 + E(b_2x_2)) \\
&= (a_2b_1x_1 + a_2E(b_2x_2)e + a_1b_1x_1 + a_1E(b_2x_2) \in \mathcal{N}e + \mathcal{B}.
\end{aligned}
$$

Hence $\vee_{a,b\in\mathcal{A}}\mathcal{R}(T_{a,b}) \subseteq \mathcal{N}e + \mathcal{B}$. Conversely, since for every $k \in \mathcal{N}$ and $b \in \mathcal{B}$, $ke = T_{k,1}(1)$ and $b = T_{e,b}(1)$, then $\mathcal{N}e + \mathcal{B} \subseteq \vee_{a,b\in\mathcal{A}}\mathcal{R}(T_{a,b})$. Now, let $T_{a,e}(x) = 0$ for all $a \in \mathcal{B}$. Take $a = e$. Then $E(x) = 0$ and so $x \in \mathcal{N}$. Conversely, if $x \in \mathcal{N}$ then $T_{a,e}(x) = aE(x) = 0$ for all $a \in \mathcal{B}$. So, $x \in \cap_{a\in\mathcal{B}}\mathcal{N}(T_{a,e})$. □

Theorem 2.8 *Let $x_0 \in \mathcal{N}$ have a right inverse and $\mathcal{N}x_0 \subseteq \mathcal{B}$. Then $\mathcal{K}_e(\mathcal{A}|\mathcal{B}; E) = \{EL_b : b \in \mathcal{A}\}$ is closed in the norm operator topology.*

Proof Let $T_n = EL_{b_n}$ and $\|T_n - T\| \to 0$ for some $T \in B(\mathcal{A})$. Then, relative to the direct sum decomposition $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}$ we have

$$
\lim_{n\to\infty} T_n = \lim_{n\to\infty} \begin{bmatrix} L_{b_{n1}} & EL_{b_{n2}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = T.
$$

where $b_{n1} = E(b_n)$ and $b_{n2} = b_n - b_{n1}$. Since for every $n \in \mathbb{N}$, $T_n(\mathcal{B}) \subseteq \mathcal{B}$ and $T_n(\mathcal{N}) \subseteq \mathcal{B}$, hence $T(\mathcal{B}) \subseteq \mathcal{B}$ and $T(\mathcal{N}) \subseteq \mathcal{B}$. Consequently $T_3 = T_4 = 0$. It is sufficient to calculate T_1 and T_2. Since $\lim_{n\to\infty} b_{n1} = \lim_{n\to\infty} E(b_n) = \lim_{n\to\infty} T_n(1) = T(1) := t \in \mathcal{B}$, then $T_1x_1 = \lim_{n\to\infty} L_{b_{n1}}x_1 = \lim_{n\to\infty} b_{n1}x_1 =$

$tx_1 = L_t x_1$. Thus $T_1 = L_t$. Now, we calculate T_2. Since x_0 has a right inverse, there exists $y_0 \in \mathcal{A}$ such that $x_0 y_0 = 1$. Then we have

$$\begin{aligned}(Tx_0)y_0 &= \lim_{n\to\infty} E(b_n x_0)y_0 = \lim_{n\to\infty}[E(b_{n1}x_0 + b_{n2}x_0)]y_0 \\ &= \lim_{n\to\infty}[b_{n1}E(x_0) + E(b_{n2}x_0)]y_0 \\ &= \lim_{n\to\infty} b_{n2}x_0 y_0 = \lim_{n\to\infty} b_{n2} \qquad (\text{since } \mathcal{N}x_0 \subseteq \mathcal{B}) \end{aligned} \tag{2.1}$$

Since $\mathcal{N}$ is a closed subspace, then $(Tx_0)y_0 := k \in \mathcal{N}$ and

$$T_2(x_2) = \lim_{n\to\infty} E(b_{n2}x_2) = E\Big(\lim_{n\to\infty} b_{n2}x_2\Big) = E(kx_2) = EL_k(x_2).$$

This implies that $T_2 = EL_k$. Put $b = t + k$, then

$$T = \begin{bmatrix} L_t & EL_k \\ 0 & 0 \end{bmatrix} = EL_{t+k} = EL_b \in \mathcal{K}_e(\mathcal{A}|\mathcal{B}; E).$$

□

Corollary 2.9 *Under assumptions of Theorem 2.8, if $a \in \mathcal{A}$ is invertible, then $\mathcal{K}_a(\mathcal{A}|\mathcal{B}; E) = \{L_a EL_b : b \in \mathcal{A}\}$ is closed in the norm operator topology.*

Proof Let $\lim_{n\to\infty} L_a EL_{b_n} = T$ for some $T \in B(\mathcal{A})$. Then $\lim_{n\to\infty} E(L_{b_n}) = L_{a^{-1}}T := T_1$. So there exists $b \in \mathcal{A}$ such that $T_1 = EL_b$ and consequently $T = L_a T_1 = L_a EL_b \in \mathcal{K}_a(\mathcal{A}|\mathcal{B}; E)$. □

Now, let $\|L_{b_n}E - T\| \to 0$ for some $T \in B(\mathcal{A})$. Then $EL_{b_n*} \to T^*$. Under assumptions of Theorem 2.8, there exists $c \in \mathcal{A}$ such that $T^* = EL_c$, and so $T = L_{c^*}E$. So we have the following corollary.

Corollary 2.10 *Under assumptions of Theorem 2.8, if $b \in \mathcal{A}$ is invertible, then $\mathcal{K}_b(\mathcal{A}|\mathcal{B}; E) = \{L_a EL_b : a \in \mathcal{A}\}$ is closed in the norm operator topology.*

Let $b \in \mathcal{B}$ and let R_b be the right multiplication operator on $\mathcal{A}$. Then for each $T_{a,c} \in \mathcal{K} = \{L_a EL_b : a, c \in \mathcal{A}\}$ we have $R_b T_{a,c}(x) = R_b(aE(cx)) = aE(cx)b$ and $T_{a,c}R_b(x) = aE(cxb) = aE(cx)b$. Thus, $R = \{R_b : b \in \mathcal{B}\} \subseteq \mathcal{K}' = \{T \in B(\mathcal{A}) : TT_{a,c} = T_{a,c}T,\ T_{a,c} \in \mathcal{K}\}$. In the following we determine the commutant of $\mathcal{K}$, under some assumptions.

One way to study operators is to see them as entries of simpler operators. Recall that $\mathcal{N} = \mathcal{N}(E) = e\mathcal{N} \oplus (1-e)\mathcal{N} = \mathcal{N}e \oplus \mathcal{N}(1-e)$ and $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}$. Then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, where $\mathcal{A}_1 = \mathcal{B}$, $\mathcal{A}_2 = \mathcal{N}e$ and $\mathcal{A}_3 = \mathcal{N}(1-e)$. Relative to this direct sum decomposition any operator T on $\mathcal{A}$ has the matrix form $(T_{i,j})$, $1 \le i, j \le 3$ where $T_{i,j} : \mathcal{A}_j \to \mathcal{A}_i$. If $e = 1$, then the matrix form of T is a 2×2 matrix.

Theorem 2.11 *Let $\mathcal{A}$ be a unital C^*-algebra with identity element 1 and $\mathcal{B}$ a proper C^*-algebra of $\mathcal{A}$ with unit e, and E a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{S}_1 = \{0\}$, then $T \in \mathcal{K}'$ if and only if T has the matricial form*

$$\begin{bmatrix} R_t & 0 & 0 \\ 0 & R_t & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$$

where $t = T(e)$ and R_t is right multiplication by t and T_{33} is arbitrary.

Proof Let $T \in \mathcal{K}'$, then for all $a, c, x \in \mathcal{A}$,

$$T(aE(cx)) = TT_{a,c}(x) = T_{a,c}T(x) = aE(cTx). \tag{2.2}$$

Also, let the matrix form of T relative to direct sum decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$ be given by

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

If $x \in \mathcal{N}$ and $c \in \mathcal{B}$, then $cx \in \mathcal{N}$ and so by (2.2), $acE(Tx) = 0$. Put $c = e$. Then for every $a \in \mathcal{A}$, we have $aE(Tx) = 0$. Thus, $E(Tx) \in \mathcal{S}_1 = \{0\}$. This implies that $T(\mathcal{N}) \subseteq \mathcal{N}$. It follows that $T_{12} = T_{13} = 0$. Now, we show that $T(\mathcal{A}_2) \subseteq \mathcal{A}_2$ where $\mathcal{A}_2 = \mathcal{N}e$. Put $T(e) = t$ and take $x = e$ and $a = c = 1$ in (2.2). Then we obtain $t = T(e) = E(T(e)) = E(t) \in \mathcal{B}$. Also, if we take $a = x = e$ in (2.2), then $T(E(c)) = E(ct)$. Again, put $c = 1$, $x = e$ in (2.2). Then we get that $T(ae) = at$, for each $a \in \mathcal{A}$. Since $te = t \in \mathcal{B}$, then for each $(E(c) - c)e \in \mathcal{A}_2$, $T\big((E(c) - c)e\big) = T\big(E(c)\big) - T(ce) = E(ct) - T(ce) = E(c)t - ct = (E(c) - c)t = (E(c) - c)te \in \mathcal{N}e$. This implies that $Tx = xt$ for every $x \in \mathcal{N}e$, and so $T(\mathcal{A}_2) \subseteq \mathcal{A}_2$. Consequently, $T_{32} = 0$ and $T_{22} = R_t$. Now, take $c = x = 1$ in (2.2) and let $a \in \mathcal{A}_1 = \mathcal{B}$. Then $T(a) = aE(t) = at = R_t(a)$, and hence $T_{11} = ETE = R_t$.

For calculation of T_{21}, T_{23}, T_{31} and T_{33}, we need the matrix form of $T_{a,b}$ relative to direct sum decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. Let $x \in \mathcal{A}$. Then $x = E(x) + (x - E(x))e + x(1 - e)$. It follows that

$$\begin{aligned} T_{a,b}(x) &= aE\big(bE(x)\big) + aE\big(b(x - E(x)e\big) + aE\big(bx(1 - e)\big) \\ &= ab_1E(x) + aE\big(b(x - E(x)\big) + aE\big(bx(1 - e)\big). \end{aligned} \tag{2.3}$$

Let $T_{a,b} = (A_{i,j})$ and P_i be a projection on $\mathcal{A}_i$ for $1 \leq i, j \leq 3$, i.e., $P_1(x) = x_1$, $P_2(x) = x_2e$ and $P_3(x) = x_2(1 - e)$. For $x = x_1 = P_1(x)$, $T_{a,b}(x_1) = ab_1x_1$ and hence

$$\begin{aligned} A_{11}(x_1) &= P_1(ab_1x_1) = a_1b_1x_1 = L_{a_1b_1}x_1; \\ A_{21}(x_1) &= P_2(ab_1x_1) = a_2b_1x_1 = L_{a_2b_1}x_1; \end{aligned}$$

$$A_{31}(x_1) = P_3(ab_1x_1) = a_2b_1x_1(1-e) = 0.$$

Now, put $x = x_2e$ in (2.3). Then $T_{a,b}(x_2e) = aE(bx_2e) = aE(b_2x_2)$. This implies that $A_{12}(x_2e) = P_1T_{a,b}(x_2e) = a_1E(b_2x_2) = L_{a_1}EL_{b_2}(x_2e)$. Similarly, $A_{22} = L_{a_2}EL_{b_2}$ and $A_{23}(x_2e) = P_3T_{a,b}(x_2e) = a_2E(b_2x_2)e(1-e) = 0$. Take $x = x_2(1-e)$ in (2.3). Then $T_{a,b}\big(x_2(1-e)\big) = 2aE(bx_2) - 2aE(bx_2e) = 0$, and hence $A_{13} = A_{23} = A_{33} = 0$. Consequently,

$$T_{a,b} = \begin{bmatrix} L_{a_1b_1} & L_{a_1}EL_{b_2} & 0 \\ L_{a_2b_1} & L_{a_2}EL_{b_2} & 0 \\ 0 & 0 & 0. \end{bmatrix};$$

$$\begin{aligned} T_{a,b}T &= \begin{bmatrix} L_{a_1b_1} & L_{a_1}EL_{b_2} & 0 \\ L_{a_2b_1} & L_{a_2}EL_{b_2} & 0 \\ 0 & 0 & 0. \end{bmatrix} \begin{bmatrix} R_t & 0 & 0 \\ T_{21} & R_t & T_{23} \\ T_{31} & 0 & T_{33}. \end{bmatrix} \\ &= \begin{bmatrix} L_{a_1b_1}R_t + L_{a_1}EL_{b_2}T_{21} & L_{a_1}EL_{b_2}R_t & L_{a_1}EL_{b_2}T_{23} \\ L_{a_2b_1}R_t + L_{a_2}EL_{b_2}T_{21} & L_{a_2}EL_{b_2}R_t & L_{a_2}EL_{b_2}T_{23} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$TT_{a,b} = \begin{bmatrix} R_tL_{a_1b_1} & R_tL_{a_1}EL_{b_2} & 0 \\ T_{21}L_{a_1b_1} + R_tL_{a_2b_1} & T_{21}L_{a_1}EL_{b_2} + R_tL_{a_2}EL_{b_2} & 0 \\ T_{31}L_{a_1b_1} & T_{31}L_{a_1}ELb_2 & 0 \end{bmatrix}.$$

Note that $a_iE(b_jx)t = a_iE(b_jxt)$ for each $x \in \mathcal{A}$, $t \in \mathcal{B}$ and $1 \leq i, j \leq 2$. Thus, $R_tL_{a_i}EL_{b_j} = L_{a_i}EL_{b_j}R_t$. This implies that $T_{23} = T_{21} = T_{31} = 0$ and T_{33} is arbitrary. Thus, we deduce that if $T \in \mathcal{K}'$, then

$$T = \begin{bmatrix} R_t & 0 & 0 \\ 0 & R_t & 0 \\ 0 & 0 & T_{33} \end{bmatrix}. \tag{2.4}$$

Conversely, if T has the form given in (2.4), direct calculation shows that $T \in \mathcal{K}'$. □

Example 2.12 Let $\mathcal{A} = \{(a_{ij})_{3\times3} : a_{ij} \in \mathbb{C}\}$ be the algebra of 3×3 matrices with complex entries and $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\}.$$

Define $E : \mathcal{A} \to \mathcal{B}$ by $E(a) = eae$. Then

$$E(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

and

$$\mathcal{N}=\left\{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{bmatrix};\ x_i, y_1, z_1 \in \mathbb{C}, 1 \leq i \leq 3\right\}.$$

Recall that $x \in \mathcal{S}_0$ if and only if for each $a \in \mathcal{A}$,

$$\begin{aligned} aex &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3. \end{bmatrix} \\ &= \begin{bmatrix} a_{12}x_2 + a_{13}x_3 & a_{12}y_2 + a_{13}y_3 & a_{12}z_2 + a_{13}z_3 \\ a_{22}x_2 + a_{23}x_3 & a_{22}y_2 + a_{23}y_3 & a_{22}z_2 + a_{23}z_3 \\ a_{32}x_2 + a_{33}x_3 & a_{32}y_2 + a_{33}y_3 & a_{32}z_2 + a_{33}z_3 \end{bmatrix} \in \mathcal{B}, \end{aligned}$$

if and only if $x_2 = x_3 = 0$, $y_2 = y_3 = 0$ and $z_2 = z_3 = 0$. Thus

$$\mathcal{S}_0 = \left\{\begin{bmatrix} x_1 & y_1 & z_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} :\ x_1, y_1, z_1 \in \mathbb{C}\right\}.$$

Also, since $\{0\} \subseteq S_1 \subseteq \mathcal{B} \cap S_0$ then $\mathcal{S}_1 = \{0\}$. Thus, $\mathcal{B}$ is of restricted type zero. Let $b = (b_{ij}) \in \mathcal{A}$. Then

$$E(bx) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{21}y_1 + b_{22}y_2 + b_{23}y_3 & b_{21}z_1 + b_{22}z_2 + b_{23}z_3 \\ 0 & b_{31}y_1 + b_{32}y_2 + b_{33}y_3 & b_{31}z_1 + b_{32}z_2 + b_{33}z_3 \end{bmatrix}.$$

If for $j_0 \in \{1, 2, 3\}$, $\{b_{2j_0}, b_{3j_0}\} \neq \{0\}$, then

$$\mathcal{N}(T_{1,b}) = \left\{\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{bmatrix};\ x_i \in \mathbb{C}, 1 \leq i \leq 3\right\} = S(\mathcal{A}|\mathcal{N}).$$

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