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SOME APPLICATIONS IN SOLVING DIFFERENTIAL EQUATIONS

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Abstract. We study a well-known problem concerning a random variable uniformly distributed between two independent random variables. A new extension has been introduced for this problem and fairly large classes of randomly weighted average distributions are identified by their generalized Stieltjes transforms. In this article we employ the Stieltjes transforms for finding the distributions of the random vector in question.

1. Introduction and preliminaries

Recently [1] introduced the notion of a random variable Z Beta distributed between two independent random variables X_1 and X_2, which arose in studying the distribution of products of random 2×2 matrices for stochastic search of global maxima studied by [2]. By letting X_1 and X_2 to have identical distributions, he derived that: (i) for X_1 and X_2 on $[-1, 1]$, Z is uniform on $[-1, 1]$ if and only if X_1 and X_2 have arcsin distribution; and (ii) Z possesses the same distribution as X_1 and

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X_2 if and only if X_1 and X_2 are degenerated or have a Cauchy distribution, but [4] extended some results as follows: They let $X_1, \cdots, X_n$ be independent, and considered for $n \geq 2$

$$S_n(R_1, ..., R_{n-1}) = R_1X_1 + R_2X_2 + \cdots + R_{n-1}X_{n-1} + R_nX_n,$$

where random proportions are $R_i = U_{(i)} - U_{(i-1)}$ (for any $i \in \{1, ..., n-1\}$) and $R_n = 1 - \sum_{i=1}^{n-1} R_i$; $U_{(1)}, ..., U_{(n-1)}$ are order statistics from a uniform distribution on $[0, 1]$, and $U_{(0)} = 0$. The R_i's $(i = 1, \ldots, n)$ are sometimes called the *spacings*; Following this [1] defined fairly large classes of randomly weighted average $(\mathsf{RWA})_n$ distributions by using new random weights. These are cuts of the interval $[0, 1]$ by $U_{(k_1)}, ..., U_{(k_{n-1})}$ for $k_0 = 0 < k_1 < \cdots < k_{n-1} < k_n = n^*$ in $\{0, 1, \ldots, n^*\}$, where

$$R_{k_j} = U_{(k_j)} - U_{(k_{j-1})}\big(j = 1, \cdots, n\big), U_{(k_0)} = 0, U_{(k_n)} = 1,$$

and

$$S_{n^*}(R_{k_1}, ..., R_{k_{n-1}}) = \sum_{j=1}^{n} R_{k_j} X_j.$$

Using the properties of the Beta distribution, [6] solved some differential equations. Indeed, the given equations may be solved by difficult methods and very long calculations. Let us recall that a Theorem of [6] identifies the distribution of (the 1-dimensional) Z from the distributions of X_i's by means of the following differential equation:

$$\frac{(-1)^{n^*-1} d^{n^*-1}}{(n^*-1)! dzn^* - 1} S(F_Z, z) = \prod_{i=1}^{n} \frac{(-1)^{m_i-1}}{(m_i - 1)!} \frac{d^{m_i-1}}{dz^{m_i-1}} S(F_{X_i}, z) \qquad (1.1)$$

where F_Z denote the cumulative distribution function of a random variable Y and $S(F_Z, z)$ is Stieljes transform defined by

$$S(H, z) = \int \frac{1}{z - x} H(dx), \ z \in \mathbf{C} \cap (\mathsf{supp} H)^c. \qquad (1.2)$$

Here, $\mathsf{supp} H$ is the support of H; see [6]. Where $m_j = k_j - k_{j-1}, \ j = 1, ..., n, \ k_n = n^*, \ \sum_{j=1}^{n} m_j = n^*$.

2. Discussion

In this section, we will complete the discussion(Page 4) in [6]. We answer the question raised in the last article and we state the next theorem like the one there. In this section, next to the emphasized method [6], The multivariate c-characteristic function (MCF) is introduced.

Definition 2.1. For random vector $u = (U_1, \cdots, U_k)$, its MCF is defined as

$$g(t_1, \cdots, t_k; u, c) = E\left\{(1 - it_1u_1 - \cdots - it_ku_k)^{-c}\right\}$$

where c is a positive real number.

See for more details, [6]. The following is the main theorem of this section that MCF of the random convex combination with respect to those of $\mathbf{X}_1, \ldots, \mathbf{X}_n$.

Theorem 2.2. *For independent and continuous random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, and F denote the distribution of $Z(= \sum_{i=1}^{n} \frac{Y_i}{\sum_{i=1}^{n} Y_i}\boldsymbol{X}_i)$, the following equality holds:*

$$g(t_1, \cdots, t_k; Z, c) = \prod_{j=1}^{n}\prod_{r=1}^{k} g(t_1, \cdots, t_k; X_{jr}, c_j). \tag{2.1}$$

where $g(.)$ is MCF.

3. Some applications in solving differential equations

From the moment generating function, it can be shown that the sum of the independent random variables are in the form of the product of the moment generating random variables, but here we can not use this property because of the presence of the random weights. Our main theorem shows that using the MCF again creates the same property.

For the notation and proof of the following corollary, we refer the reader to [6].

Corollary 3.1. *: For bounded support, independent and continuous random variables $X_1, \ldots, X_n$, and F denote the distribution of Z, the following equality holds:*

$$\frac{(-1)^{n^*-1}}{(n^*-1)!}\frac{d^{n^*-1}}{dz^{n^*-1}}S(F_Z, z) = \prod_{i=1}^{n}\frac{(-1)^{m_i-1}}{(m_i-1)!}\frac{d^{m_i-1}}{dz^{m_i-1}}S(F_{X_i}, z),$$

where $S(F, z)$ is a Stieltjes transformation.

Application

3.1. **In statistics.** Some results from the last few decades show that some researchers are interested in obtaining the distribution of $\mathbf{Z}$, the random convex combination, and in more detail, it can be seen that they are still interested in finding the distribution of $\mathbf{X}_i$ while assuming the distribution of $\mathbf{Z}$ is known, for example see [6].

By considering our main theorem and equation 2 in [1], one can obtain the distribution of $\mathbf{Z}$ or $\mathbf{X}_i$'s by placing MCF and solving a differential equation, of course if can solve it. This equation can play a fundamental role in obtaining the distribution of $\mathbf{Z}$ provided that an according differential equation can be solved.

3.2. **In mathematics.** The main theorem may be useful to solve some differential equations and some interesting mathematical facts which may be very difficult to solve.
In this subsection, we change our view of this theorem.
First, we assume equation 2 in [1], then we recognize the solutions of the equation from the main theorem of [1], so we were able to use the results of the main theorem to obtain the answer of equation 2 in [1].

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