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A NOTE ON PRODUCT OF RANDOM STOCHASTIC MATRICES

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ABSTRACT. In this article, we have proposed a new model for a real lifetime, which can be used in topics such as vehicle speed, asphalt, etc. We discuss the distributional properties of this model, and it has been used to the generalization of Nadarajah and Kotz distribution.

1. INTRODUCTION

Multiple stochastic matrices have long been the focus of many researchers in applied and theoretical disciplines, and significant results have been published in books and journals, but recently the close relationship between this discussion and average dynamics has been suggested by some authors. This has intensified attention to the issue. They see average dynamics as an essential role in studying changes in distributed systems and algorithms, as well as providing examples. On the other hand, paralleled, in the last few decades, authors have also

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argued about finding the distribution of a mixed random variable that does not seem to be irrigated to multiplication of random matrices. Example given in these works are lifetime. So the results will undoubtedly be in line with assumptions that are consistent with the examples. In this article, by changing our perspective on the interesting work of [5] and accepting their assumptions, we have discussed and obtained the results for multiplying random matrices of aspects which that are very different from their work and those of others.

In this paper $\mathbf{S}$ is a random combination of random matrices

$$\mathbf{S} = \sum_{j=1}^n \mathbf{Y}_j \mathbf{X}_j$$

where Y_j , $j = 1, \dots, n$ has exponential distribution with parameter (α_i) , and $\mathbf{X}_j$'s are Dirichlet distribution with parameter $\tilde{\beta}_j$, and $\bar{\mathbf{S}}$ is randomly weighted averages of random matrices

$$\bar{\mathbf{S}} = \sum_{j=1}^n \mathbf{W}_j \mathbf{X}_j.$$

Where $(W_1, \dots, W_n)$ has a Dirichlet distribution with parameter $(\alpha_1, \dots, \alpha_n)$. (for symbols see [1], [2], [5] [3])

2. PRODUCT OF RANDOM STOCHASTIC MATRICES

Theorem 2.1. *Let $\mathbf{X}$ be any random matrix with bounded support and Y be independent random variable of $\mathbf{X}$ with $Ga(\sum_{j=1}^n \alpha_j)$ distribution. If*

$$\mathbf{S}(\alpha_1, \dots, \alpha_n; \tilde{\beta}_1, \dots, \tilde{\beta}_n) \stackrel{d}{=} Y \mathbf{X},$$

then $\mathbf{X}$ and $\bar{\mathbf{S}}(\alpha_1, \dots, \alpha_n; \tilde{\beta}_1, \dots, \tilde{\beta}_n)$ have identical distribution.

Proof. At first we define $Y^+ = \sum_{i=1}^n Y_i$ (and $\alpha^+ = \sum_{j=1}^n \alpha_j$), which has $Ga(\alpha^+)$ distribution, then by use of one of assumptions we have

$$Y^+ \cdot \frac{\sum_{i=1}^n Y_i \mathbf{X}_i}{Y^+} \stackrel{d}{=} Y \mathbf{X},$$

the fraction $\frac{\sum_{i=1}^n Y_i \mathbf{X}_i}{Y^+}$ has the same distribution as $\bar{\mathbf{S}}(\alpha_1, \dots, \alpha_n; \tilde{\beta}_1, \dots, \tilde{\beta}_n)$, so we can rewrite the mentioned expression in the form of

$$Y^+ \bar{\mathbf{S}} \stackrel{d}{=} Y \mathbf{X},$$

then both sides have the same moments.

$$E((Y^+)^{k_1} S_{11}^{k_{11}} \cdot (Y^+)^{k_{12}} S_{12}^{k_{12}} \dots Y^{+k_n} S_{nn}^{k_{nn}}) = E(Y^{k_{11}} X_{11}^{k_{11}} \cdot Y^{k_{12}} X_{12}^{k_{12}} \dots Y^{k_{nn}} X_{nn}^{k_{nn}})$$

and so $E((Y^+)^{k^+})E(S_{11}^{k_{11}}.S_{12}^{k_{12}}\dots S_{nn}^{k_{nn}}) = E(Y^{k^+})E(X_{11}^{k_{11}}.X_{12}^{k_{12}}\dots X_{nn}^{k_{nn}})$ where $k^+ = \sum_{j=1}^n \sum_{i=1}^n k_{ij}$.

Considering the same distribution of Y^+ and Y , we can omit the first expectations from both sides of the equation

$$E((Y^+)^{k^+})E(S_{11}^{k_{11}}.S_{12}^{k_{12}}\dots S_{nn}^{k_{nn}}) = E(Y^{k^+})E(X_{11}^{k_{11}}.X_{12}^{k_{12}}\dots X_{nn}^{k_{nn}})$$

As a result of having bounded support variables, the equation of the same moments of two variables conduces to the same distribution, so the proof is completed and $\mathbf{X}$ and $\bar{\mathbf{S}}(\alpha_1, \dots, \alpha_n; \tilde{\beta}_1, \dots, \tilde{\beta}_n)$ have identical distribution. $\square$

$\square$

Theorem 2.2. *If random coefficient environment $\mathbf{X}_i$'s are identically distributed independent random matrices, then components*

$$\mathbf{S}(\mathbf{n}\alpha, \dots, \mathbf{n}\alpha; \tilde{\beta}_1, \dots, \tilde{\beta}_n)$$

have independent gamma(α) distribution if and only if $\tilde{\beta}_1 = \dots = \tilde{\beta}_n = (\alpha, \dots, \alpha)$.

Proof.

$$\begin{aligned} E(e^{t'_1 \mathbf{S}^{(1)} + t'_2 \mathbf{S}^{(2)} + \dots + t'_n \mathbf{S}^{(n)}}) &= E(e^{t'_1 \sum_{i=1}^n Y_i \mathbf{X}_i^{(1)}}) \dots E(e^{t'_n \sum_{i=1}^n Y_i \mathbf{X}_i^{(n)}}) \\ &= \prod_{i=1}^n E(e^{t'_1 Y_i \mathbf{X}_i^{(1)}}) \prod_{i=1}^n \dots \prod_{i=1}^n E(e^{t'_n Y_i \mathbf{X}_i^{(n)}}) \\ &= \prod_{i=1}^n E(E(e^{t'_1 Y_i \mathbf{X}_i^{(1)}} | \mathbf{X}_i)) \prod_{i=1}^n \\ &\quad \dots \prod_{i=1}^n E(E(e^{t'_n Y_i \mathbf{X}_i^{(n)}} | \mathbf{X}_i)) \\ &= \prod_{i=1}^n E\left(\left(\frac{1}{1 - t'_1 \mathbf{X}_i^{(1)}}\right)^\alpha\right) \prod_{i=1}^n \dots \prod_{i=1}^n E\left(\left(\frac{1}{1 - t'_n \mathbf{X}_i^{(n)}}\right)^\alpha\right) \\ &= \left(E\left(\left(\frac{1}{1 - t'_1 \mathbf{X}_i^{(1)}}\right)^\alpha\right)\right)^n \dots \left(E\left(\left(\frac{1}{1 - t'_n \mathbf{X}_i^{(n)}}\right)^\alpha\right)\right)^n \end{aligned}$$

The last statement is the Stieltjes transformation which is unique, so leads to the prove of both if part and only if part. $\square$

Theorem 2.3. $\bar{\mathbf{S}}(\sum_{i=1}^k \alpha_i^{(1)}, \dots, \sum_{i=1}^k \alpha_i^{(n)}; \underline{\alpha}^{(1)}, \dots, \underline{\alpha}^{(n)})$ has the

$$MDirichlet\left(\sum_{j=1}^n \alpha_1^{(j)}, \dots, \sum_{j=1}^n \alpha_k^{(j)}\right)$$

matrix distribution, where $\underline{\alpha}^{(j)} = \langle \alpha_1^{(j)}, \dots, \alpha_k^{(j)} \rangle$ ($j = 1, \dots, n$).

Theorem 2.4. $\bar{\mathbf{S}}(\mathbf{k}\alpha, \dots, \mathbf{k}\alpha; \tilde{\beta}_1, \dots, \tilde{\beta}_n)$ has $MDirichlet(n\alpha, \dots, n\alpha)$ distribution if and only if $\tilde{\beta}_1 = \dots = \tilde{\beta}_n = (\alpha, \dots, \alpha)$.

3. CONCLUSIONS

In this paper, a novel method for obtaining the distribution of the randomly weighted averages on random matrix is presented, which is simpler and more elementary than the others. Beside that one can obtain the distribution of $T = \sum \mathbf{X}_i Y_i$ by that method, which is left to be done in the future. In case this distribution appears to be complicated, we will approximate it by simulation, and we will study some distributional properties of T in general. The four examples illustrated in [6] (Cauchy Composition Test, Real Lifetime, Solving Some Differential Equations, and Random Convex Combination) are some visible applications of the research in this paper.

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