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ON THE CO-DEDEKINDIAN FINITE p-GROUPS WITH NON-CYCLIC ABELIAN SECOND CENTRE*

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Abstract. A group G is called *co-Dedekindian* if every subgroup of G is invariant under all central automorphisms of G. In this paper we give some necessary conditions for certain finite p-groups with non-cyclic abelian second centre to be co-Dedekindian. We also classify 3-generator co-Dedekindian finite p-groups which are of class 3, having non-cyclic abelian second centre with $|\Omega_1(G^p)| = p$.

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1. Introduction. Let G be a group, and let $Z(G)$ denote the centre of G. An automorphism α of G is called *central* if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The set of all central automorphisms of G, denoted by $Aut_c(G)$, is a normal subgroup of the full automorphism group of G. A group G is called *co-Dedekindian* ($\mathscr{C}$-group for short) if every subgroup of G is invariant under all central automorphisms of G. In [**1**], Deaconescu and Silberberg give a Dedekind-like structure theorem for the non-nilpotent $\mathscr{C}$-groups with trivial Frattini subgroup and by reducing the finite nilpotent $\mathscr{C}$-groups to the case of p-groups they obtain the following theorem.

THEOREM 1.1. *Let G be a p-group. If G is a non-abelian $\mathscr{C}$-group, then $Z_2(G)$ is a Dedekindian group. If $Z_2(G)$ is non-abelian, then $G \cong Q_8$. If $Z_2(G)$ is cyclic, then $G \cong Q_{2^n}$, $n \geq 4$, where Q_{2^n} is the generalized quaternion group of order 2^n.*

In [**1**], the authors notice that non-abelian p-groups with abelian non-cyclic second centre and which are $\mathscr{C}$-groups do exist. They show that if G is a non-abelian $\mathscr{C}$-group of order p^4, with $Z_2(G)$ abelian and non-cyclic, then $p = 3$ and

$$G = \langle a, b \mid a^9 = 1,\ b^3 = a^6,\ [a,b]^3 = 1,\ [a,[a,b]] = a^3,\ [b,[a,b]] = 1 \rangle.$$

The purpose of this paper is (1) to find some necessary conditions for certain p-groups with abelian non-cyclic second centre to be $\mathscr{C}$-groups and (2) to classify the

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3-generator $\mathcal{C}$-groups G satisfying these conditions with the additional condition $c\ell(G) = 3$.

Finally we show that given any natural number $m \geq 3$, there is a 2-group with abelian non-cyclic second centre which is a $\mathcal{C}$-group of class m.

Our notation is standard. We refer in particular to [**6**].

2. General results. In this section we first give some results that will be used later. Throughout the paper G will stand for a finite non-abelian p-group. If $\alpha \in Aut_c(G)$, we shall denote $F_\alpha = \{x \in G | \alpha(x) = x\}$ and $K_\alpha = \langle x^{-1}\alpha(x) | x \in G\rangle$. Also we put $F = \bigcap_{\alpha \in A} F_\alpha$, where $A = Aut_c(G)$, and $K = \langle K_\alpha | \alpha \in Aut_c(G)\rangle$. We now collect some information about the subgroups F and K of G.

LEMMA 2.1. *Let G be a $\mathcal{C}$-group.*
(i) $\Omega_1(G) \leq F \leq \Phi(G)$;
(ii) *if* $|G : \Phi(G)| > |G^p \cap \Omega_1(G)|$, *then G is not regular.*

Proof. (i) By [**1**, Lemma 3.1], we have $\Omega_1(G) \leq F$. Now let M be any maximal subgroup of G, and let z be an element of order p in $M \cap Z(G)$. Let $x \notin M$, and define $\alpha : G \to G$ by $\alpha(x^i m) = x^i m z^i$, where $i \in \{0, 1, ..., p-1\}$ and $m \in M$. It is easy to see that $\alpha \in Aut_c(G)$ and $F_\alpha = M$. Hence $F \leq \Phi(G)$.

(ii) Since $\Omega_1(G) \leq \Phi(G)$, we have $\Omega_1(G)G^p \leq \Phi(G) \underset{\neq}{\leq} G$. This shows that $|\Omega_1(G)||G^p| \leq |\Phi(G)||G^p \cap \Omega_1(G)| < |G|$. Hence G is not regular by [**6**, Chapter 4, Theorem 3.14(iv)] . □

PROPOSITION 2.2. *Let G be a finite non-abelian p-group. If G is a $\mathcal{C}$-group, then $Z(G)$ is cyclic and $Z(G) \leq \Phi(G)$.*

Proof. Let M be any maximal subgroup of G and let u be an element of order p in $Z(G)$ and $x \notin M$. By considering the central automorphism α defined in the proof of Lemma 2.1(i), we have $\alpha(x) = xu$. Since G is a $\mathcal{C}$-group, $u \in \langle x\rangle$. Hence $|\Omega_1(Z(G))| = p$, from which we conclude that $Z(G)$ is cyclic. Next we let g be an element of $G\backslash(F \cup Z(G))$. Since G is a $\mathcal{C}$-group and $g \notin F$, there is an $l \in \mathbb{N}$ such that $g^{p^l} \neq 1$ and $g^{p^l} \in Z(G)$. We define l_g to be the least positive integer such that $g^{p^{l_g}} \in Z(G)$. We then have $g^{p^{l_g}} = z^{p^{k_g}}$, where z is a generator of $Z(G)$ and k_g is a non-negative integer. We claim that $l_g > k_g$ for some element g of $G\backslash(F \cup Z(G))$. Denying this assertion, we may write $(g^{-1}z^{p^{k_g - l_g}})^{p^{l_g}} = 1$. Now as $g^{-1}z^{p^{k_g - l_g}} \notin Z(G)$, we must have $g^{-1}z^{p^{k_g - l_g}} \in F$: for, put $a = g^{-1}z^{p^{k_g - l_g}}$ and assume $a \notin F$; this implies $a^{p^{l_g - 1}} \in Z(G)$ and so $g^{p^{l_g - 1}} \in Z(G)$, which is contrary to the minimality assumption. Hence $g \in Z(G)F$, showing that $G = Z(G)F$. It follows that G/F is cyclic; so is $G/\Phi(G)$, giving a contradiction. Now $l_g > k_g$ leads to $z^{-1}g^{p^{l_g - k_g}} \in F$, by a similar argument. However, $g^{p^{l_g - k_g}} \in \Phi(G)$, which implies that $z \in \Phi(G)$. □

LEMMA 2.3. *Let G be a finite non-abelian p-group with $|\Omega_1(G^p)| = p$. If G is a $\mathcal{C}$-group, then G^p is cyclic. Moreover, if $Z_2(G)$ is non-cyclic and abelian, then p is odd.*

Proof. It is clear that if p odd, then G^p is cyclic. Now suppose that $p = 2$. We have $\Phi(G) = G^2$ and hence $\Omega_1(G) \leq G^2$, by Lemma 2.1(i). It follows that $G \cong \mathcal{Q}_{2^n}$, as

$|\Omega_1(G^2)| = 2$. Therefore G^2 is cyclic. However, in this case $Z_2(G)$ is cyclic or non-abelian, completing the proof. □

THEOREM 2.4. *Let G be a finite non-abelian p-group with a non-cyclic abelian second centre $Z_2(G)$. Suppose that $|\Omega_1(G^p)| = p$. If G is a $\mathscr{C}$-group, then $G^p = Z(G)$ and $Z_2(G) \leq \Phi(G)$.*

Proof. By Lemma 2.3, G^p is cyclic and p is odd. Let $G^p = \langle a \rangle$. We first show that $G^p = Z(G)$. The proof is divided into three steps.

Step 1. If $g \in G \backslash \Phi(G)$, then $G^p = \langle g^p \rangle$.

Suppose that $g^p = a^{lp^i}$, where $(l, p) = 1$ and i is a positive integer. Since $[a^l, g^{-1}] \in [G^p, g^{-1}]$ and $[G^p, g^{-1}]$ is properly contained in G^p, we have $[a^l, g^{-1}] \in \langle a^p \rangle$ and, consequently, $[a^{lp^{i-1}}, g^{-1}] \in \langle a^{p^i} \rangle \leq Z(\langle a, g^{-1} \rangle)$. Thus $(a^{lp^{i-1}} g^{-1})^p = a^{lp^i} g^{-p} [a^{lp^i}, g^{-1}]^{(p-1)/2} = 1$. We now have $a^{lp^{i-1}} g^{-1} \in \Omega_1(G)$, from which we get $g \in \Phi(G)$, a contradiction.

Step 2. $G^p \leq Z(G)$.

Suppose that G^p is not contained in $Z(G)$, so that $aZ(G) \neq Z(G)$. For any minimal generating set $\{y_i Z(G)\}$ of $G/Z(G)$, we have $y_i \notin \Phi(G)$ for each i. Hence, by Step 1, $y_i^{pn_i} = a$ for some positive integer n_i. Thus for each i, $y_i^{pn_i} Z(G) = aZ(G)$, contrary to [**5**, 3.2.10]. Hence $G^p \leq Z(G)$.

Step 3. If $g \in G \backslash \Phi(G)$ then $Z(G) = \langle g^p \rangle$, and hence $G^p = Z(G)$.

If $g^p = z^p$ for some $z \in Z(G)$, then gz^{-1} has order p and so $gz^{-1} \in \Phi(G)$. It follows, by Proposition 2.2, that $g \in \Phi(G)$, a contradiction. Hence $G^p = Z(G)$.

To prove the second part of the theorem, we assume that $x \in Z_2(G) \backslash \Phi(G)$. Thus $y^p = x^{lp}$ for each $y \in G \backslash \Phi(G)$, where $(l, p) = 1$ (because in view of Step 1, x^p and y^p are generators of G^p.) Hence

$$(yx^{-l})^p = y^p x^{-lp} [x^{-l}, y]^{p(p-1)/2} = [x^{-l}, y^p]^{(p-1)/2} = 1.$$

Therefore $yx^{-l} \in \Phi(G)$, whence $G/\Phi(G)$ is cyclic, a contradiction. □

The following result will be used throughout the sequel.

LEMMA 2.5. *Let G be a metabelian group. If x, y are elements of G and $n \in \mathbb{N}$, then*

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2} [\eta_2, x][\eta_1, y],$$

for some $\eta_1, \eta_2 \in G'$.

Proof. This is a special case of P. Hall's formula and is easily proved by using the identity $xy = yx[x, y]$. □

THEOREM 2.6. *Let G be a finite metabelian p-group with a non-cyclic abelian second centre $Z_2(G)$. Suppose that $|\Omega_1(G^p)| = p$. If G is a $\mathscr{C}$-group, then $|Z(G)| = p$. Hence $\Phi(G) = G'$ and $Z_2(G)$ is elementary abelian.*

Proof. According to Theorem 2.4, $Z(G) = G^p$. We first suppose that there exists an element x in $\Phi(G)$ such that $Z(G) = \langle x^p \rangle$. Then we may choose an element y in $G \backslash \Phi(G)$ with $x^p = y^p$. Since $x \in \Phi(G)$ and $\Phi(G)$ is abelian, we have

$$(yx^{-1})^p = y^p x^{-p}[x^{-1}, y]^{p(p-1)/2}[\eta, y] = [\eta, y],$$

where $\eta \in G'$. It follows that $(yx^{-1})^{p^2} = 1$. Now since $yx^{-1} \notin \Phi(G)$, we see that $(yx^{-1})^p \neq 1$ and $Z(G) = \langle (yx^{-1})^p \rangle$. Hence $|Z(G)| = p$.

Now suppose that for each $g \in \Phi(G)$, $\langle g^p \rangle$ is a proper subgroup of $Z(G)$. Then, by choosing x, y in $G \backslash \Phi(G)$ with $x^p = y^p$ and $xy^{-1} \notin \Phi(G)$, we have

$$(yx^{-1})^p = [x^{-1}, y]^{p(p-1)/2}[\eta_1, x^{-1}][\eta_2, y],$$

where $\eta_1, \eta_2 \in G'$. By Theorem 2.4, $(yx^{-1})^p$ is a generator of G^p; put $a = (yx^{-1})^p$. By our assumption, $[x^{-1}, y]^{p(p-1)/2} = a^{pl}$ for some $l \in \mathbb{N}$. Now since $[\eta_2, x^{-1}]$ and $[\eta_1, y]$ are of order p, we get $a^p = a^{lp^2}$, and so $a^p = 1$. Hence, $|Z(G)| = p$. Obviously $\Phi(G) = G'$.

For the final part of Theorem, we let $x \in Z_2(G)$. If $x^p \neq 1$, then x^p is a generator of $Z(G)$ and, as before, there is an element y in $G \backslash \Phi(G)$ such that $x^p = y^p$, and so $(yx^{-1})^p = y^p x^{-p}[x^{-1}, y]^{p(p-1)/2} = 1$, because $[x^{-1}, y] \in Z(G)$. Consequently, $yx^{-1} \in \Phi(G)$ and we have $y \in \Phi(G)$, a contradiction. □

COROLLARY 2.7. *Let G be a finite p-group of class 3 with non-cyclic abelian second centre $Z_2(G)$. Suppose that $|\Omega_1(G^p)| = p$. If G is a $\mathscr{C}$-group, then*
(i) $Z(G) = G^p$ *and* $|Z(G)| = p$;
(ii) $\Phi(G) = G' = Z_2(G)$, *and* $\exp(\Phi(G)) = p$;
(iii) $p = 3$.

Proof. In view of Theorem 2.4, $Z(G) = G^p$, and $Z_2(G) \leq \Phi(G)$. Since $G' \leq Z_2(G)$, we have $G' = \Phi(G) = Z_2(G)$ and $|Z(G)| = p$. Now G is not regular, by Lemma 2.1(ii) and so $p \leq c\ell(G) = 3$ using [**6**, Chapter 4, 3.13(ii)]. Hence $p = 3$. □

3. An application. In this section we classify the finite 3-generator p-groups G that are $\mathscr{C}$-groups with the following properties:
(i) $Z_2(G)$ is abelian and non-cyclic,
(ii) $|\Omega_1(G^p)| = p$,
(iii) $c\ell(G) = 3$.
There is one family of such groups consisting of four non-isomorphic groups.

We also give an example of a 2-group with abelian non-cyclic second centre and arbitrarily large nilpotency class that is a $\mathscr{C}$-group.

From now on G will stand for a finite p-group in $\mathscr{C}$ satisfying the conditions (i)–(iii).

LEMMA 3.1. *If a, b and c belong to a minimal generating set of G, then*
(i) $\{a, b\} \nsubseteq \mathscr{C}_G([a, b])$,
(ii) $Z(G)$ *intersects* $\langle [a, b], [a, c], [b, c] \rangle$ *trivially*.

Proof. (i) Assume that $\{a, b\} \subseteq \mathscr{C}_G([a, b])$. Then we have $(ab)^3 = a^3 b^3$ and $(ab^{-1})^3 = a^3 b^{-3}$. Since a^3 and b^3 are generators of $Z(G)$ and $|Z(G)| = 3$, $a^3 = b^3$ or $a^3 = b^{-3}$. Thus either $(ab)^3 = 1$ or $(ab^{-1})^3 = 1$. Consequently either $ab \in \Phi(G)$ or $ab^{-1} \in \Phi(G)$, a contradiction.

(ii) Assume that $Z(G)$ intersects $\langle [a, b], [a, c], [b, c] \rangle$ non-trivially. Since G' is elementary abelian, we may suppose that

$$[a, b][a, c]^{\varepsilon_1}[b, c]^{\varepsilon_2} \in Z(G),$$

for some $\varepsilon_i \in \{0, \pm 1\}$, $i = 1, 2$. Clearly $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$ by (i). If $\varepsilon_2 = 0$, then $[a, bc^{\epsilon_1}] \in Z(G)$, because $Z_2(G) = G'$. This is impossible, since a, bc^{ε_1} belong to a minimal generating set of G. Similarly $\varepsilon_1 = 0$ is impossible. We now suppose that $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$. If $\varepsilon_1 = \varepsilon_2$, then $[ab, bc^{\epsilon_1}] \in Z(G)$. But ab, b and bc^{ϵ_1} belong to a minimal generating set, contrary to (i). Also if $\epsilon_1 = -\epsilon_2$, then $[ab^{-1}, bc^{\epsilon_1}] \in Z(G)$, again a contradiction. □

In what follows $d(G)$ denotes the minimal number of generators of G.

Corollary 3.2. *$|G| = 3^4$ if $d(G) = 2$, and $|G| = 3^7$ if $d(G) = 3$.*

Proof. We prove the second part of the Corollary; the first part is established similarly. Let $d(G) = 3$ and $G = \langle a, b, c \rangle$. Since G' is elementary abelian, we have

$$G' = \langle [a, b], [a, c], [b, c] \rangle \times Z(G),$$

by Lemma 3.1(ii), so that $|G'| = 3^4$. Now $G' = \Phi(G)$ shows that $|G| = 3^7$. □

Lemma 3.3. *Let a, b and c be elements of G.*
(i) $(ab)^3 = a^3 b^3 [a, [a, b]]\, [b, [a, b]]^{-1}$.
(ii) $(abc)^3 = a^3 b^3 c^3 [a, x][a, y][b, x]^{-1}[b, y]^{-1}[b, z][c, x]^{-1}[c, y]^{-1}[c, z]^{-1}$, *where* $x = [a, b]$, $y = [a, c]$ *and* $z = [b, c]$.
(iii) $[b, [a, c]] = [a, [b, c]]\, [c, [a, b]]$.
(iv) *If a and b are elements of a minimal generating set of G such that $[b, [a, b]] = 1$, then $b^6 = [a, [a, b]]$.*

Proof. The first two parts are easily checked. (iii) is most conveniently proved by using the identity $((ab)c)^3 = (a(bc))^3$. To prove (iv), we observe that $(ab^{-1})^3 = a^3 b^{-3} [a, [a, b]]^{-1}$, by (i). Now since $(ab)^3$ and $(ab^{-1})^3$ are generators of $Z(G)$, $(ab)^3(ab^{-1})^3 = 1$ or $(ab)^3 = (ab^{-1})^3$. The former shows that $a^3 = 1$, which is impossible. The result is now settled by using the latter. □

Proposition 3.4. *If $d(G) \geq 3$, then G has a minimal generating set containing three elements a, b and c such that*
(i) $[a, [a, b]] = a^3$, $[b, [a, b]] = 1$,
(ii) $[b, [b, c]] = b^3$, $[c, [b, c]] = 1$.

Proof. Suppose that a, b and c are elements of a minimal generating set of G. Without loss of generality, we may assume that $[a, [a, b]] \neq 1$, by Lemma 3.1(i). Since $|Z(G)| = 3$ and $a^3 \in Z(G)$, it follows that $[a, [a, b]] = a^3$ or a^6. In the latter case, if we replace b by b^2, we get $[a, [a, b]] = a^3$, as required. Now if $[b, [a, b]] \neq 1$, then we have

$$[a, [a, b]]^{\varepsilon}[b, [a, b]] = 1,$$

for some $\varepsilon = \pm 1$. Therefore, by setting $b' = a^{\varepsilon} b$, we find that $[b', [a, b']] = 1$. Here we still have $[a, [a, b']] = a^3$ and consequently (i) holds.

Now if $[b,[b,c]] \neq 1$, then we may repeat the above process to obtain the relations $[b,[b,c]] = b^3$ and $[c,[b,c]] = 1$ for a suitable c. We suppose that $[b,[b,c]] = 1$, which implies that $[b,[c^\varepsilon a, b]] = 1$ for every $\varepsilon \in \{-1,0,1\}$. Therefore, $[c^\varepsilon a,[c^\varepsilon a, b]] \neq 1$ by Lemma 3.1(i) which, together with the assumption $[c,[c,a]] \neq 1$, enables us to perform the above process with $a' = c$, $b' = c^\varepsilon a$ and $c' = b$ in order to obtain the desired generators. Hence it suffices to show that $[c,[c,a]] \neq 1$. To see this, we consider the central elements $(abc)^3$ and $(abc^{-1})^3$. If $(abc)^3(abc^{-1})^3 = 1$ then it follows, from the relations of (i) and Lemma 3.3, that $a^3[c,[a,c]] = 1$, which gives us $[c,[c,a]] \neq 1$. Now we assume that $(abc)^3 = (abc^{-1})^3$. Then $c^3[a,y] = [b,y][c,x]$, and hence $(ab^{-1}c)^3 = c^6[a,y]^{-1}[c,y]^{-1}$. If $(ab^{-1}c)^3(abc)^3 = 1$, then $a^3[c,[a,c]] = c^3[a,[a,c]]$, and hence $[c,[a,c]] = (ac^{-1})^3 \neq 1$. Also $(ab^{-1}c)^3 = (abc)^3$ leads to $a^3c^3[a,[a,c]] = 1$, which shows that $[c,[a,c]] = (ac)^{-3} \neq 1$, completing the proof. □

THEOREM 3.5. *Let G be a 3-generator finite p-group of class 3 with non-cyclic abelian second centre $Z_2(G)$ and let $|\Omega_1(G^p)| = p$. If G is a $\mathscr{C}$-group, then G is generated by the elements a, b, c, x, y and z, subject to the following defining relations:*

$a^9 = b^9 = c^9 = x^3 = y^3 = z^3 = 1,\ a^3 = b^6 = c^3,$

$[a^3,b] = [a^3,c] = [x,y] = [x,z] = [y,z] = [b,x] = [c,z] = 1,$

$x = [a,b],\ y = [a,c],\ z = [b,c],$

$[c,y] = 1,\ a^3 = [a,x],\ b^3 = [b,z],$

$[a,y] = a^{6(m-1)(m-2)},\ [b,y] = a^{3(m-n)},\ [a,z] = a^{3(m+n)},\ [c,x] = a^{3n},$

where $m,n \in \{0,1,2\}$. Furthermore, if we denote the above group G by $G(m,n)$ then $G(0,0) \cong G(2,0)$, $G(0,1) \cong G(2,2)$, $G(1,1) \cong G(2,1)$ and $G(0,2) \cong G(1,0) \cong G(1,2)$.

Proof. According to Corollary 2.7, G satisfies the conditions (i)–(iii) of the corollary. Now, by Proposition 3.4, we may choose a minimal generating set $\{a,b,c\}$ in such a way that

$$[a,[a,b]] = a^3,\quad [b,[a,b]] = 1,\quad [b,[b,c]] = b^3,\quad [c,[b,c]] = 1.$$

By Lemma 3.3(iv), we have $a^3 = b^6 = c^3$. For convenience, we set $x = [a,b]$, $y = [a,c]$ and $z = [b,c]$. We now consider the central elements $(abc)^3$, $(abc^{-1})^3$ of G. We claim that $(abc)^3 \neq (abc^{-1})^3$. If this is not the case, in view of Lemma 3.3(iii) and the above relations, we shall have $[a,y] = [b,y][c,x]$. Thus $(abc)^3 = a^3[c,y]^{-1}$, and so $[c,y] \neq a^3$. It follows that $(ac)^3 = (ac^{-1})^3$, by Lemma 3.3(i), and hence $[a,y] = c^6 \neq 1$. Now since $(ab^{-1}c)^3 = a^6[c,y]^{-1}$, we find that $(ab^{-1}c)^3(abc)^3 = 1$, and so $[c,y] = 1$. But $(a^{-1}bc)^3 = [c,y]$, a contradiction. Therefore we must have $(abc)^3(abc^{-1})^3 = 1$. In this case, $[c,y] = a^3 \neq 1$ and hence $(ac)^3 = a^3[a,y]$. Now we obtain

$$(a^{-1}bc)^3 = [a,y][b,y][c,x].$$

We first suppose that $(a^{-1}bc)^3(abc)^3 = 1$. In this case, $[a,y] = 1$ and so $[b,y][c,x] \neq 1$. As before, exactly one of $[a,z]$, $[b,y]$, $[c,x]$ is the identity element (otherwise, $[b,y] = [c,x]^{-1}$ by Lemma 3.3(iv).) Therefore we may assume that

$$[b,y] = a^{3(m-n)},\quad [a,z] = a^{3(m+n)},\quad [c,x] = a^{3n},$$

where $m \in \{1,2\}$ and $n \in \{0,1,2\}$.

We next suppose that $(a^{-1}bc)^3 = (abc)^3$. Then $[b, y][c, x] = 1$ and so we have $[a, z] = [c, x]$ and $[a, y] \neq 1$, which implies that $[a, y] = [c, y]$ (otherwise $(ac^{-1})^3 = 1$.) Therefore, in this case the following defining relations are obtained for G:

$$[b, y] = a^{-3n},\ [a, z] = a^{3n},\ [c, x] = a^{3n},$$

where $n \in \{0, 1, 2\}$.

We are now able to write down a single presentation for G in both cases. On the other hand by using GAP [**4**], one can easily check that each group $G(m, n)$ is a $\mathscr{C}$-group of order 3^7 and that $G(0, 0)$, $G(0, 1)$, $G(0, 2)$ and $G(1, 1)$ are the only non-isomorphic groups among the groups $G(m, n)$ where $m, n \in \{0, 1, 2\}$, as required. □

Deaconescu and Silberberg [**1**] have proved that a finite p-group with non-abelian or cyclic second centre is a $\mathscr{C}$-group if and only if $G \cong Q_{2^n}$ for some n. It seems reasonable to ask whether there are finite 2-groups with non-cyclic abelian second centre that are $\mathscr{C}$-groups. The following example shows that given any positive integer $m \geq 3$, there exists a finite 2-group G with non-cyclic abelian second centre that is a $\mathscr{C}$-group of class m.

Example. Let n be a positive integer, and let

$$G_n = < a, b \mid b^4 = 1, b^2 = a^{2^{n+1}}, b^{-1}a^2b = a^{-2}, [a, b]^{2^n} = 1 > .$$

It is easy to check that the following relations hold in G_n:

$$[a, b]^b = [a, b]^{-1},\ [a, b]^a = a^{-4}[a, b]^{-1},\ [a^2, [a, b]] = 1.$$

Taking $x = a^2$, $y = [a, b]$, and $L = < x, y >$, we observe that L is an abelian subgroup of G_n with $|G_n : L| = 4$. Using the procedure described in [**3**], a presentation on the generators x and y is obtained for L as follows:

$$L = < x, y \mid x^{2^{n+1}} = y^{2^n} = [x, y] = 1 > .$$

Hence G_n is of order 2^{2n+3}, $|a| = 2^{n+2}$ and $|b| = 4$. Next we put $H = < a^4, [a, b] >$ and see that H is an abelian normal subgroup of G_n and that $|G_n : H| = 8$. As G_n/H is abelian and $|G_n/G_n'| = 8$, we have $G_n' = H$. Now by considering the normal subgroup $K = < a^2 >$ of G_n, we find that

$$G_n/K = < \overline{a}, \overline{b} \mid \overline{a}^2 = \overline{b}^2 = 1,\ [\overline{a}, \overline{b}]^{2^n} = 1 > \cong D_{2^{n+2}},$$

where $\overline{g} = Kg$ for any $g \in G_n$.

Hence $Z(G_n/K) = < K[a, b]^{2^{n-1}} >$, and we see that if $z \in Z(G_n)\backslash K$ then $z = k[a, b]^{2^{n-1}}$, where $k \in K$ (because $Z(G_n)K/K \leq Z(G_n/K)$). Therefore,

$$1 = [a, z] = [a, [a, b]^{2^{n-1}}] = [a, [a, b]]^{2^{n-1}} = a^{2^{n+1}}[a, b]^{2^n} = b^2.$$

Since $b^2 \neq 1$, we get $Z(G_n) \leq K$. Now we suppose that z is a generator of $Z(G_n)$, and $z = (a^2)^i$. Then $(a^2)^i = (a^{2i})^b = a^{-2i}$, and hence $(a^4)^i = 1$, which shows that $i = 2^n$. It follows that $z = a^{2^{n+1}} = b^2$, and $Z(G_n) \leq G_n'$.

Finally we show that $c\ell(G_n) = n + 2$. Obviously $\Gamma_2(G_n) = H$. Now since H is abelian, $\Gamma_3(G_n) = [G_n, H] =< a^4, [a, b]^2 >$. Inductively one can show that $\Gamma_i(G_n) =< a^{2^{i-1}}, [a, b]^{2^{i-2}} >$ for $i \geq 3$. Hence $\Gamma_{n+2}(G_n) =< b^2 >$ and $\Gamma_{n+3}(G_n) = 1$, proving that $c\ell(G_n) = n + 2$. Also, since $\Gamma_{n+1}(G_n) \leq Z_2(G_n)$, we see that $Z_2(G_n)$ has a subgroup of type $\mathbb{Z}_2 \times \mathbb{Z}_4$. In fact an easy calculation within G_n shows that $Z_2(G_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Now using the relation $ba^2b^{-1} = a^{-2}$, we observe that for each w in $G_n \backslash \Omega_1(G_n)$, w^2 has one of the following forms: b^2, $(ab)^2$, $(ba)^2$, and a^l, where l is an even positive integer. On the other hand, by using $(ab)^2 = a^2b^2[a, b]$, we get $(ab)^{2^{n+1}} = a^{2^{n+1}} = b^2$, from which we conclude that $b^2 \in< w >$. Hence, if α is a central automorphism of G_n, then $\alpha(w) = wb^{2m} \in< w >$, where $m \in \{0, 1\}$. Also α fixes $\Omega_1(G_n)$ elementwise. This proves that G_n is a $\mathscr{C}$-group.

It is worth noting that $Aut_c(G_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by [**2**]. In fact, $Aut_c(G_n) = \langle\alpha, \beta\rangle$, where $\alpha(a) = a$, $\alpha(b) = b^{-1}$ and $\beta(a) = ab^2$, $\beta(b) = b$. □

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