The Extended Abstracts Of
The 55 th Annual Iranian Mathematics Conference
14-16 August 2024, Ferdowsi University of Mashhad, Iran
03240-74315

GROUPS WITH A SOLUBLE MAXIMAL SUBGROUP

HAMID MOUSAVI*

Abstract. Let G be an infinite group. Assume that M is a soluble maximal subgroup of G with normal supplement N such that $M \cap N \leqslant \mathrm{Core}_G(M)$. We find some necessary condition for solubility of G.

1. Introduction

A subgroup H of G is c-normal if it has a normal supplement N such that $H \cap N \leqslant \mathrm{Core}_G(H)$. Group G is c-simple if it has no nontrivial c-normal subgroup.

Wang [4] introduced the notion of c-normal subgroups and used the c-normality of a maximal subgroup to establish certain criteria for the solubility and supersolubility of a finite group. Many authors research on relationship between c-normality and solubility of group G. In this article, we extend the previous results on groups with a soluble c-normal maximal subgroup to infinite groups.

2. Preliminaries and main results

Let G be a finite group. There are some results that express the relationship between the solubility of G and c-normal maximal subgroups of G.

Theorem 2.1. [4, Theorem 3.1] *Let G be a finite group. Then G is solvable if and only if every maximal subgroup of G is c-normal in G.*

1991 *Mathematics Subject Classification.* 20E28, 20F19.
Key words and phrases. Infinite group, soluble group, maximal subgroup.
* Speaker.

Theorem 2.2. [4, Theorem 3.4] *Let G be a finite group. Then G is soluble if and only if there exists a solvable c-normal maximal subgroup M of G.*

Theorem 2.3. [4, Theorem 3.5] *Let G be a finite group. Then G is soluble if and only if M is c-normal in G for every maximal subgroup M of composite index.*

Let $\mathfrak{X}$ be a class of group. A group G is said locally $\mathfrak{X}$-group if, and only if, every finite subset of G generates a $\mathfrak{X}$-group. If $\mathfrak{X}$ is a finite, nilpotent, and solvable, then G is said to be locally finite, locally nilpotent, and locally soluble.

Definition 2.4. [2, Page 19] Let G be a group and p be a prime number. The set of the p-subgroups of G is inductive with respect to the relation of inclusion, thus every p-subgroup of G is contained in some maximal p-subgroup. The maximal p-subgroups of G are called Sylow p-subgroups of G and their set will be denoted in the next by $\mathrm{Syl}_p(G)$.

All Sylow p-subgroups of a finite group are conjugated by Sylow's theorem. However, this is not necessarily true in infinite groups and requires additional assumptions. Classically known are the two following theorems; the second of them appeared for the first time in a work by Dietzmann-Kurosch-Uzkov [1] published in far 1938.

Theorem 2.5. [2, Theorem 1.2.3] *Let G be a locally finite group having a finite Sylow p-subgroup. Then the Sylow p-subgroups of G are conjugate.*

Theorem 2.6. [2, Theorem 1.2.4] *If there exists a $P \in \mathrm{Syl}_p(G)$ having only finite many conjugates, then the Sylow p-subgroups of G with be conjugate and their number will be $\equiv 1 \pmod p$.*

The following theorem represents one of the generalizations of the Schur-Zassenhaus splitting theorem for locally finite groups.

Theorem 2.7. [2, Theorem 2.2.4] *Let N be a Hall normal subgroup of a locally finite group G, with G/N at most countably infinite. Then G splits over N.*

Lemma 2.8. [2, Theorem 1.1.2]

(1) *Locally soluble chief factors of a locally finite group are elementary abelian.*
(2) *The chief factors of a locally soluble group are abelian.*
(3) *The chief factors of a locally polycyclic group are elementary.*

(4) *The chief factors of a locally supersoluble group are cyclic of prime orders.*
(5) *The chief factors of a locally nilpotent group are central and of prime orders.*

Theorem 2.9 ([4], Theorem 3.4)**.** *Let G be a finite group, and let M be a soluble maximal subgroup of G. If M has a normal supplement N such that $N \cap M \leqslant \mathrm{Core}_G(M)$, then G is soluble.*

We prove the above theorem again with a similar method but it can be extended to infinite groups

Proof. We can assume that $\mathrm{Core}_G(M) = 1$, since solubility of $G/\mathrm{Core}_G(M)$ implies the solubility of G. Let L be the minimal normal subgroup of M and $P \in \mathrm{Syl}_p(G)$ such that $L \leqslant P$, where $p \mid |L|$. Since $\mathcal{C}_N(L)$ and $\mathcal{C}_{P\cap N}(L)$ are subgroups of $\mathcal{N}_G(L) = M$, so $\mathcal{C}_N(L) = 1$ and $P \cap N = 1$, therefore $p \nmid |N|$ (see [3, Ch-6, Theorem 2.3]). By [3, Ch-6, Theorem 2.2(i)], we can assume that there exist $Q \in \mathrm{Syl}_q(N)$ such that $Q^L = Q$, where q is prime divisor of $|N|$. Since for any $m \in M$, $(Q^m)^L = (Q^L)^m = Q^m$, so by [3, Ch-6, Theorem 2.2(ii)], $Q = Q^m$. Therefore $G = QM$ and so G is soluble. $\square$

Corollary 2.10. *Let G be a finite group and M be a maximal soluble subgroup of G. If M has a normal supplement N such that $M \cap N \leqslant \mathrm{Core}_G(M)$, then $M = G$ is soluble.*

Proof. Assume that $M \neq G$. Since G is finite, we can chose a subgroup $M_1 \leqslant G$, such that M is a maximal subgroup of M_1. By assumption, $M_1 = (N \cap M_1)M$ and $N \cap M \leqslant \mathrm{Core}_G(M) \leqslant \mathrm{Core}_{M_1}(M)$. According to Theorem 2.9, M_1 is soluble, a contradiction. $\square$

Corollary 2.11. *Let G be a group with soluble maximal subgroup M of finite index. If M has a normal supplement N such that $M \cap N \leqslant \mathrm{Core}_G(M)$, then G is soluble.*

Proof. Since $\bar{G} = G/\mathrm{Core}_G(M)$ is finite with a soluble maximal subgroup $\bar{M}$. Since $\bar{N} \cap \bar{M} = 1$, thus $\bar{G}$ and so G is soluble. $\square$

Theorem 2.12. *Let G be a locally finite group with a core free locally soluble maximal subgroup M and N be a normal complement of M. Assume that,*

(i) *a minimal normal subgroup L of M, is finite;*
(ii) *any Sylow p-subgroup of $G^* = NL$ satisfies the normalizer condition, where $\pi(L) = \{p\}$;*

Then $G = QM$, where $Q \trianglelefteq G$ is an elementary abelian Sylow q-subgroup and $q \neq p$. Furthermore, if M is soluble then G is soluble.

Proof. Let L be a minimal normal subgroup of M. Since M is locally soluble, L is elementary abelian for some prime $p \in \pi(M)$ (by Lemma 2.8-(1)). As $\mathcal{C}_N(L) \leqslant \mathcal{N}_G(L) = M$, so $\mathcal{C}_N(L) = 1$. Set $G^* = NL$ and assume that $P \in \mathrm{Syl}_p(G^*)$, such that $L \leqslant P$. If $P \cap N \neq 1$, then $L < L(P \cap N) \leqslant P$. By assumption L is proper subgroup of its normalizer in G^*, so $L \trianglelefteq G$, which is contract to $\mathrm{Core}_G(M) = 1$. Therefor $p \notin \pi(N)$.

Step 1: N contains a L-invariant Sylow q-subgroup for some prime $q \in \pi(N)$.

Assume that $Q \in \mathrm{Syl}_q(N)$, then $G^* = N\mathcal{N}_{G^*}(Q)$ by Frattini argument. Since $(\pi(G^*/N), p) = 1$, so $\mathcal{N}_{G^*}(Q)$ spited over $\mathcal{N}_N(Q)$ (by Theorem 2.7) and hence $\mathcal{N}_{G^*}(Q) = \mathcal{N}_N(Q)S$ for some p-subgroup S, therefore, $G^* = NS$. Since for some $y \in N$, $L = S^y$, (by Theorem 2.5), so Q^y is L-invariant.

Step 2: Any L-invariant Sylow q-subgroups Q of N is M-invariant.

Let $Q \in \mathrm{Syl}_q(N)$ be L-invariant. For any $m \in M$,

$$(Q^m)^L = (Q^L)^m = Q^m.$$

Now for some $y \in N$, $Q = Q^{my}$. As Q^{my} is L^y-invariant, so Q is L and L^y-invariant. Thus L and L^y both are Sylow p-subgroups of $\mathcal{N}_{G^*}(Q)$, hence for some $x \in \mathcal{N}_{G^*}(Q)$, $L = L^{yx}$. Therefore $[L, xy] \in L \cap N = 1$ and so $xy \in \mathcal{C}_N(L) = 1$. Then $y = x^{-1}$ and $Q^m = Q^x = Q$.

Therefore $G = QM$ and $Q \trianglelefteq G$. Since Q is locally nilpotent and M is maximal subgroup of G, so Q is elementary abelian q-group. □

In Theorem 2.12, if Q is finite, then G is finite, for $G/\mathcal{C}_M(Q) \hookrightarrow \mathrm{Aut}(Q)$ and $\mathcal{C}_M(Q) \leqslant \mathrm{Core}_G(M) = 1$.

Corollary 2.13. *Let G be a locally finite group with a soluble maximal subgroup M. Assume that N be a normal supplement of M such that $N \cap M \leqslant \mathrm{Core}_G(M)$. If*

(i) *a minimal normal subgroup L/C of M/C, is finite, where $C = \mathrm{Core}_G(M)$;*
(ii) *any Sylow p-subgroup of $G^* = NL/C$ satisfies the normalizer condition, where $\pi(L) = \{p\}$.*

Then G is soluble.

Proof. With lose of generality we can assume that $\mathrm{Core}_G(M) = 1$. So $G = NM$ and N is complement of M in G. By Theorem 2.12, the result is achieved. □

References

[1] A.P. Dietzmann, A.G. Kubosch, A.I. Uzkov, *Sylowsche untergruppen von unendlichen gruppen,* Mat. Sbornik, 3 (1938), 179–185.

[2] M. Curzio, *Some problems of Sylow type in locally finite,* London Academic Press, Vol 5, 1979.

[3] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.

[4] Y. Wang, *c-Normality of groups and its properties,* J. Algebra 180 (1996) 954–965.

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

E-mail address: hmousavi@tabrizu.ac.ir